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Compactness and Connectedness Concept in Intuitionistic Fuzzy Topological Spaces

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COMPACTNESS AND CONNECTEDNESS CONCEPT IN INTUITIONISTIC FUZZY TOPOLOGICAL SPACES

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Ph. D. Thesis

Submitted

By

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By

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Declaration

This is to confirm that the research presented in this dissertation entitled **“COMPACTNESS AND CONNECTEDNESS CONCEPT IN INTUITIONISTIC FUZZY TOPOLOGICAL SPACES”** has been carried out by me. All the findings reported in this dissertation are absolutely based on my own work. Neither the dissertation nor any part of it has been submitted for any degree/diploma or any other academic award anywhere before.

December, 2020 (Md. Aman Mahbub)

Author

Recommendation of supervisors

This is to certify that the present work entitled **"COMPACTNESS AND CONNECTEDNESS CONCEPT IN INTUITIONISTIC FUZZY TOPOLOGICAL SPACES"** has been carried out by Md. Aman Mahbub under our supervision at the Department of Mathematics, University of Rajshahi, Bangladesh. We recommend that the prepared contents of this dissertation can be accepted in partial fulfillment of the requirements for the degree, Doctor of Philosophy (PhD) at the Department of Mathematics, under the faculty of Science, University of Rajshahi, Bangladesh. The results explained in this dissertation have not been submitted elsewhere for the award of any degree or diploma. His dissertation is worthy of presentation to the University of Rajshahi for the award of the degree of Doctor of Philosophy in Mathematics.

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Published Papers

(1) **Md. Aman Mahbub**, Md. Sahadat Hossain and M. Altab Hossain, 'Some Properties of Compactness in Intuitionistic Fuzzy Topological Spaces', *Intern. J. Fuzzy Mathematical Archive,* Vol. 16, No. 2, pp.39-48, 2018.

(2) **Md. Aman Mahbub**, Md. Sahadat Hossain, M. Altab Hossain, 'Separation Axioms in Intuitionistic Fuzzy Compact Topological Spaces', *Journal of Fuzzy Set Valued Analysis*, 2019(1), pp. 14-23, 2019.

(3) **Md. Aman Mahbub**, Md. Sahadat Hossain, M. Altab Hossain, 'ON Q-COMPACTNESS IN INTUITIONISTIC FUZZY TOPOLOGICAL SPACES', *J. Bangladesh Acad. Sci.*, Vol. 43, No. 2, pp.197-203, 2019.

Abstract

The fundamental concept of a fuzzy set and fuzzy set operations was first introduced by L. A. Zadeh (Zadeh, 1965) in 1965 and it provides a natural foundation for treating mathematically the fuzzy phenomena, which exists pervasively in our real world and for building new branches of fuzzy mathematics. This also provides a natural frame work for generalizing various branches of mathematics such as fuzzy topology, fuzzy group, fuzzy rings, fuzzy vector spaces, fuzzy number, fuzzy system, fuzzy function, fuzzy relation, fuzzy logic and fuzzy computation. The concepts of fuzzy topology was introduced by C. L. Chang (Chang, 1968) in 1968 based on fuzzy set. Ming and Ming (Pao-Ming & Ying-Ming, 1980) (Pao-Ming & Ying-Ming, 1980), Khedr (Khedr et al., 2001), Hutton (Hutton, 1975), Azad (Azad, 1981), Ali (Ali, 1992) (Ali et al., 1990), Lowen (Lowen, 1976) etc. discussed various properties of fuzzy topology using fuzzy sets and fuzzy topology.

Fuzzy compactness occupies a very important place in fuzzy topological spaces and so does some of its forms. Fuzzy compactness first discussed by C. L. Chang [C. L. Chang, Fuzzy Topological Spaces, *J. Math. Anal. Appl.*, **24**(1968), 182–190], T. E. Gantner et al. [T. E. Gantner, R. C. Steinlage and R. H. Warren, Compactness in Fuzzy Topological Spaces, *J. Math. Anal. Appl.*, **62**(1978), 547–562] introduced α -compactness, A. D. Concilio and G. Gerla [A. D. Concilio and G. Gerla, Almost Compactness in Fuzzy Topological Spaces, *Fuzzy Sets and Systems*, **13**(1984), 187–192] discussed almost compact spaces and M. N. Mukherjee and A. Bhattacharyya [M. N. Mukherjee and A. Bhattacharyya, α -Almost Compactness for Crisp Subsets in a Fuzzy Topological Spaces, *J. Fuzzy Math*, **11**(1) (2003), 105–113] discussed almost α -compact spaces. After two decades, in 1983, Atanassov (K. T. Atanassov, “Intuitionistic Fuzzy Sets,” *VII ITKR`s*

Session, (V, Sgurev, Ed.), Sofia (1983), Bulgaria) introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets which looks more accurately to uncertainty quantification and provides the opportunity to precisely model the problem based on the existing knowledge and observations. An intuitionistic fuzzy set (A-IFS), developed by Atanassov (K. T. Atanassov, “Intuitionistic Fuzzy Sets,” Theory and Applications, *Springer-Verlag* (1999), Heidelberg, New York & K. T. Atanassov, “Intuitionistic Fuzzy Sets,” *Fuzzy Sets and Systems* (1986), vol. 20, 87 – 96) is a powerful tool to deal with vagueness. A prominent characteristic of A-IFS is that it assigns to each element a membership degree and a non-membership degree, and thus, A-IFS constitutes an extension of Zadeh’s fuzzy set. He added a new component (which determines the degree of non-membership) in the definition of fuzzy set. The fuzzy sets give the degree of membership of an element in a given set (and the non-membership degree equals one minus the degree of membership), while intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership which are more-or-less independent from each other, the only requirement is that the sum of these two degrees is not greater than 1. In the last few years various concepts in fuzzy sets were extended to intuitionistic fuzzy sets. Intuitionistic fuzzy sets have been applied in a wide variety of fields including computer science, engineering, mathematics, medicine, chemistry and economics (K. P. Huber and M. R. Berthold, “Application of Fuzzy Graphs for Metamodeling”, *Proceedings of the 2002 IEEE Conference*, 640 644). In 1997, Coker [D. Coker, “An Introduction to Intuitionistic Fuzzy Topological Space,” *Fuzzy Sets and Systems* (1997), vol. 88, 81 –89] introduced the concept of intuitionistic fuzzy topological spaces. S. Bayhan and D. Coker, “On Fuzzy Separation Axioms in Intuitionistic Fuzzy Topological Space,” *BUSEFAL* (1996), vol. 67, 77 –87, D. Coker and A. Es. Hyder, “On Fuzzy Compactness in Intuitionistic Fuzzy Topological Spaces,” *The Journal of Fuzzy*

Mathematics (1995), vol. 3, no. 4, 899 –909, S. Ozcag and D. Coker, "On Connectedness in Intuitionistic Fuzzy Special Topological Spaces," *Int. J. Math. Math. Sciences* (1998), vol. 21, no. 1, 33 –40] gave some other concepts of intuitionistic fuzzy topological spaces, such as fuzzy continuity, fuzzy compactness, fuzzy connectedness, fuzzy Hausdorff space and separation axioms in intuitionistic fuzzy topological spaces. After this, many concepts in fuzzy topological spaces are being extended to intuitionistic fuzzy topological spaces.

Recently many fuzzy topological concepts such as fuzzy compactness [D. Coker and A. Es. Hyder, "On Fuzzy Compactness in Intuitionistic Fuzzy Topological Spaces," *The Journal of Fuzzy Mathematics* (1995), vol. 3, no. 4, 899 –909], fuzzy connectedness [N. Turanli and D. Coker, "Fuzzy Connectedness in Intuitionistic Fuzzy Topological Spaces," *Fuzzy Sets and Systems* (2000), vol. 116, no. 3, 369 –375], fuzzy separation axioms[S. Bayhan and D. Coker, "On Separation Axioms in Intuitionistic Topological Space," *Int. J. of Math. Sci*. (2001), vol. 27, no. 10, 621 –630], fuzzy continuity [H. Gurcay, D. Coker and A. Es. Hayder, "On Fuzzy Continuity in Intuitionistic Fuzzy Topological Spaces," *The Journal of Mathematics of Fuzzy Mathematics* (1997), vol. 5, 365 –378], fuzzy g-closed sets[S. S. Thakur and Rekha Chaturvedi, "Generalized Closed Set in Intuitionistic Fuzzy Topology," *The Journal of Fuzzy Mathematics* (2008), vol. 16, no. 3, 559 –572] and fuzzy g-continuity[S. S. Thakur and Rekha Chaturvedi, "Generalized Continuity in Intuitionistic Fuzzy Topological Spaces," *Notes on Intuitionistic Fuzzy Set* (2006), vol. 12 no. 1, 38 –44] have been generalized for intuitionistic fuzzy topological spaces.

Deschrijver and Kerre [G. Deschrijver and E. E. Kerre, "On the Relationship between Some Extensions of Fuzzy Set Theory," *Fuzzy Sets and Systems* (2003), vol. 133, 227–235], Goguen [J. Goguen, "L-fuzzy Sets," *J. Math. Anal. Applicat.* (1967), vol. 18, 145–

174] established the relationships between IFSs, L-fuzzy sets, interval-valued fuzzy sets, and interval-valued IFSs.

Hausdorffness in an intuitionistic fuzzy topological space has been introduced earlier by Coker[D. Coker, “An Introduction to Intuitionistic Fuzzy Topological Space,” *Fuzzy Sets and Systems* (1997), vol. 88, 81 –89]. Lupianez [F. G. Lupianez. “Hausdorffness in Intuitionistic Fuzzy Topological Spaces,” *Mathware and Soft Computing* (2003), vol. 10, 17 –22] has also defined new notions of Hausdorffness in the intuitionistic fuzzy sense and obtained some new properties in particular in convergence.

Separation axioms is very impotent in any kind of topological space. Bayhan and Coker (Bayhan & Coker, 1996) introduced fuzzy separation axioms in intuitionistic fuzzy topological spaces. Singh and Srivastava (Singh & Srivastava, 2012), Yue and Fang (Yue & Fang, 2006), Bhattacharjee and Bhaumik (Bhattacharjee & Bhaumik, 2012) also studied separation axioms in intuitionistic fuzzy topological spaces.

The purpose of this thesis is to suggest new definitions of compactness and connectedness axioms in intuitionistic fuzzy topological spaces. We have studied several features of these definitions and the relations among them. We have also shown ‘good extension’ properties of all these spaces. Our criteria for definitions have been preserved as much as possible the relations between the corresponding separation properties for intuitionistic fuzzy topological spaces.

The materials of this thesis have been divided into six chapters. A brief scenario of which we have presented as follows:

Chapter one incorporates some of the basic definitions and results of general sets, fuzzy sets, intuitionistic sets, intuitionistic fuzzy sets and topologies based on such sets. In this chapter, subspace of topological space, product space and mapping in topological spaces has been discussed, which are to be used as references for understanding the next chapters. Most of the results are quoted from various research papers and books.

Our main works start from chapter two. In this chapter, we give seven new notions of intuitionistic fuzzy compact (in short, IF-Compact) space and investigate some relationship among them. At first we show that all these notions satisfy ‘good extension’ property. Furthermore, it proves that these intuitionistic fuzzy compact spaces are hereditary and productive. Finally, we observe that all concepts are preserved under one-one, onto and continuous mapping.

In chapter three, we have introduced Q-compactness in intuitionistic fuzzy compact topological spaces. Furthermore, we have established some theorems and examples of Q-compactness in intuitionistic fuzzy topological spaces and discussed different characterizations of Q-compactness.

Also we have defined $\delta - Q$ compactness, $Q - \sigma$ compactness and $\delta - Q - \sigma$ compactness in intuitionistic fuzzy topological spaces and found different properties between Q-compactness and $\delta - Q$ compactness, $Q - \sigma$ compactness and $\delta - Q - \sigma$ compactness in intuitionistic fuzzy topological spaces.

In fourth chapter, we discusses various type of compactness in intuitionistic fuzzy topological spaces. Almost compact fuzzy sets was first constructed by Concilio and Gerla which is local property. Here we give wo new possible notions of almost compactness in intuitionistic fuzzy topological spaces are studied and investigated some of their properties. We show that these notions satisfy hereditary and productive property of intuitionistic fuzzy topological spaces. Under some conditions it is shown that image and preimage preserve intuitionistic fuzzy topological spaces. Also we give three new notions of I -compactness, C -compactness and $I-C$ -compactness in intuitionistic fuzzy topological spaces and investigate some relations between our notionss.

At last we give three new notions of paracompactness and one new notion of σ-compactness in intuitionistic fuzzy topological spaces and established some properties of them.

In chapter five, we give some new notions of separated, connectedness and totally connectedness and one notions of T_1-space in intuitionistic fuzzy topological space and investigate some relationship among them. Also we find a relation about classical topology and intuitionistic fuzzy topology. Further, we show that connectedness in intuitionistic fuzzy topological spaces are productive.

In the chapter six, we have introduced (r, s)-connectedness in intuitionistic fuzzy topological spaces. Furthermore, we have established some theorems and examples of (r, s)-connectedness in intuitionistic fuzzy topological spaces and discussed different characterizations of (r, s)-connectedness.

Notations

X, Y	: Non empty set
$\lambda\,,\ \mu, \nu$	: Fuzzy set
$\underline{0}$	: Empty fuzzy set
$\underline{1}$	: Hole fuzzy set
T	: Topology
$\mathcal{T}$	: Intuitionistic topology
t	: Fuzzy topology
τ	: Iintuitionistic fuzzy topology
$A = (A_1, A_2)$	: Iintuitionistic set
$\phi_{\sim} = (\phi, X)$	: Iintuitionistic empty set
$X_{\sim} = (X, \phi)$	: Iintuitionistic hole set
$A = (\ \mu_A, \nu_A)$	: Iintuitionistic fuzzy set
$0_{\sim} = (\underline{0}, \underline{1})$	: Iintuitionistic fuzzy empty set
$1_{\sim} = (\underline{1}, \underline{0})$.	: Iintuitionistic fuzzy hole set
$\mathrm{Cl(A)},\ \overline{\mathrm{A}}$	: Closure of *A*
$\mathrm{Int(A)},\ \mathrm{A}^{\mathrm{o}}$	: Interior of *A*
A^{c}	: Complement of *A*
$1_{\mathrm{A}}(\mathrm{x})$	: Characteristic function of *A*

Acronyms

FCS	: Fuzzy Closed Set
FOS	: Fuzzy Open Set
FS	: Fuzzy Set
FT	: Fuzzy Topology
FTS	: Fuzzy Topological Space
ICS	: Intuitionistic Closed Set
IF	: Intuitionistic Fuzzy
IFCS	: Intuitionistic Fuzzy Closed Set
IFOS	: Intuitionistic Fuzzy Open Set
IFS	: Intuitionistic Fuzzy Set
IFT	: Intuitionistic Fuzzy Topology
IFTS	: Intuitionistic Fuzzy Topological Space
IOS	: Intuitionistic Open Set
IS	: Intuitionistic Set
IT	: Intuitionistic Topology
ITS	: Intuitionistic Topological Space
TS	: Topological Space

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Chapter One

Preliminary

This chapter contains various results and concepts of the general sets, fuzzy sets, intuitionistic sets, intuitionistic fuzzy sets, general topology, fuzzy topology, intuitionistic topology, intuitionistic fuzzy topology, subspaces of general topological space, subspace of fuzzy topological space, subspace of intuitionistic fuzzy topological space, fuzzy product topological spaces, intuitionistic fuzzy product topological spaces, compactness and connectedness in different topological spaces and its properties which are to be used as ready references for understanding the subsequent chapters. Most of the results are quoted from various research papers and books.

1.1 Classical Topology

The concept of a set is fundamental in Mathematics and intuitively can be described as a collection of objects possibly linked through some properties. A classical set has clear boundaries, i.e., $x \in A$ or $x \notin A$ exclude any other possibility.

Definition 1.1.1. (Lipschutz, 1965): Suppose that to each element of a set X there is assigned a unique element of a set Y. The collection f of such assignment is called a function or mapping from X into Y and is written as $f: X \rightarrow Y$.

The unique element in Y assigned to $x \in X$ by this function f is called the value of f at x or the image of x under f and is denoted by $f(x)$. The set X is called the domain of f and Y is called the co-domain of f. The set of image points of Y is called the range of f.

Definition 1.1.2. (Zadeh, 1965): Let X be a set and A be a subset of X ($A \subseteq X$). Then the function $1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$ is called the characteristic function of the set A in X.

Classical sets and their operations can be represented by their characteristic functions. Let us consider the union $A \cup B = \{x \in X : x \in A \text{ or } x \in B\}$. Its characteristic function

$$1_{A \cup B}(x) = \max\{1_A(x), 1_B(x)\}.$$

For the intersection $A \cap B = \{x \in X : x \in A \text{ and } x \in B\}$ the characteristic function is

$$1_{A \cap B}(x) = \min\{1_A(x), 1_B(x)\}.$$

If we consider the complement of A in X, $A^c = \{x \in X : x \notin A\}$ it has the characteristic function $1_{A^c}(x) = 1 - 1_A(x)$.

Definition 1.1.3. (Lipschutz, 1965): A function $f: X \to Y$ is called one-one (or one-to-one or 1-1 or injective**)** if distinct elements in X have distinct images, i.e., if

$$f(a) = f(b) \Rightarrow a = b$$

Definition 1.1.4. (Lipschutz, 1965): A function $f: X \to Y$ is called onto (or Surjective**)** if every element y in Y is the image of some element x in X, i.e., if $y \in Y \Rightarrow \exists x \in X$ such that $f(x) = y$. Hence f is onto if $f(A) = \{y \in Y : y = f(x)\}$ for some $x \in X = B$.

Definition 1.1.5. If the function $f: X \to Y$ is both one-one and onto then the function $g: Y \to X$ is called the inverse function of f if g(y) = x when $f(x) = y$. This inverse function is denoted by $f^{-1}: Y \to X$.

Definition 1.1.6. (Lipschutz, 1965): Let X be a non-empty set. A class T of subsets of X is called a topology on X if T satisfy the following conditions.

(1) $X, \phi \in T$,

(2) $A \cap B \in T$ for all $A, B \in T$,

(3) $\cup A_i \in T$ for any class $A_i \in T$.

The members of T are called open sets, their complements are called closed sets and the set X together with the topology T, i.e. the pair (X, T) is called a topological space (TS, in short).

Definition 1.1.7. (Lipschutz, 1965): Let A be a subset of a topological space X. The closure of A is denoted by $\overline{A}$ is the intersection of all closed superset of A. i.e. $\overline{A} = \cap \{F: F \text{ is closed and } A \subset F\}$

Observe that $\overline{A}$ is the smallest closed super set A and if A is closed then $A = \overline{A}$.

Definition 1.1.8. (Lipschutz, 1965): Let A be a subset of a topological space X. The interior of A is denoted by A^o is the union of all open subset of A. i.e.

$$A^o = \cup \{G: G \in T \text{ and } G \subset A\}$$

Observe that A^o is the largest open subset A and if A is open then $A = A^o$.

Definition 1.1.9. (Lipschutz, 1965): Let (X, T) be a topological space and $A \subset X$. The class T_A of all intersections of A with members of T, i.e. $T_A = \{A \cap G: G \in T\}$ is a topology on A relative to T. The topological space (A, T_A) is called subspace of (X, T).

Definition 1.1.10. (Lipschutz, 1965): Let (X, T) and (Y, T^*) be topological spaces and $f: (X, T) \to (Y, T^*)$, i.e. $f: X \to Y$ is a function.

(1) f is called an open function if image of every open set is open.

(2) f is called a closed function if image of every closed set is closed.

(3) f is called a continuous function if pre-image of every open set is open or equivalently pre-image of every closed set is closed.

Definition 1.1.11. (Lipschutz, 1965): Two topological spaces X and Y are called homeomorphic or topologically equivalent if there exist a bijective (i.e. one-one and onto) function $f: X \to Y$ such that f and f^{-1} are continuous.

Definition 1.1.12. (Lipschutz, 1965): A property P of sets is called topological property if whenever a topological space (X, T) has property P, then every topological space homeomorphic to (X, T) also has P.

1.2 Fuzzy Set and Fuzzy Topological Spaces

Definition 1.2.1. (Zimmermann, 1992)**:** Let X be a non-empty set and I is the closed unit interval [0,1]. A fuzzy set (FS, in short) in X is a set of ordered pairs $\{(x, u(x)): x \in X\}$ where $u: X \to I$ which assigns to every element $x \in X$. $u(x)$ denotes the degree (or the grade) of membership of x. The set of all fuzzy sets in X is denoted by I_X. A member of I_X may also be called a fuzzy subset of X. The fuzzy set $\{(x, u(x)): x \in X\}$ is usually denoted by u.

Remark 1.2.2. Every subset A of X may be consider as an FS in X by its characteristic function 1_A.

Definition 1.2.3. (Pao-Ming & Ying-Ming, 1980)**:** A fuzzy set is empty if and only if its grade of membership is identically zero in X. We denote it by $\underline{0}$.

Definition 1.2.4. (Pao-Ming & Ying-Ming, 1980)**:** A fuzzy subset is whole if and only if its grade of membership is identically 1 in X. We denote it by $\underline{1}$.

Definition 1.2.5. (Pao-Ming & Ying-Ming, 1980): A fuzzy singleton or fuzzy point x_r is a fuzzy set in X defined by

$$x_r(y) = \begin{cases} 0 \text{ if } y \neq x \\ r \text{ if } y = x \end{cases}$$

Here x is called the support of the fuzzy point x_r. Two fuzzy singletons are said to be distinct if their supports are distinct. A fuzzy point x_r is said to belongs to a fuzzy set u if $r < u(x)$,

Definition 1.2.6. (Chang, 1968)**:** Let u and v be two fuzzy sets in X. Then we define

(1) $u = v$ iff $u(x) = v(x)$ for all $x \in X$.

(2) $u \subset v$ iff $u(x) \leq v(x)$ for all $x \in X$.

(3) $u = v^c$ iff $u(x) = v^c(x) = 1 - v(x)$ for all $x \in X$.

obviously $(u^c)^c = u$.

(4) $\lambda = u \cup v$ iff $\lambda(x) = (u \cup v)(x) = \max(u(x), v(x))$ for all $x \in X$.

In general if $\{u_i\}$ is a family of fuzzy sets in X then $\cup\, u_i(x) = \sup(u_i(x))$ for all $x \in X$.

(5) $\lambda = u \cap v$ iff $\lambda(x) = (u \cap v)(x) = \min(u(x), v(x))$ for all $x \in X$.

In general if $\{u_i\}$ is a family of fuzzy sets in X then $\cap\, u_i(x) = \inf(u_i(x))$ for all $x \in X$.

Definition 1.2.7. (Chang, 1968)**:** Let X be a non empty set. A family t of fuzzy sets in X is called a fuzzy topology on X if the following conditions hold.

(1) $\underline{0}\,, \underline{1} \in t$,

(2) $\lambda \cap \mu \in t$ for all $\lambda, \mu \in t$,

(3) $\cup\, \lambda_j \in t$ for any arbitrary family $\{\lambda_j \in t\,, j \in J\}$.

The pair (X, t) is called a fuzzy topological space (FTS, in short) and any members of t is called fuzzy open set (FOS, in short). The complement of an FOS is called fuzzy closed set (FCS, in short). i.e. a fuzzy set v in X is closed iff $1 - v \in t$.

We know that every subset A of X may be regarded as a fuzzy set in X. So we have the following theorem

Theorem 1.2.8. Let (X, T) be a topological space. Then (X, t) is a fuzzy topological space where $t = \{1_A : A \in T\}$.

Proof: Since T is a topology on X, then $\phi, X \in T$. But $1_\phi = \underline{0}$ and $1_X = \underline{1}$. Therefore $\underline{0}, \underline{1} \in t$.

Let $1_A, 1_B \in t$, Then clearly $A, B \in T$. Since T is a topology on X, then $A \cap B \in T$. By the definition of t, it is clear that $1_{A\cap B} \in t$. But $1_A \cap 1_B = 1_{A\cap B}$. So $1_A \cap 1_B \in t$.

Again let $1_{A_j} \in t$ for $j \in J$. So clearly $A_j \in T$ for each $j \in J$. Since T is a topology, then $\cup A_j \in T$. By the definition of t, $1_{(\cup A_j)} \in t$. Now $\cup 1_{A_j} = 1_{(\cup A_j)}$. So $\cup 1_{A_j} \in t$. Therefore t is a fuzzy topology on X, i.e. (X, t) is a fuzzy topological space.

Example 1.2.9. Let $X = \{x, y\}$. Then $t = \{\underline{0}, \underline{1}, \lambda, \mu, u, v\}$ is a fuzzy topology on X where $\underline{0} = \{(x, 0), (y, 0)\}$, $\underline{1} = \{(x, 1), (y, 1)\}$, $\lambda = \{(x, 0.6), (y, 0.3)\}$, $\mu = \{(x, 0.2), (y, 0.7)\}$, $u = \{(x, 0.2), (y, 0.3)\}$, $v = \{(x, 0.6), (y, 0.7)\}$.

Definition 1.2.10. (Pao-Ming & Ying-Ming, 1980): Let u be a fuzzy set in (X, t). The interior of u is defined as the union of all t-open fuzzy sets contained in u. It is denoted by u^o, i.e. $u^o = \cup\{\lambda: \lambda \in t \text{ and } \lambda \subset u\}$. Evidently u^o is the largest open fuzzy set contained in u and $(u^o)^o = u^o$. If u is open then $u^o = u$.

Definition 1.2.11. (Pao-Ming & Ying-Ming, 1980): Let v be a fuzzy set in (X, t). The closure of v is defined as the intersection of all t-closed fuzzy sets containing v. It is denoted by $\bar{v}$, i.e. $\bar{v} = \cap\{\lambda: \lambda^c \in t \text{ and } \lambda \supset v\}$. Evidently $\bar{v}$ is the smallest closed fuzzy set containing v and $\overline{(\bar{v})} = \bar{v}$. If v is closed then $\bar{v} = v$.

Definition 1.2.12. (Pao-Ming & Ying-Ming, 1980)**:** Let (X, t) be a fuzzy topological space and A be an ordinary subset of X. The class $t_A = \{u|A: u \in t\}$ determines a fuzzy topology on A where $u|A$ is a fuzzy set in A defined by $u|A(a) = u(a)$ for all $a \in A$ and (A, t_A) is a fuzzy topological space. This space is called a subspace of (X, t).

Definition 1.2.13. (Chang, 1968)**:** Let f be a mapping from a set X into a set Y, i.e. $f: X \rightarrow Y$ and u be a fuzzy set in X. Then f and u induce a fuzzy set v in Y defined by

$$f(u)(x) = v(x) = \begin{cases} \sup\limits_{x \in f^{-1}(y)} u(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

Definition 1.2.14. (Chang, 1968)**:** Let f be a mapping from a set X into a set Y and v be a fuzzy set in X. Then the inverse of v, written as $f^{-1}(v)$ is a fuzzy set in X and is defined by $f^{-1}(v)(x) = v(f(x))$ for all $x \in X$.

Theorem 1.2.15. Let f be a mapping from X into Y. Then

(1) for any fuzzy v in Y, $f^{-1}(v^c) = \left(f^{-1}(v)\right)^c$ and $f(f^{-1}(v)) \subset v$.

(2) for any fuzzy u in X, $f(u^c) \supset \left(f(u)\right)^c$ and $u \subset (f^{-1}(f(u))$.

(3) for any fuzzy sets v_1 and v_2 in Y, $v_1 \subset v_2 \Rightarrow f^{-1}(v_1) \subset f^{-1}(v_2)$.

(4) for any fuzzy sets u_1 and u_2 in X, $u_1 \subset u_2 \Rightarrow f(u_1) \subset f(u_2)$.

Definition 1.2.16. (Pao-Ming & Ying-Ming, 1980)**: A** function $f: (X, t) \rightarrow (Y, \delta)$ is called fuzzy closed if and only if the image of every fuzzy closed set is fuzzy closed, i.e. iff $u^c \in t \Rightarrow \left(f(u)\right)^c \in \delta$.

Definition 1.2.17. (Malghan & Benchalli, 1994)**: A** function $f: (X, t) \rightarrow (Y, \delta)$ is called fuzzy open (or open) if and only if the image of every fuzzy open set is fuzzy open, i.e. iff $u \in t \Rightarrow f(u) \in \delta$.

Definition 1.2.18. (Pao-Ming & Ying-Ming, 1980)**: A** function $f: (X, t) \rightarrow (Y, \delta)$ is called fuzzy continuous (or continuous) if and only if the pre-image of an open fuzzy set is open, i.e. if $v \in \delta \Rightarrow f^{-1}(v) \in t$. The function f is called fuzzy homeomorphism if and only if f is bijective and both f and f^{-1} are fuzzy continuous.

Proposition 1.2.19. (Pao-Ming & Ying-Ming, 1980)**:** Let $(X, t)$and (Y, δ) be two fuzzy topological spaces and $f: (X, t) \to (Y, \delta)$ be a continuous function, then the following properties hold:

(a) If v is closed in (Y, δ), then $f^{-1}(v)$ is closed in (X, t).

(b) For any fuzzy set u in X, $f(\bar{u}) \subset \overline{f(u)}$.

(c) For any fuzzy set v in Y, $\overline{f^{-1}(v)} \subset f^{-1}(\bar{v})$.

Proposition 1.2.20. (Malghan & Benchalli, 1994)**:** Let $(X, t)$and (Y, δ) be two fuzzy topological spaces and $f: (X, t) \to (Y, \delta)$ be an open function, then the following properties hold:

(a) For any fuzzy set u in X, $f(u^o) \subset \left(f(u)\right)^o$.

(d) For any fuzzy set v in Y, $\left(f^{-1}(v)\right)^o \subset f^{-1}(v^o)$.

Proposition 1.2.21. (Malghan & Benchalli, 1994)**:** Let $(X, t)$and (Y, δ) be two FTSs and $f: (X, t) \to (Y, \delta)$ be a function, then f is closed if and only if $\overline{f(u)} \subset f(\bar{u})$ for each fuzzy set u in X.

Definition 1.2.22. (Ghamin & Kerre, 1984)**:** A fuzzy topological space (X, t) is called fuzzy regular if for any fuzzy point $x_\alpha \in X$ and fuzzy closed set λ in X with $x_\alpha \notin \lambda$ there exists $u, v \in t$ such that $x_\alpha \in u$, $\lambda \subset v$ and $u \cap v = \underline{0}$.

Definition 1.2.23. (Hutton, 1975)**:** A fuzzy topological space (X, t) is called normal if for all closed fuzzy sets m and open fuzzy set u with $m \subset u$, there exist an open fuzzy set v such that $m \subset v \subset \overline{v} \subset u$ where $\overline{v}$ is the closer of v.

1.3 Intuitionistic Set and Intuitionistic Topological Spaces

Definition 1.3.1. (Coker, 1996): Suppose X is a non empty set. An intuitionistic set (IS, in short) A on X is an object having the form $A = (X, A_1, A_2)$ where A_1 and A_2 are subsets of X satisfying $A_1 \cap A_2 = \phi$. The set A_1 is called the set of member of A while A_2 is called the set of non-member of A. In this thesis, we use the simpler notation $A = (A_1, A_2)$ instead of $A = (X, A_1, A_2)$ for an intuitionistic set.

Remark 1.3.2. (Coker, 1996): Every subset A of a nonempty set X may obviously be regarded as an intuitionistic set having the form $A = (A, A^c)$ where $A^c = X \setminus A$, the complement of A.

Definition 1.3.3. (Coker, 1996)**:** Let the intuitionistic sets A and B in X be of the forms $A = (A_1, A_2)$ and $B = (B_1, B_2)$ respectively. Furthermore, let $\{A_j, j \in J\}$ be an arbitrary family of intuitionistic sets in X, where $A_j = \left(A_j^{(1)}, A_j^{(2)}\right)$. Then

(a) $A \subseteq B$ if and only if $A_1 \subseteq B_1$ and $A_2 \supseteq B_2$,

(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$,

(c) $A^c = (A_2, A_1)$, denotes the complement of A,

(d) $\cap A_j = (\cap A_j^{(1)}, \cup A_j^{(2)})$,

(e) $\cup A_j = (\cup A_j^{(1)}, \cap A_j^{(2)})$,

(f) $\phi_\sim = (\phi, X)$ and $X_\sim = (X, \phi)$.

Definition 1.3.4. (Coker, 1996)**:** Let X be a non empty set and $p \in X$ a fixed element in X. Then the intuitionistic set $p_\sim$ defined by $p_\sim = (\{p\}, \{p\}^c)$ is called an intuitionistic point. The intuitionistic point $p_\sim$ contained in the intuitionistic set $A = (A_1, A_2)$ if $p \in A_1$.

Definition 1.3.5. (Coker & Bayhan, 2001)**:** Let X be a non empty set. A family $\mathcal{T}$ of intuitionistic sets in X is called an intuitionistic topology on X if the following conditions hold.

(1) $\phi_\sim$, $X_\sim \in \mathcal{T}$,

(2) $A \cap B \in \mathcal{T}$ for all $A, B \in \mathcal{T}$,

(3) $\cup A_j \in \mathcal{T}$ for any arbitrary family $\{A_j \in \mathcal{T}, j \in J\}$.

The pair $(X, \mathcal{T})$ is called an intuitionistic topological space (ITS, in short), members of $\mathcal{T}$ are called intuitionistic open sets (IOS, in short) in X and their complements are called intuitionistic closed sets (ICS, in short) in X.

Definition 1.3.6. (Coker, 1996)**:** Let X and Y be two nonempty sets and $f: X \to Y$ a function, $A = (A_1, A_2)$ and $B = (B_1, B_2)$ are intuitionistic sets in X and Y respectively. Then the image of A under f denoted by f(A) is the intuitionistic set in Y defined by $f(A) = (f(A_1), f\_(A_2))$ where $f(A_2) = \left(f(A_2^c)\right)^c$ and the pre-image of B under f denoted by $f^{-1}(B)$ is the intuitionistic set in X defined by $f^{-1}(B) = (f^{-1}(B_1), f^{-1}(B_2))$.

Proposition 1.3.7. (Coker, 1996)**:** Let X and Y be two nonempty sets and $f: X \to Y$ a function. If A, B are intuitionistic sets in X and C, D are intuitionistic sets in Y. Then

(a) $A \subset B \Rightarrow f(A) \subset f(B)$.

(b) $C \subset D \Rightarrow f^{-1}(C) \subset f^{-1}(D)$.

(c) $A \subset f^{-1}(f(A))$ and if f is one-one, then $A = f^{-1}(f(A))$.

(d) $f^{-1}(f(B)) \subset B$ and if f is onto, then $B = f^{-1}(f(B))$.

(e) $f(A \cup B) = f(A) \cup f(B)$.

(f) $f(A \cap B) \subset f(A) \cap f(B)$ and if f is one-one, then $f(A \cap B) = f(A) \cap f(B)$.

(g) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$.

(h) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

(i) $f(X_{\sim}) = Y_{\sim}$, if f is onto.

(j) $f(\phi_{\sim}) = \phi_{\sim}$

(k) $f^{-1}(Y_{\sim}) = X_{\sim}$.

(l) $f^{-1}(\phi_{\sim}) = \phi_{\sim}$.

(m) If f is onto, then $\left(f(A)\right)^c \subset f(A^c)$; and if, furthermore, f is one-one, we have $\left(f(A)\right)^c = f(A^c)$.

(n) $f^{-1}(B^c) = \left(f^{-1}(B)\right)^c$.

Definition 1.3.8. (Chu, 2009): Let $(X, \mathcal{T})$ and $(Y, \mathcal{T}')$ be two ITSs and $f: X \to Y$. Then f is called continuous if preimage of open set is open, i.e. if $B \in \mathcal{T}' \Rightarrow f^{-1}(B) \in \mathcal{T}$ or equivalently if pre-image of closed set is closed, i.e. if $B^c \in \mathcal{T}' \Rightarrow \left(f^{-1}(B)\right)^c \in \mathcal{T}$.

Definition 1.3.9. (Chu, 2009): Let $(X,\mathcal{T})$ and $(Y,\mathcal{T}')$ be two ITSs and $f: X \to Y$. Then f is called open if image of open set is open, i.e. if $A \in \mathcal{T} \Rightarrow f(A) \in \mathcal{T}'$.

Definition 1.3.10. (Chu, 2009): Let $(X,\mathcal{T})$ and $(Y,\mathcal{T}')$ be two ITSs and $f: X \to Y$. Then f is called closed if image of closed set is closed, i.e. if $A^c \in \mathcal{T} \Rightarrow \left(f(A)\right)^c \in \mathcal{T}'$.

Definition 1.3.11. (Chu, 2009): Let $(X,\mathcal{T})$ ITS and $A = (A_1, A_2) \in X$. Then the closure of A is the intersection of all closed superset of A, i.e.

$$\mathrm{cl}(A) = \bigcap\{K: K^c \in \mathcal{T}, A \subset K\}$$

And the interior of A is the union of all open subset of A i.e.

$$\mathrm{int}(A) = \bigcup\{K: K \in \mathcal{T}, K \subset A\}$$

Observe that $\mathrm{cl}(A)$ is the smallest closed IS containing A and $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$. If A is closed then $\mathrm{cl}(A) = A$. Again $\mathrm{int}(A)$ is the largest open IS contained in A and $\mathrm{int}(\mathrm{int}(A)) = \mathrm{int}(A)$. If A is open then $\mathrm{int}(A) = A$.

1.4 Intuitionistic Fuzzy Set

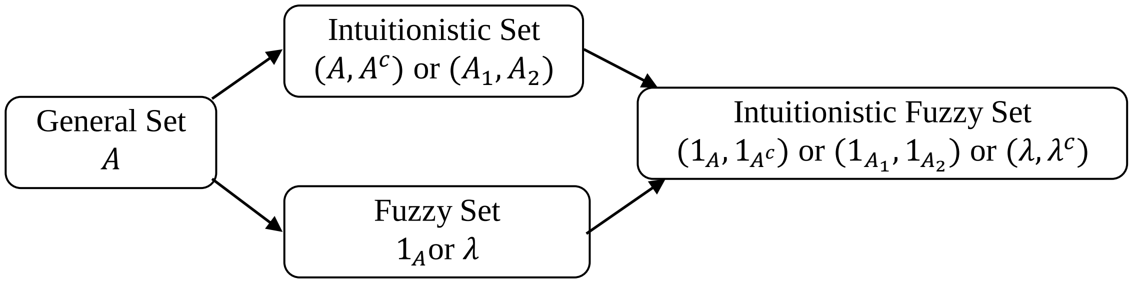
Definition 1.4.1. (Atanassov, 1986)**:** Let X be a non empty set. An intuitionistic fuzzy set A (IFS, in short) in X is an object having the form $A = \{(x, \mu_A(x), \nu_A(x)): x \in X\}$, where μ_A and ν_A are fuzzy sets in X denote the degree of membership and the degree of non- membership respectively subject to the condition that $\mu_A(x) + \nu_A(x) \leq 1$.

Throughout this thesis, we use the simpler notation $A = (\mu_A, \nu_A)$ instead of $A = \{(x, \mu_A(x), \nu_A(x)): x \in X\}$ for IFS.

Remark 1.4.2. (Ying-Ming & Mao-Kang, 1997)**:** Let X be a non empty set and $A \subseteq X$, then the set A may be regarded as a fuzzy set in X by its characteristic function $1_A : X \to \{0,1\}$ which is defined by

$$1_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A, \text{i.e., if } x \in A^c \end{cases}$$

Again we know that a fuzzy set λ in X may be regarded as an intuitionistic fuzzy set by $(\lambda, 1-\lambda) = (\lambda, \lambda^c)$. So every sub set A of X may be regarded as intuitionistic fuzzy set by $(1_A, 1-1_A) = (1_A, 1_{A^c})$. Therefore we have the following relation.



Definition 1.4.3. (Atanassov, 1986): Let X be a nonempty set and IFSs A, B in X be given by $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ respectively, then

(a) $A \subseteq B$ if $\mu_A(x) \le \mu_B(x)$ and $\nu_A(x) \ge \nu_B(x)$ for all $x \in X$,

(b) $A = B$ if $A \subseteq B$ and $B \subseteq A$,

(c) $\overline{A} = (\nu_A, \mu_A)$,

(d) $A \cap B = (\mu_A \cap \mu_B, \nu_A \cup \nu_B)$,

(e) $A \cup B = (\mu_A \cup \mu_B, \nu_A \cap \nu_B)$.

Definition 1.4.4. (Coker, 1997): Let $\{A_j = \left(\mu_{A_j}, \nu_{A_j}\right), j \in J\}$ be an arbitrary family of IFSs in X. Then

(a) $\cap A_j = (\cap \mu_{A_j}, \cup \nu_{A_j})$,

(b) $\cup A_j = (\cup \mu_{A_j}, \cap \nu_{A_j})$,

(c) $0_\sim = (\underline{0}, \underline{1}), 1_\sim = (\underline{1}, \underline{0})$.

Definition 1.4.5. (Singh & Srivastava, 2012): Let $\alpha, \beta \in [0,1]$ and $\alpha + \beta \leq 1$. An intuitionistic fuzzy point $x_{(\alpha,\beta)}$ in X is an intuitionistic fuzzy set in X define by

$$x_{(\alpha,\beta)}(y) = \begin{cases} (\alpha, \beta) \text{ if } y = x \\ (0.1) \text{ if } y \neq x \end{cases}$$

An intuitionistic fuzzy point $x_{(\alpha,\beta)}$ is said to belong to an intuitionistic fuzzy set $A = (\mu_A, \nu_A)$ if $\alpha < \mu_A(x)$ and $\beta > \nu_A(x)$.

Definition 1.4.6. (Atanassov, 1986)**:** Let X and Y be two nonempty sets and $f: X \to Y$ be a function. If $A = \{(x, \mu_A(x), \nu_A(x)): x \in X\}$ and $B = \{(y, \mu_B(y), \nu_B(y)): y \in Y\}$ are IFSs in X and Y respectively, then the pre image of B under f, denoted by $f^{-1}(B)$ is the IFS in X defined by

$$f^{-1}(B) = \left\{\left(x, (f^{-1}(\mu_B))(x), (f^{-1}(\nu_B))(x)\right): x \in X\right\}$$

$$=\{(x, \mu_B(f(x)), \nu_B(f(x))): x \in X\}$$

and the image of A under f, denoted by $f(A)$ is the IFS in Y defined by

$$f(A)= \left\{\left(y, (f(\mu_A))(y), (f(\nu_A))(y)\right): y \in Y\right\},$$

where for each $y \in Y$

$$(f(\mu_A))(y) = \begin{cases} \sup_{x \in f^{-1}(y)} \mu_A(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

$$(f(\nu_A))(y) = \begin{cases} \inf_{x \in f^{-1}(y)} \nu_A(x) & \text{if } f^{-1}(y) \neq \phi \\ 1 & \text{otherwise} \end{cases}$$

Definition 1.4.7. (Bayhan & Coker, 1996): Let $A = (x, \mu_A, \nu_A)$ and $B = (y, \mu_B, \nu_B)$ be IFSs in X and Y respectively. Then the product of IFSs A and B denoted by $A \times B$ is defined by $A \times B = \left\{(x, y), \mu_A \overset{\times}{\underset{\cdot}{}} \mu_B, \nu_A \underset{\times}{\overset{\cdot}{}} \nu_B)\right\}$ where for all $(x, y) \in X \times Y$.

$$\left(\mu_A \overset{\times}{\underset{\cdot}{}} \mu_B\right)(x, y) = \min\{\mu_A(x), \mu_B(y)\}$$

And

$$\left(\nu_A \underset{\times}{\overset{\cdot}{}} \nu_B\right)(x, y) = \max\{\nu_A(x), \nu_B(y)\}$$

Obviously $0 \leq \left(\mu_A \overset{\times}{\underset{\cdot}{}} \mu_B\right) + \left(\nu_A \underset{\times}{\overset{\cdot}{}} \nu_B\right) \leq 1$. This definition can be extended to an arbitrary family of IFSs.

1.5 Intuitionistic Fuzzy Topological Spaces

Definition 1.5.1. (Coker, 1997): An intuitionistic fuzzy topology (IFT, in short) on a nonempty set X is a family τ of IFSs in X, satisfying the following axioms:

(1) $0_\sim, 1_\sim \in \tau$,

(2) $A \cap B \in \tau$, for all $A, B \in \tau$,

(3) $\cup A_j \in \tau$ for any arbitrary family $\{A_j \in \tau, j \in J\}$.

The pair (X, τ) is called an intuitionistic fuzzy topological space (IFTS, in short), members of τ are called intuitionistic fuzzy open sets (IFOS, in short) in X, and their complements are called intuitionistic fuzzy closed sets (IFCS, in short) in X.

We know that every subset A of X may be regarded as an intuitionistic fuzzy set in X. So we have the following theorem:

Theorem 1.5.2. Let (X, T) be a topological space. Then (X, τ) is an IFTS where $\tau = \left\{(1_{A_j}, 1_{A_j}^c), j \in J : A_j \in T\right\}$.

Proof : The proof is obvious.

Note: Above τ is the corresponding intuitionistic fuzzy topology of T.

Again we know that every fuzzy set in X may be regarded as an intuitionistic fuzzy set in X. So we have the following theorem:

Theorem 1.5.3. Let (X, t)be a fuzzy topological space. Then (X, τ) is an IFTS where $\tau = \{(\lambda_j, \lambda_j^c), j \in J : \lambda_j \in t\}$.

Proof : The proof is obvious.

Note: Above τ is the corresponding intuitionistic fuzzy topology of t.

Again we know that every intuitionistic set in X may be regarded as an intuitionistic fuzzy set in X. So we have the following theorem:

Theorem 1.5.4. Let $(X, \mathcal{T})$ be an intuitionistic topological space. Then (X, τ) is an intuitionistic fuzzy topological space where

$$\tau = \left\{ \left(1_{A_{j1}}, 1_{A_{j2}} \right), j \in J : A_j = \left(A_{j1}, A_{j2} \right) \in \mathcal{T} \right\}.$$

Proof : The proof is obvious.

Note: Above τ is the corresponding intuitionistic fuzzy topology of $\mathcal{T}$.

Definition 1.5.5. (Coker, 1997): Let (X, τ) be an IFTS and $A = (\mu_A, \nu_A)$ be an IFS in X. Then the interior and closure of A are defined by

$$\mathrm{cl}(A) = \cap \{K: K \text{ is an IFCS in X and } A \subset K),$$

$$\mathrm{int}(A) = \cup \{G: G \text{ is an IFOS in X and } G \subset A\}.$$

It can be also shown that $\mathrm{cl}(A)$ is an IFCS with $A \subset \mathrm{cl}(A)$ and $\mathrm{int}(A)$ is an IFOS in X with $\mathrm{int}(A) \subset A$, and

(a) A is an IFCS in X iff $\mathrm{cl}(A) = A$;

(b) A is an IFOS in X iff $\mathrm{int}(A) = A$.

Definition 1.5.6. (Bayhan & Coker, 1996): Let (X_j, τ_j), $j = 1,2$ be two IFTSs. The product topology $\tau_1 \times \tau_2$ on $X_1 \times X_2$ is the IFT generated by $\left\{ \rho_j^{-1}(U_j): U_j \in \tau_j, j = 1,2 \right\}$, where $\rho_j: X_1 \times X_2 \to X_j$, $j = 1,2$ are the projection maps and IFTS $(X_1 \times X_2, \tau_1 \times \tau_2)$ is called the product IFTS of (X_j, τ_j), $j = 1,2$. In this case $\mathcal{S} = \left\{ \rho_j^{-1}(U_j), j \in J: U_j \in \tau_j \right\}$ is a sub base and $\mathcal{B} = \left\{ U_1 \times U_2: U_j \in \tau_j, j = 1,2 \right\}$ is a base for $\tau_1 \times \tau_2$ on $X_1 \times X_2$.

Definition 1.5.7. (Coker, 1997)**:** Let (X, τ) and (Y, δ) be IFTSs. A function $f: X \to Y$ is called

(a) continuous if pre-image of an open set is open, i.e. if $f^{-1}(B) \in \tau$ for all $B \in \delta$ or equivalently pre-image of a closed set is closed, i.e. if $(f^{-1}(B))^c \in \tau$ for all $B^c \in \delta$.

(b) open if image of an open set is open, i.e. if $f(B) \in \delta$ for all $B \in \tau$.

(c) closed if image of a closed set is closed, i.e. if $f(A)^c \in \delta$ for all $A^c \in \tau$.

Homeomorphism if f is bijective, open and continuous..

1.6 Compactness

Definition: 1.6.1. (Lipschutz, 1965) A subset A of a topological space X is compact if every open cover of A is reducible to a finite cover.

Definition: **1.6.2.** (Lipschutz, 1965) Let A be a subset of a topological space (X, T). Then A is compact with respect to T if and only if A is compact with respect to the relative topology T_A on A.

Definition: 1.6.3. A subset A of a topological space X is limit point compact if for any infinite subset A of X, there is a cluster point of A in X.

Bolzano- Weierstran's property: A metric space X is said to be Bolzano-Weierstran's property if every infinite subset of X has a limit point in X.

Definition: 1.6.4. A fuzzy topological space (X, t) is called compact if and only if for every family u of fuzzy open sets of X and for every a∈I such that $\vee\{U : U \in u\} \geq \bar{a}$ and for every ε∈ (0, a] there exists a finite subfamily u_1 of u such that $\vee\{U : U \in u_1\} \geq \overline{a - \varepsilon}$.

Definition: 1.6.5. (Srivastava and Srivastava, 1985). A fuzzy topological space (X, t) is called a fuzzy Hausdorff space or T_2- space if for any pair of distinct fuzzy points (i.e. fuzzy points with distinct supports) x_t and y_t, there exist fuzzy open sets U and V such that $x_t \in U$, $y_t \in V$ and $U \cap V = 0_X$.

Definition: 1.6.6. (Wong, 1974). A fuzzy topological space (X, t) is said to be fuzzy locally compact if and only if for every fuzzy point x_t in X there exists a fuzzy open set U∈T such that $x_t \in U$ and U is fuzzy compact, i.e. , each fuzzy open cover of U has a finite subcover.

Note: Each fuzzy compact space is fuzzy locally compact.

Definition: 1.6.7. Let $\{A_n, n\in N\}$ be a net of fuzzy sets in a fuzzy topological space Y. Then by F- $\overline{\lim_N}(A_n)$, we denote the fuzzy upper limit of the net $\{A_n, n\in N\}$ in I^Y , that is, the fuzzy set which is the union of all fuzzy points p_x^a in Y such that for every $n_0 \in N$ and for every fuzzy open Q – neighbourhood U of p_x^a in Y there exists an element n ∈N for which $n \geq n_0$ and A_n q U. In other cases we set F- $\overline{\lim_N}(A_n) = \overline{0}$.

Finaly for the notions of : (α) upper limit of a net of a subsets in a topological space X, (β) compact topological spaces, (γ) Ω - compact topological spaces and (δ) (α, β) – compact topological spaces.

Let X be a non-empty set. Then by |X| we denote the cardinality of X. Also, throughout this paper the words “fuzzy space” means “fuzzy topological space”.

Definition 1.6.8. (Coker, 1997): Let (X, τ) be an IFTS.

(a) If a family $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle : i \in J\}$ of IFOS in X satisfy the condition $\cup \{\langle x, \mu_{G_i}, \nu_{G_i}\rangle : i \in J\} = 1_\sim$ then it is called a fuzzy open cover of X. A finite subfamily of fuzzy open cover $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle : i \in J\}$ of X, which is also a fuzzy open cover of X is called a finite subcover of $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle : i \in J\}$.

(b) A family $\{\langle x, \mu_{K_i}, \nu_{K_i}\rangle : i \in J\}$ of IFCS's in X satisfies the finite intersection property iff every finite subfamily $\{\langle x, \mu_{K_i}, \nu_{K_i}\rangle : i = 1,2, \dots, n\}$ of the family satisfies the condition $\bigcap_{i=1}^{n}\{\langle x, \mu_{K_i}, \nu_{K_i}\rangle\} \neq 0_{\sim}$.

Definition 1.6.9. (Coker, 1997): An IFTS (X, τ) is called fuzzy compact iff every fuzzy open cover of X has a finite subcover.

Definition 1.6.10. (Coker, 1997): (a) Let (X, τ) be an IFTS and A be an IFS in X. If a family $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle : i \in J\}$ of IFOS's in X satisfies the condition $A \subseteq \cup \{\langle x, \mu_{G_i}, \nu_{G_i}\rangle : i \in J\}$, then it is called a fuzzy open cover of A. A finite subfamily of the fuzzy open cover $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle : i \in J\}$ of A, which is also a fuzzy open cover of A, is called a finite subcoverof $\{\langle x, \mu_{G_i}, \nu_{G_i}\rangle : i \in J\}$.

(b) An IFS $A = \langle x, \mu_{G_i}, \nu_{G_i}\rangle$ in an IFTS (X, τ) is called fuzzy compact iff every fuzzy open cover of A has a finite subcover.

Definition 1.6.11. (Ramadan, Abbas & El-Latif, 2005): An IFTS (X, τ) is called (α, β)-intuitionistic fuzzy compact (resp., (α, β)-intuitionistic fuzzy nearly compact and (α, β)-intuitionistic fuzzy almost compact) if and only if for every family $\{G_i : i \in J\}$ in $\{G : G \in \zeta^X, \tau(G) > \langle \alpha, \beta \rangle\}$ such that $\bigcup_{i \in J} G_i = 1_{\sim}$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$, there exists a finite subset J_0 of J such that $\bigcup_{i \in J_0} G_i = 1_{\sim}$ (resp., $\bigcup_{i \in J_0} int_{\alpha,\beta}\left(cl_{\alpha,\beta}(G_i)\right) = 1_{\sim}$ and $\bigcup_{i \in J_0} cl_{\alpha,\beta}(G_i) = 1_{\sim}$).

Definition 1.6.12. (Ramadan, Abbas & El-Latif, 2005): Let (X, τ) be an IFTS and A be an IFS in X. A is said to be (α, β)-intuitionistic fuzzy compact if and only if every family $\{G_i : i \in J\}$ in $\{G: G \in \zeta^X, \tau(G) > \langle\alpha, \beta\rangle\}$ such that $A \subseteq \bigcup_{i \in J_0} G_i$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$.

Definition 1.6.13. (Ramadan, Abbas & El-Latif, 2005): A family $\{K_i : i \in J\}$ in $\{K: K \in \zeta^X, \tau^*(K) > \langle\alpha, \beta\rangle\}$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$ has the finite intersection property (FIP) if and only if for any finite subset J_0 of J, $\bigcap_{i \in J_0} K_i \neq 0_\sim$.

Definition 1.6.14. (Ramadan, Abbas & El-Latif, 2005): An IFTS (X, τ) is called (α, β)-intuitionistic fuzzy regular if and only if for each IFS A in X such that $\tau(A) > \langle\alpha, \beta\rangle$, where $\alpha \in I_0, \beta \in I_1$ with $\alpha + \beta \leq 1$, can be written as $A = \cup\{B: B \in \zeta^X, \tau(B) \geq \tau(A), cl_{\alpha,\beta}(B) \subseteq A\}$.

1.7 Connectedness

Definition: 1.7.1. (Lipschutz, 1965) A subset A of a topological space X is disconnected if there exist open subsets G and H of X such that A∩G and A∩H are disjoint non-empty sets whose union is A. In this case, G∪H is called a disconnection of A. A set is connected if it is not disconnected.

Observe that, $A = (A \cap G) \cup (A \cap H)$ iff $A \subset G \cup H$ and $\phi = (A \cap G) \cap (A \cap H)$ iff $G \cap H \subset Ac$. Therefore G∪H is a disconnection of A if and only if $\overline{A} \cap G \neq \phi$, $A \cap H \neq \phi$, $A \subset G \cup H$, and $G \cap H \subset A^C$.

Example 1.7.2. Consider the following topology on X = {a, b, c, d, e}:

T = {X, ϕ, {a, b, c}, {c, d, e}, {e}}

Now A = {a, d, e} is disconnected. For let G = {a, b, c} and

H = {c, d, e}; then A∩G = {a} and A∩H = {d, e} are non-empty disjoint sets whose union is A.

Definition: **1.7.3.** (Lipschutz, 1965) A topological space X is connected if and only if

(i) X is not the union of two non-empty disjoint open sets,

(ii) X and ϕ are the only subsets of X which are both open and closed.

Definition: 1.7.4. (Fatteh & Bassam, 1985) A fuzzy topological space X is said to be fuzzy connected if it has no proper fuzzy clopen set. (A fuzzy set λ in X is proper if λ ≠ 0 and λ ≠ 1, clopen means closed-open.)

The pair (X, τ) is called an intuitionistic fuzzy topological space (IFTS, in short), members of τ are called intuitionistic fuzzy open sets (IFOS, in short) in X, and their complements are called intuitionistic fuzzy closed sets (IFCS, in short) in X.

Definition 1.7.5. (Srivastava & Singh, 2011): Two disjoint non-empty intuitionistic fuzzy subsets $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ of an IFTS (X, τ) are said to be separated if there exist $U_i \in \tau\ (i = 1,2)$ such that $U_1 \supseteq A, U_2 \supseteq B$ and $U_1 \cap A = U_2 \cap B = 0_{\sim}$.

Definition 1.7.6. (Srivastava & Singh, 2011): Let (X,τ) be an IFTS and A be an IFS in X which is strictly positive i.e. $A(x) \gg 0_{\sim}$ (i.e. $\mu_A(x) > 0, \nu_A(x) < 1, \forall x \in X$). A pair $U_1, U_2 \in \tau$ is called $(C_1) -$ separation of A if $U_1 \neq A$, $U_2 \neq A$, $U_1 \cup U_2 = A$ and $U_1 \cap U_2 = 0_{\sim}$.

Definition 1.7.7. (Sethupathy & Lakshmivarahan, 1977) A fuzzy topological space X is said to be disconnected if $X = A \cup B$, where A and B are non-empty open fuzzy sets in X such that $A \cap B = \emptyset$. Hence a fuzzy topological space X cannot be represented as the union of two non-empty, disjoint open fuzzy sets on X.

CHAPTER TWO

Compactness in IFTS

Fuzzy compact space was first introduced by Chang (Chang, 1968) in fuzzy topological spaces and mentioned some properties which are global property. Later Dogen Coker (Cocker et al. 1996, 1997, 2001, 2003) introduced the basic definitions and properties of intuitionistic fuzzy topological spaces and fuzzy compactness in intuitionistic fuzzy topological spaces.. After then A. A. Ramadan, S. E. Abbas, A. A. Abd El-Latif (Ramadan et al. 2005) and M. A. Mahbub (Mahbub et al.2018) introduced compactness in intuitionistic fuzzy topological spaces.

In this chapter, we give seven new notions of intuitionistic fuzzy compact (in short, IF-Compact) space and investigate some relationship among them. At first we show that all these notions satisfy 'good extension' property. Furthermore, it proves that these intuitionistic fuzzy compact spaces are hereditary and productive. Finally, we observe that all concepts are preserved under one-one, onto and continuous mapping.

2.1 Definition and Properties

Definition 2.1.1. Let (X, τ) be an intuitionistic fuzzy topological space. A family $\{(\mu_{G_i}, \nu_{G_i}): i \in J\}$ of IFOS in X is called open cover of X if $\cup \mu_{G_i} = 1$ and $\cap \nu_{G_i} = 0$. If every open cover of X has a finite subcover then X is said to be intuitionistic fuzzy compact (IF-compact, in short).

Example 2.1.2. Let $X = \{1,2\}$ and τ be an intuitionistic fuzzy topology on X generated by$\{G_n\}_{n\in\mathbb{N}}$, where $G_n = \{x, \left(\frac{1}{\frac{n+1}{n+2}}, \frac{2}{\frac{n+2}{n+3}}\right), (\frac{1}{\frac{1}{n+3}}, \frac{2}{\frac{1}{n+4}})\}$. Note that $\bigcup_{n\in\mathbb{N}} G_n$ is an open cover for X but this cover has no finite subcover.

Consider, $G_1 = \{x, \left(\frac{1}{0.66}, \frac{2}{0.75}\right), (\frac{1}{0.25}, \frac{2}{0.2})\}$

$$G_2 = \{x, \left(\frac{1}{0.75}, \frac{2}{0.8}\right), (\frac{1}{0.2}, \frac{2}{0.16})\}$$

$$G_3 = \{x, \left(\frac{1}{0.8}, \frac{2}{0.83}\right), (\frac{1}{0.16}, \frac{2}{0.14})\}$$

and observe that $G_1 \cup G_2 \cup G_3 = G_3$. So, for any finite collection $\{G_{n_i}: i \in I\}$, where I is a finite subset of $\mathbb{N}$, $\bigcup_{n_i\in I} G_{n_i} = G_m \neq (1, 0)$, where $m = \max\{n_i: n_i \in I\}$. Therefore the IFTS (X, τ) is not compact.

Definition 2.1.3. A family $\{(\mu_{G_i}, \nu_{G_i}): i \in J\}$ of IFOS in X is called (α, β)-level open cover of X if $\cup\, \mu_{G_i} \geq \alpha$ and $\cap\, \nu_{G_i} \leq \beta$ with $\alpha + \beta \leq 1$. If every (α, β)-level open cover of X has a finite subcover then X is said to be (α, β)-level IF-compact.

Example 2.1.4. Let $X = I$ and consider the IFSs $\{G_n: n = 2,3,4, \dots\}$ as follows:

$$\mu_{G_n} = \begin{cases} 0.9 & x = 0 \\ nx & 0 < x \leq \frac{1}{n^2} \\ 1 & \frac{1}{n^2} < x \leq 1 \end{cases}$$

$$\nu_{G_n} = \begin{cases} 0.1 & x = 0 \\ 1 - nx & 0 < x \leq \frac{1}{n^2} \\ 0 & \frac{1}{n^2} < x \leq 1 \end{cases}$$

$$\mu_G = \begin{cases} 0.9 & x = 0 \\ 1 & otherwise \end{cases}$$

$$\nu_G = \begin{cases} 0.1 & x = 0 \\ 0 & otherwise \end{cases}$$

The intuitionistic fuzzy sets $G_n = \{x, \mu_{G_n}, \nu_{G_n}\}, n = 2,3,4, \dots$ is (α, β)-level IF-compact for $\alpha = 0.75$ and $\beta = 0.2$.

Theorem 2.1.5. Show that the following statements are equivalent: (i) X is IF-compact, (ii) For every $\{F_i\}$ where $F_i = (\nu_{F_i}, \mu_{F_i})$ of closed subset of X with $\cap F_i = (0, 1)$ implies $\{F_i\}$ contains finite subclass $\{F_{i1}, F_{i2}, \dots, F_{im}\}$ with $F_{i1} \cap F_{i2} \cap \dots \cap F_{im} = (0, 1)$.

Proof: (i) $\Longrightarrow$ (ii). Suppose $\cap F_i = (0, 1)$ then by De Morgan's law

$$(\cap F_i)^c = \left((0, 1)\right)^c$$

$$\Longrightarrow \cup {F_i}^c = (1, 0)$$

$$\Longrightarrow \cup \left(\nu_{F_i}, \mu_{F_i}\right)^c = (1, 0)$$

$$\Longrightarrow \cup \left(\mu_{F_i}, \nu_{F_i}\right) = (1, 0)$$

$$\Longrightarrow \left(\cup \mu_{F_i}, \cap \nu_{F_i}\right) = (1, 0).$$

So, $\{{F_i}^c\}$, $({F_i}^c = \left(\mu_{F_i}, \nu_{F_i}\right))$ is an open cover of X. Since X is IF-compact hence $\exists {F_{i1}}^c, {F_{i2}}^c, \dots, {F_{im}}^c \epsilon \{{F_i}^c\}$ such that ${F_{i1}}^c \cup {F_{i2}}^c \cup \dots \cup {F_{im}}^c = (1, 0)$. Then $(0, 1) = (1, 0)^c = ({F_{i1}}^c \cup {F_{i2}}^c \cup \dots \cup {F_{im}}^c)^c = ({F_{i1}}^c)^c \cap ({F_{i2}}^c)^c \cap \dots \cap ({F_{im}}^c)^c$ (By De Morgan's law) $= F_{i1} \cap F_{i2} \cap \dots \cap F_{im}$, so we have shown that (i) $\Longrightarrow$ (ii).

(ii) $\Longrightarrow$ (i). Let $\{G_i\}$ be an open cover of X where $G_i = (\mu_{G_i}, \nu_{G_i})$, i.e. $\bigcup_i G_i = (1, 0)$. By De Morgan's law, $(0, 1) = (1, 0)^c = (\bigcup_i G_i)^c = \bigcap_i {G_i}^c$. Since each G_i is open, so $\{{G_i}^c\}$ is a class of closed sets and by (ii) $\exists {G_{i1}}^c, {G_{i2}}^c, \dots, {G_{im}}^c \epsilon \{{G_i}^c\}$ such that ${G_{i1}}^c \cap {G_{i2}}^c \cap \dots \cap {G_{im}}^c = (0, 1)$. So by De Morgan's law $(1, 0) = (0, 1)^c = ({G_{i1}}^c \cap$

${G_{i2}}^c \cap \ldots \cap {G_{im}}^c)^c = G_{i1} \cup G_{i2} \cup \ldots \cup G_{im}$, hence X is IF-compact. So, we have shown that (ii) ⟹(i).

Theorem 2.1.6. Let (X, τ) be an IFTS. If $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are IFS in X. If A and B are IF-compact in (X, τ) then $A \cup B$ is also IF-compact in (X, τ).

Proof: Let $\mathcal{M} = \{A_i = (\mu_{A_i}, \nu_{A_i}) : i \in J\}$ be an open cover of $A = (\mu_A, \nu_A)$ and $\aleph = \{B_i = (\mu_{B_i}, \nu_{B_i}) : i \in J\}$ be an open cover of $B = (\mu_B, \nu_B)$ in (X, τ). Hence $A \subseteq \bigcup_{i=1}^{m} A_i$ and $B \subseteq \bigcup_{i=1}^{n} B_i$.

Now $A \cup B \subseteq \bigcup_{i=1}^{m} A_i \cup \bigcup_{i=1}^{n} B_i$

$$= \begin{cases} \bigcup_{i=1}^{n}((A_i \cup B_i) \cup (\bigcup_{i=n+1}^{m} A_i)) & if \quad m > n \\ \bigcup_{i=1}^{m}((A_i \cup B_i) \cup (\bigcup_{i=m+1}^{n} B_i)) & if \quad n > m \end{cases}$$

$\Rightarrow A \cup B \subseteq \cup (A_i \cup B_i)$

i.e. $\{A_i \cup B_i : i \in J\}$ is a cover of $A \cup B$.

Again, as A is IF-compact in (X, τ) then A has finite subcover i.e. there exist $A_{ik} \in \{A_i\}, k \in j_n$ such that $\subseteq \bigcup_{k=1}^{n} A_{ik}$. Also as B is IF-compact in (X, τ) then B has finite subcover i.e. there exist $B_{ik} \in \{B_i\}, k \in J_n$ such that $B \subseteq \bigcup_{k=1}^{n} B_{ik}$.

Now from $A \subseteq \bigcup_{k=1}^{n} A_{ik}$ and $B \subseteq \bigcup_{k=1}^{n} B_{ik}$ gives

$A \cup B \subseteq \bigcup_{k=1}^{n} A_{ik} \cup \bigcup_{k=1}^{m} B_{ik}$

$$= \begin{cases} \bigcup_{k=1}^{m}((A_{ik} \cup B_{ik}) \cup (\bigcup_{i=m+1}^{n} A_{ik})) & if \quad n > m \\ \bigcup_{k=1}^{n}((A_{ik} \cup B_{ik}) \cup (\bigcup_{i=n+1}^{m} B_{ik})) & if \quad m > n \end{cases}$$

$\Rightarrow A \cup B \subseteq \cup (A_{ik} \cup B_{ik})$

i.e. $\{A_{ik} \cup B_{ik} : k \in J_n\}$ is a subcover of $A \cup B$.

Hence $A \cup B$ is IF-compact in (X, τ).

2.2 Good Extension Property

Theorem 2.2.1. Let (X, T) be a topological space and (X, τ) be its corresponding IFTS, where$\tau = \left\{(1_{A_j}, 1_{A_j}^{c}), j \in J : A_j \in T\right\}$. Then (X, T) is compact if and only if (X, τ) is IF-compact.

Proof: Let (X, T) be compact. Consider $\{G_i | i\epsilon J\}$be the open cover of X, i.e. $\cup G_i = X$ ….(i). Since X is compact then $\exists G_{i1}, G_{i2}, \dots, G_{in} \in T$ such that$G_{i1} \cup G_{i2} \cup \dots \cup G_{in} = X$…..(ii). Now it is clear that $(1_{G_i,} 1_{G_i}{}^{c}) \in \tau$ (by the definition).

Also we have, $\cup\, (1_{G_i,} 1_{G_i}{}^{c})$ = $(\cup\, 1_{G_i,} \cap 1_{G_i}{}^{c})$

$$= (1_{\cup G_i,} 1_{\cap G_i}{}^{c})$$

$$= (1_X, 1_{\cap G_i}{}^{c})$$

But we have, $1_X + 1_{\cap G_i}{}^{c} \leq 1$ then it must be $1_{\cap G_i}{}^{c} = 0$. Therefore we get,

$\cup\, (1_{G_i,} 1_{G_i}{}^{c})$ = $(1_X, 0)$.

Also by (ii) we get, $(1_X, 0)$ = $(1_{G_{i1} \cup G_{i2} \cup \dots \cup G_{in}}, 0)$

$$= (\cup_{j=1}^{n} 1_{G_{ij}}, 0)$$

$$= \cup\, (1_{G_{ij}}, 0)$$

Hence it is clear that the IFTS (X, τ) is IF-compact.

Conversely, let (X, τ) is IF-compact and $\{1_{H_i} | i\epsilon J\}$be the open cover of X. Then by the definition $(1_{H_i,} 1_{H_i}{}^{c}) \in \tau$, where $\cup\, (1_{H_i,} 1_{H_i}{}^{c}) = (1, 0)$

$$\Rightarrow (\cup\, 1_{H_i,} \cap 1_{H_i}{}^{c}) = (1, 0)$$

$$\Rightarrow (1_{\cup H_i,} 1_{\cap H_i}{}^{c}) = (1, 0)$$

$$\Rightarrow 1_{\cup H_i} = 1 = 1_X \text{ and } 1_{\cap H_i}{}^{c} = 0 = 1_{\emptyset}$$

$$\Rightarrow \cup H_i = X, \ \cap {H_i}^C = \emptyset$$

Again, since X is compact then $\exists\big((1_{H_{i1}}, {1_{H_{i1}}}^c), (1_{H_{i2}}, {1_{H_{i2}}}^c), \dots, (1_{H_{in}}, {1_{H_{in}}}^c) \in \tau$ such that $\bigcup_{j=1}^{n}(1_{H_{ij}}, 1_{H_{ij}^C}) = (1, 0) \Rightarrow \left(\cup\, 1_{H_{ij}}, \cap\, 1_{H_{ij}^C}\right) = (1, 0)$

$$\Rightarrow \left(1\ \cup_{H_{ij}}, 1_{\cap H_{ij}^C}\right) = (1, 0)$$

$$\Rightarrow 1_{\cup H_{ij}} = 1 = 1_X \text{ and } {1_{\cap H_{ij}}}^c = 0 = 1_\emptyset$$

$$\Rightarrow \cup H_{ij} = X, \ \cap {H_{ij}}^C = \emptyset$$

Hence, (X, T) is compact.

Corrolary 2.2.2. Let (X, T) be a topological space and (X, τ) be its corresponding IFTS, where$\tau = \left\{(1_{A_j}, 1_{A_j}^c), j \in J : A_j \in T\right\}$. Then (X, T) is compact if (X, τ) is (α, β)-level IF-compact.

Proof: Here it is clear that for any $\alpha, \beta \epsilon I$ with $\alpha + \beta \leq 1 => 1 \geq \alpha$ and $\beta \geq 0$. So, (X, τ) is (α, β)-level IF-compact.

Theorem 2.2.3. Let $(X, \mathcal{T})$ be an intuitionistic topological space and (X, τ) be its corresponding IFTS, where $\tau = \left\{1_{A_j} = \left(1_{A_{j1}},\ 1_{A_{j2}}\right), j \in J :\ A_j = \left(A_{j1}, A_{j2}\right) \in \mathcal{T}\right\}$. Then $(X, \mathcal{T})$ is intuitionistic compact iff (X, τ) is IF-compact.

Proof: Let $(X, \mathcal{T})$ be an intuitionistic compact space, we shall prove that (X, τ) is IF-compact. Consider $\{1_{A_k}\}$ be an open cover of τ, i.e. $\cup\, 1_{A_k} = (1{,}0)$, where $(1, 0)$ is intuitionistic fuzzy set.

Now $1_{A_k} = \left(1_{A_{k1}},\ 1_{A_{k2}}\right) \Rightarrow \cup\, 1_{A_k} = \cup \left(1_{A_{k1}},\ 1_{A_{k2}}\right)$

$$\Rightarrow \cup\, 1_{A_k} = \left(\cup\, 1_{A_{k1}},\ \cap\, 1_{A_{k2}}\right)$$

$$\Rightarrow 1_{\cup A_k} = \left(1_{\cup A_{k1}},\ 1_{\cap A_{k2}}\right)$$

$$\Rightarrow 1_{\cup A_k} = (1_X, 0)$$

$$\Rightarrow 1_{\cup A_k} = (1, 0)$$

By the given definition $\{A_k \in \mathcal{T}\}$, $k \in \Lambda$ is the open cover of X, since $\cup A_k$= (X, ∅).

But we have $(X, \mathcal{T})$ is compact then $\exists\ A_{k_{i1}}, A_{k_{i2}}, \dots, A_{k_{in}} \in \mathcal{T}$ such that

$$\cup_{j=1}^{n} A_{k_{ij}} = (X, \emptyset)$$

$$\Rightarrow \cup_{j=1}^{n}(A_{k_{ij_1}}, A_{k_{ij_2}}) = (X, \emptyset)$$

$$\Rightarrow (\cup_{j=1}^{n} A_{k_{ij_1}},\ \cap_{j=1}^{n} A_{k_{ij_2}}) = (X, \emptyset)$$

$$\Rightarrow \left(1_{\cup_{j=1}^{n} A_{k_{ij_1}}}, 1_{\cap_{j=1}^{n} A_{k_{ij_2}}}\right) = (1_X, 1_\emptyset)$$

$$\Rightarrow \left(1_{\cup_{j=1}^{n} A_{k_{ij_1}}}, 1_{\cap_{j=1}^{n} A_{k_{ij_2}}}\right) = (1,0)$$

Hence (X, τ) is IF-compact.

Conversely, suppose (X, τ) is IF-compact. Consider $\{1_{A_k}\}$be an open cover of τ,

i.e. $\cup\ 1_{A_k}$= (1, 0)

$$\Rightarrow \cup \left(1_{A_{k1}},\ 1_{A_{k2}}\right) = (1,0)$$

$$\Rightarrow \left(\cup\ 1_{A_{k1}},\ \cap\ 1_{A_{k2}}\right) = (1,0)$$

$$\Rightarrow \left(1_{\cup A_{k1}},\ 1_{\cap A_{k2}}\right) = (1,0)$$

$$\Rightarrow 1_{\cup A_{k1}} = 1 = 1_X \text{ and } 1_{\cap A_{k2}} = 0 = 1_\emptyset$$

$$\Rightarrow \cup A_{k1} = X \text{ and } \cap A_{k2} = \emptyset$$

Again, as (X, τ) is compact then

$\exists\ \left(1_{A_{k_{i1_1}}}, 1_{A_{k_{i1_2}}}\right), \left(1_{A_{k_{i2_1}}}, 1_{A_{k_{i2_2}}}\right), \dots, (1_{A_{k_{in_1}}}, (1_{A_{k_{in_2}}}) \in \tau$ such that

$$\cup_{j=1}^{n}(1_{A_{k_{ij1}}}, 1_{A_{k_{ij2}}}) = (1,0)$$

$$\Longrightarrow \left(\bigcup_{j=1}^{n} 1_{A_{k_{ij_1}}},\ \bigcap_{j=1}^{n} 1_{A_{k_{ij_2}}}\right) = (1,0)$$

$$\Longrightarrow \left(1_{\bigcup_{j=1}^{n} A_{k_{ij_1}}}, 1_{\bigcap_{j=1}^{n} A_{k_{ij_2}}}\right) = (1,0)$$

$$\Longrightarrow 1_{\bigcup_{j=1}^{n} A_{k_{ij_1}}} = 1 = 1_X \text{ and } 1_{\bigcap_{j=1}^{n} A_{k_{ij_2}}} = 0 = 1_{\emptyset}$$

$$\Longrightarrow \bigcup_{j=1}^{n} A_{k_{ij_1}} = X \text{ and } \bigcap_{j=1}^{n} A_{k_{ij_2}} = \emptyset$$

Hence, $(X, \mathcal{T})$ is compact.

Theorem 2.2.4. Let (X, t) be a fuzzy topological space and (X, τ) be its corresponding IFTS, where $\tau = \{(\lambda, \lambda^C) : \lambda \in t\}$. Then (X, t) is compact if and only if (X, τ) is IF-compact.

Proof: Let (X, t) be a fuzzy compact space, we shall prove that (X, τ) is IF-compact. Consider $\{\lambda_i | i \epsilon J\}$be the open cover of X, i.e. $\bigcup \lambda_i = 1$ ….(i). Since X is compact then $\exists \lambda_{i1}, \lambda_{i2}, \dots, \lambda_{in} \in t$ such that $\lambda_{i1} \cup \lambda_{i2} \cup \dots \cup \lambda_{in} = 1$…..(ii). Now it is clear that $(\lambda_{i,} \lambda_i{}^c) \in \tau$ (by the definition).

Also we have, $\cup\, (\lambda_{i,} \lambda_i{}^c) = (\cup\, \lambda_{i,} \cap \lambda_i{}^c)$. But we have,$\cup\, \lambda_i + \cap\, \lambda_i{}^c \leq 1$ then it must be $\cap\, \lambda_i{}^c = 0$ as $\bigcup \lambda_i = 1$. Therefore we get, $\cup\, (\lambda_{i,} \lambda_i{}^c) = (1, 0)$.

Also by (ii) we get, $(1, 0) = (\lambda_{i1} \cup \lambda_{i2} \cup \dots \cup \lambda_{in}, 0)$

$$= (\textstyle\bigcup_{j=1}^{n} \lambda_{ij}, 0)$$

$$= \textstyle\bigcup_{j=1}^{n} (\lambda_{ij}, 0)$$

Hence it is clear that, the IFTS (X, τ) is IF-compact.

Conversely, let (X, τ) is IF-compact and $\{\lambda_i | i \epsilon J\}$be the open cover of X. Then by the definition$(\lambda_{i,} \lambda_i{}^c) \in \tau$, where $\cup\, (\lambda_{i,} \lambda_i{}^c) = (1, 0)$

$$\Longrightarrow (\cup\, \lambda_{i,} \cap \lambda_i{}^c) = (1, 0)$$

$$\Rightarrow \cup \lambda_i = X$$

Again, since (X,τ) is compact then $\exists ((\lambda_{i1},\lambda_{i1}{}^c),(\lambda_{i2},\lambda_{i2}{}^c),\ldots,(\lambda_{in},\lambda_{in}{}^c) \in \tau$ such that $\cup_{j=1}^{n}(\lambda_{ij},\lambda_{ij}{}^c) = (1,0) \Rightarrow (\cup \lambda_{ij}, \cap, \lambda_{ij}{}^c) = (1,0)$

$$\Rightarrow \cup \lambda_{ij} = 1$$

Hence, (X,t) is compact.

2.3 Mapping in IF-Compact Space

Theorem 2.3.1. Let (X,τ) and (Y,δ) be IFTSs and $f: X \to Y$ is bijective, open and continuous. Then (Y,δ) is IF-compact $\Rightarrow$ (X,τ) is IF-compact.

Proof: Let $A_i = (\mu_i, \nu_i) \in \tau$ with $\cup A_i = (1,0)$. Now $A_i \in \tau \Rightarrow f(A_i) \in \delta$ with $\cup f(A_i) = (1,0)$. For $y \in Y$, $f(A_i)(y) = (y, f(\mu_{A_i})(y), f(\nu_{A_i})(y))$, where $f(\mu_{A_i})(y) = \sup_{x \in f^{-1}(y)} \mu_{A_i}(x) = \mu_{A_i}(x)$. Similarly we get, $f(\nu_{A_i})(y) = \nu_{A_i}(x)$. Now $\cup f(A_i) = \cup (f(\mu_{A_i}), f(\nu_{A_i})) = (\cup f(\mu_{A_i}), \cap f(\nu_{A_i}))$, i.e. $\cup f(\mu_{A_i})(y) = \cup \mu_{A_i}(x) = 1$ and $\cap f(\nu_{A_i})(y) = \cap \nu_{A_i}(x) = 0$, so $\cup f(A_i) = (1,0)$. Since f is open then $\{f(A_i)\}$ is an open cover of Y. Again Y is compact then there exist $f(A_{1i}), f(A_{2i}), \ldots, f(A_{ni}) \in \delta$ such that $\cup_{j=1}^{n} f(A_{ji}) = (1,0)$ $\Rightarrow f(\cup_{j=1}^{n} A_{ji}) = (1,0) \Rightarrow f^{-1}(f(\cup_{j=1}^{n} A_{ji})) = f^{-1}(1,0)$ $\Rightarrow f^{-1}(1,0) \subseteq \cup_{j=1}^{n} A_{ji}$ (since from Chang $\mu \subseteq f^{-1}(f(\mu))$). Therefore $\cup_{j=1}^{n} A_{ji} = (1,0)$. Hence (X,τ) is IF-compact.

Theorem 2.3.2. Let (X,τ) and (Y,δ) be IFTSs and $f: X \to Y$ is one-one, onto and continuous. Then (X,τ) is IF-compact $\Rightarrow$ (Y,δ) is IF-compact.

Proof: Let $A_i = (\mu_i, \nu_i) \in \delta$ with $\cup A_i = (1, 0)$. Since δ is a topology so $\cup A_i \in \delta \Rightarrow f^{-1}(\cup A_i) \in \tau$ with $f^{-1}(\cup A_i) = (1, 0)$ (as f is continuous) $\Rightarrow \cup f^{-1}(A_i) = (1, 0)$. But

$\cup f^{-1}(A_i) = \cup f^{-1}(\mu_i, \nu_i) = \cup(f^{-1}(\mu_i), f^{-1}(\nu_i)) \in \tau$ with $\cup(f^{-1}(\mu_i), f^{-1}(\nu_i)) =$ (1,0). Since (X, τ) is IF-compact then $\exists\, A_{i1}, A_{i2}, \ldots, A_{im} \in \delta$ where $(f^{-1}(\mu_{i1}), f^{-1}(\nu_{i1}))$, $(f^{-1}(\mu_{i2}), f^{-1}(\nu_{i2}))$, …, $(f^{-1}(\mu_{im}), f^{-1}(\nu_i m)) \in \tau$ such that $(f^{-1}(\mu_{i1}), f^{-1}(\nu_{i1})) \cup (f^{-1}(\mu_{i2}), f^{-1}(\nu_{i2})) \cup \ldots \cup \left(f^{-1}(\mu_{im}), f^{-1}(\nu_i m)\right) =$ $(1, 0)$

$\Rightarrow \cup_{j=1}^{m}(f^{-1}(\mu_{ij}), f^{-1}(\nu_{ij})) = (1, 0)$

$\Rightarrow (\cup_{j=1}^{m} f^{-1}(\mu_{ij}),\ \cap_{j=1}^{m} f^{-1}(\nu_{ij})) = (1, 0)$

$\Rightarrow f(\cup_{j=1}^{m} f^{-1}(\mu_{ij}),\ \cap_{j=1}^{m} f^{-1}(\nu_{ij})) = f(1, 0)$

$\Rightarrow (\cup_{j=1}^{m} f(f^{-1}(\mu_{ij})),\ \cap_{j=1}^{m} f(f^{-1}(\nu_{ij}))) = (1, 0)$, since f is one-one and onto, so $f(1, 0) = (1, 0)$. Therefore $(\cup_{j=1}^{m} \mu_{ij},\ \cap_{j=1}^{m} \nu_{ij}) = (1,0)$, i.e. $\cup_{j=1}^{m}(\mu_{ij}, \nu_{ij}) = (1,0)$.

Hence $(Y, \delta)$is IF-compact.

2.4 Subspace and Product Space of IF-Compact Space

Theorem 2.4.1. Any closed IF-subspace of an IF-compact space is IF-compact.

Proof. Let A be a closed IF-subspace of an IF-compact space X and let $\mathcal{F} = \{F_i : i \in J\}$ where $F_i = \left(\mu_{F_i}, \nu_{F_i}\right)$ be an open cover of A, i.e. $A \subseteq \cup_{i \in J} F_i$. So, $X = (\cup_i G_i) \cup A^c$, that is $\mathcal{F}^* = \{F_i\} \cup \{A^c\}$ is a cover of X. But A^c is open since A is closed, so $\mathcal{F}^*$ is an open cover of X. By hypothesis, X is IF-compact, hence $\mathcal{F}^*$ is reducible to a finite subcover of X, say $X = F_{i_1} \cup F_{i_2} \cup \ldots \cup F_{i_n} \cup A^c, F_{i_k} \in \mathcal{F}, k = 1,2, \ldots, n$. But A and A^c are disjoint, hence $\subset F_{i_1} \cup F_{i_2} \cup \ldots \cup F_{i_n}, F_{i_k} \in \mathcal{F}, k = 1,2, \ldots, n$. We have just shown that any open cover of A contains a finite subcover, i.e. A is IF-compact.

Theorem 2.4.2. Let (X,τ)be an IFTS and (V,τ_V) be a subspace of (X,τ) with (X,τ) is IF-compact. Let $f:(X,\tau)\to(V,\tau_V)$ be continuous, open and onto, then (V,τ_V) is IF-compact.

Proof: Let $\mathcal{M}=\{B_i : i\epsilon J\}$ be an open cover of (V,τ_V) with $\cup B_i=(1_V,0)$. By the definition of subspace topology, let $B_i=U_i|V$, where $U_i\epsilon\tau$. Since f is continuous then $f^{-1}(B_i)\epsilon\,\tau$ implies that $f^{-1}(U_i|V)\epsilon\,\tau$. As, (X,τ) is IF-compact then $\bigcup_{i\epsilon J} f^{-1}(U_i|V)(x)=(1_X\,,0)$. Thus we see that, $\{f^{-1}(U_i|V): i\in J\}$ is an open cover of (X,τ). Hence there exist $f^{-1}(U_{i1}|V), f^{-1}(U_{i2}|V),\dots, f^{-1}(U_{in}|V)\in\{f^{-1}(U_i|V)$ such that $\bigcup_{k=1}^{n} f^{-1}(U_{ik}|V)=(1_X,0)$. Put $B_{ik}=U_{ik}|V$, then it is clear that $B_{ik}\in\tau_V$ with

$$\bigcup_{k=1}^{n} f^{-1}(B_{ik})=(1_X,0)$$

$$\Rightarrow f\left(\bigcup_{k=1}^{n} f^{-1}(B_{ik})\right)=f(1_X,0)$$

$$\Rightarrow \bigcup_{k=1}^{n} f(f^{-1}(B_{ik}))=(f(1_X),0)$$

$$\Rightarrow \bigcup_{k=1}^{n} B_{ik}=(1_V,0) \text{ as } f \text{ is open.}$$

Hence (V,τ_V) is IF-compact.

Theorem 2.4.3. Let the IFTS's (X_1,τ_1) and (X_2,τ_2) be IF-compact. Then the product IFT $\tau_1\times\tau_2$ on $X_1\times X_2$ is IF-compact.

Proof. Consider, (X_1,τ_1) and (X_2,τ_2) is IF-compact. Let $A_i=(\mu_{A_i,}\nu_{A_i})\epsilon\tau_1$ with $\cup$ $A_i=(1,0)$ and $B_i=(\mu_{B_i,}\nu_{B_i})\epsilon\tau_2$ with $\cup B_i=(1,0)$.

Now $A_i\times B_i=\left(\mu_{A_i,}\nu_{A_i}\right)\times\left(\mu_{B_i,}\nu_{B_i}\right)=(\mu_{A_i}\times\mu_{B_i},\nu_{A_i}\times\nu_{B_i})$

where $\left(\mu_{A_i} \dot{\times} \mu_{B_i}\right)(x,y) = \min\left(\mu_{A_i}(x), \mu_{B_i}(y)\right)$, where $x \epsilon X_1,\ y \epsilon X_2$

$$= \min(1,1)$$

$$= 1.$$

Similarly, $(\nu_{A_i} \dot{\times} \nu_{B_i})(x,y) = \max(\nu_{A_i}(x), \nu_{B_i}(y))$

$$= \max(0,0)$$

$$= 0.$$

So, $A_i \times B_i = (1,0)$. But by the definition of product topology, $A_i \times B_i \,\epsilon\, \tau_1 \times \tau_2$, i.e. $\{A_i \times B_i\}$ is a family of intuitionistic fuzzy open set in $X_1 \times X_2$. Choose $\cup (A_i \times B_i) = (1,0)$. Since (X_1, τ_1) is IF-compact, then $\{A_i\}$ has finite subclass$\{A_{ij}\}$ such that $\bigcup_{j=1}^{n} A_{ij} = (1,0)$. Similarly, since (X_2, τ_2) is IF- compact, then $\{B_i\}$ has finite subclass$\{B_{ik}\}$ such that $\bigcup_{k=1}^{m} B_{ik} = (1,0)$. Therefore $\bigcup_{j=1}^{n} A_{ij} \times \bigcup_{k=1}^{m} B_{ik} = (1,0)$

$$\Rightarrow \bigcup_{j=1}^{n}\left(\mu_{A_{ij}}, \nu_{A_{ij}}\right) \times \bigcup_{k=1}^{m}\left(\mu_{B_{ik}}, \nu_{B_{ik}}\right) = (1,0)$$

$$\Rightarrow \left(\bigcup_{j=1}^{n} \mu_{A_{ij}},\ \bigcap_{j=1}^{n} \nu_{A_{ij}}\right) \times \left(\bigcup_{k=1}^{m} \mu_{B_{ik}},\ \bigcap_{k=1}^{m} \nu_{B_{ik}}\right) = (1,0).$$

Hence there exist four cases:

Case-I: If $\bigcup_{j=1}^{n} \mu_{A_{ij}} = 1$, $\bigcup_{k=1}^{m} \mu_{B_{ik}} = 1$

Case-II: If $\bigcup_{j=1}^{n} \mu_{A_{ij}} = 1$, $\bigcap_{k=1}^{m} \nu_{B_{ik}} = 0$

Case-III: If $\bigcap_{j=1}^{n} \nu_{A_{ij}} = 0$, $\bigcup_{k=1}^{m} \mu_{B_{ik}} = 1$

Case-IV: If $\bigcap_{j=1}^{n} \nu_{A_{ij}} = 0$, $\bigcap_{k=1}^{m} \nu_{B_{ik}} = 0$

Here from four cases, we see that the product topology $(X_1 \times X_2, \tau_1 \times \tau_2)$ is IF-compact.

2.5 Separation Axioms in IF-Compact Space

Definition 2.5.1. An IFTS (X,τ) is called Hausdorff iff $x_{(m,n)}, y_{(r,s)} \in X$, where $m,n,r,s \in I$ and $x_{(m,n)} \neq y_{(r,s)}$ imply that there exist $G_1 = \langle x, \mu_{G_1}, \nu_{G_1}\rangle$, $G_2 = \langle y, \mu_{G_2}, \nu_{G_2}\rangle \in \tau$ with $\mu_{G_1}(x_{(m,n)}) = 1, \nu_{G_1}(x_{(m,n)}) = 0, \mu_{G_2}(y_{(r,s)}) = 1, \nu_{G_2}(y_{(r,s)}) = 0$ and $G_1 \cap G_2 = 0_\sim$.

Example 2.5.2. Let $X = \{x, y\}$ and τ be an intuitionistic fuzzy topology on X generated by $A = \{(x,1,0),(y,0,1)\}$, $B = \{(x,0,1),(y,1,0)\}$ then clearly (X,τ) is IF-Hausdorff space.

Theorem 2.5.3. Let A be an IF-copmact subset of an IF-Hausdorff space X and suppose $p_{(r,s)} \in A^C$, where $r,s \in I$. Then there exists open sets $G_{1,x}$ and $G_{2,p}$ such that $p_{(r,s)} \in G_{2,p}$, $A \subset G_{1,x}$ and $G_{1,x} \cap G_{2,p} = 0_\sim$.

Proof. Let $x_{(m,n)}$ be an IF-singleton and $x_{(m,n)} \in A$. Since $p_{(r,s)} \in A^C$, where $r,s \in I$ implies $x_{(m,n)} \neq p_{(r,s)}$. By hypothesis, X is an IF-Hausdorff space $\exists G_{1,x_{(m,n)}} = (\mu_{G_1}, \nu_{G_1})$ and $G_{2,p_{(r,s)}} = (\mu_{G_2}, \nu_{G_2})$ such that $\mu_{G_1}(x_{(m,n)}) = 1, \nu_{G_1}(x_{(m,n)}) = 0, \mu_{G_2}(p_{(r,s)}) = 1, \nu_{G_2}(p_{(r,s)}) = 0$ and $G_{1,x_{(m,n)}} \cap G_{2,p_{(r,s)}} = 0_\sim$. Hence we have $A \subset \cup \{G_{1,x_{(m,n)}} : x_{(m,n)} \in A\}$, i.e. $\{G_{1,x_{(m,n)}} : x_{(m,n)} \in A\}$ is an open cover of A but A is IF-compact so it has a finite subcover $G_{1,x_{11}}, G_{1,x_{12}}, \dots, G_{1,x_{1n}} \in \{G_{1,x_{(m,n)}}\}$ such that $A \subset G_{1,x_{11}} \cup G_{1,x_{12}} \cup \dots \cup G_{1,x_{1n}}$. Now let $G_{1,x} = G_{1,x_{11}} \cup G_{1,x_{12}} \cup \dots \cup G_{1,x_{1n}}$ and

$G_{2,p} = G_{2,p_{11}} \cap G_{2,p_{12}} \cap \dots \cap G_{2,p_{1n}}$. Then $G_{1,x}$ and $G_{2,p}$ are open since they are the union and finite intersection of open sets respectively. Furthermore, $A \subset G_{1,x}$ and $p_{(r,s)} \in G_{2,p}$, since $p_{(r,s)}$ belongs to each $G_{2,p_{1j}}$ individually. Lastly we claim that $G_{1,x} \cap G_{2,p} = 0_{\sim}$. Note first that, $G_{1,x_{1j}} \cap G_{2,p_{1j}} = 0_{\sim}$ implies that $G_{1,x_{1j}} \cap G_{2,p} = 0_{\sim}$. Thus by distributive law,

$$G_{1,x} \cap G_{2,p} = (G_{1,x_{11}} \cup G_{1,x_{12}} \cup \dots \cup G_{1,x_{1n}}) \cap G_{2,p}$$

$$= \left(G_{1,x_{11}} \cap G_{2,p}\right) \cup \left(G_{1,x_{12}} \cap G_{2,p}\right) \cup \dots \cup (G_{1,x_{1n}} \cap G_{2,p})$$

$$= 0_{\sim} \cup 0_{\sim} \cup \dots \cup 0_{\sim}$$

$$= 0_{\sim}.$$

Theorem 2.5.4. Let A be an IF-compact subset of an IF-Hausdorff space X and $p_{(r,s)} \notin A$, then there exists an IFO-set $G = (\mu_G, \nu_G)$ such that $p_{(r,s)} \in G \subseteq A^C$.

Proof. In the above theorem 3.1, if we put $G = G_{2,p}$, then $G_{1,x} \cap G = 0_{\sim} \Rightarrow G \subseteq {G_{1,x}}^C$. Again we have $A \subset G_{1,x} \Rightarrow {G_{1,x}}^C \subseteq A^C$. So, $p_{(r,s)} \in G \subseteq {G_{1,x}}^C \subseteq A^C$.

Definition 2.5.5. A point $p_{(r,s)}$ is called interior point of A if we find an IFO-set G such that $p_{(r,s)} \in G \subseteq A$. Again if every point of A is interior of A, then A is open.

T**heorem 2.5.6.** Let A be an IF-compact subset of an IF-Hausdorff space X. Then A is closed.

Proof. We shall prove that A is closed IFS i.e. A^C is interior IFO-set. Consider $x_{(m,n)} \in A^C$ be any arbitrary IF-singleton, it is enough to prove that $x_{(m,n)}$ is interior point of A^C. Let $p_{(r,s)}$ be another IF-singleton and $p_{(r,s)} \in A$. By hypothesis, X is an

IF-Hausdorff space $\exists G_{1,x_{(m,n)}} = (\mu_{G_1}, \nu_{G_1})$ and $G_{2,p_{(r,s)}} = (\mu_{G_2}, \nu_{G_2})$ such that $\mu_{G_1}(x_{(m,n)}) = 1, \nu_{G_1}(x_{(m,n)}) = 0, \mu_{G_2}(p_{(r,s)}) = 1, \nu_{G_2}(p_{(r,s)}) = 0$ and $G_{1,x_{(m,n)}} \cap G_{2,p_{(r,s)}} = 0_{\sim}$. Hence we have $A \subset \cup \{G_{2,p_{(r,s)}}: p_{(r,s)} \in A\}$, i.e. $\{G_{2,p_{(r,s)}}: p_{(r,s)} \in A\}$ is an open cover of A but A is IF-compact so it has a finite subcover $G_{2,p_{11}}, G_{2,p_{12}}, \dots, G_{2,p_{1n}} \in \{G_{2,p_{(r,s)}}\}$ such that $A \subset G_{2,p_{11}} \cup G_{2,p_{12}} \cup \dots \cup G_{2,p_{1n}}$. Now let $G_{2.p} = G_{2,p_{11}} \cup G_{2,p_{12}} \cup \dots \cup G_{2,p_{1n}}$ and $G_{1,x} = G_{1,x_{11}} \cap G_{1,x_{12}} \cap \dots \cap G_{1,x_{1n}}$. Then $G_{1,x}$ and $G_{2,p}$ are open since they are the finite intersection and union of open sets respectively. Furthermore, $A \subset G_{2,p}$ and $x_{(m,n)} \in G_{1,x}$, since $p_{(r,s)}$ belongs to each $G_{1,x_{1j}}$ individually. Lastly we have to show that $G_{1,x} \cap G_{2,p} = 0_{\sim}$. Note first that, $G_{1,x_{1j}} \cap G_{2,p_{1j}} = 0_{\sim}$ implies that $G_{1,x} \cap G_{2,p_{1j}} = 0_{\sim}$. Thus by distributive law,

$$
\begin{aligned}
G_{1,x} \cap G_{2,p} &= G_{1,x} \cap (G_{2,p_{11}} \cup G_{2,p_{12}} \cup \dots \cup G_{2,p_{1n}}) \\
&= (G_{1,x} \cap G_{2,p_{11}}) \cup (G_{1,x} \cap G_{2,p_{12}}) \cup \dots \cup (G_{1,x} \cap G_{2,p_{1n}}) \\
&= 0_{\sim} \cup 0_{\sim} \cup \dots \cup 0_{\sim} \\
&= 0_{\sim}.
\end{aligned}
$$

$\Rightarrow G_{1,x} \subseteq {G_{2,p}}^C$

Again we have, $A \subseteq G_{2,p} \Rightarrow {G_{2,p}}^C \subseteq A^C$.

Hence, $x_{(m,n)} \in G_{1,x} \subseteq {G_{2.p}}^C \subseteq A^C$. So, $x_{(m,n)}$ is an interior point of A^C, since $x_{(m,n)}$ is an arbitrary so every point of A^C is interior of A^C. So, A^C is open i.e. A is closed.

Definition 2.5.7. An IFTS (X, τ) is called IF-normal if $F_1 = (\mu_{F_1}, \nu_{F_1})$ and $F_2 = (\mu_{F_2}, \nu_{F_2})$ be two closed set with $F_1 \cap F_2 = (0,1)$, then there exists $G = (\mu_G, \nu_G), H = (\mu_H, \nu_H) \in \tau$ such that $F_1 \subseteq G, F_2 \subseteq H$ and $G \cap H = (0,1)$.

Theorem 2.5.8. Let A and B be disjoint IF-compact closed subsets of an IF-Hausdorff space X. Then there exist disjoint open sets $G_{1,x}$ and $G_{2,y}$ such that $A \subset G_{1,x}$ and $B \subset G_{2,y}$.

i.e. Each IF-compact Hausdorff space is IF-normal.

Proof. Let $x_{(m,n)} \in A$ and $y_{(r,s)} \in B$ implies $x_{(m,n)} \neq y_{(r,s)}$ and $x_{(m,n)} \notin B$ for A and B are disjoint. By hypothesis, X is IF-Hausdorff Space and B is IF-compact then by the previous theorem 3.1 $\exists G_{1,x_{(m,n)}} = (\mu_{G_1}, \nu_{G_1})$ and $G_{2,y_{(r,s)}} = (\mu_{G_2}, \nu_{G_2})$ such that $x_{(m,n)} \in G_{1,x_{(m,n)}}$, $B \subset G_{2,y_{(r,s)}}$ and $G_{1,x_{(m,n)}} \cap G_{2,y_{(r,s)}} = 0_\sim$. Since $x_{(m,n)} \in G_{1,x_{(m,n)}}$, $\{G_{1,x_{(m,n)}} : x_{(m,n)} \in A\}$ is an open cover of A. Since A is IF-compact, we can select a finite number of open sets $G_{1,x_{11}}, G_{1,x_{12}}, \dots, G_{1,x_{1n}}$ so that $A \subset G_{1,x_{11}} \cup G_{1,x_{12}} \cup \dots \cup G_{1,x_{1n}}$. Furthermore, $B \subset G_{2,y_{21}} \cap G_{2,y_{22}} \cap \dots \cap G_{2,y_{2n}}$, since B is a subset of each individually. Now let $G_{1,x} = G_{1,x_{11}} \cup G_{1,x_{12}} \cup \dots \cup G_{1,x_{1n}}$ and $G_{2,y} = G_{2,y_{21}} \cap G_{2,y_{22}} \cap \dots \cap G_{2,y_{2n}}$. Observe by the above that $A \subset G_{1,x}$ and $B \subset G_{2,y}$. In addition, $G_{1,x}$ and $G_{2,y}$ are open since they are respectively the union and finite intersection of open sets. We have to show that $G_{1,x}$ and $G_{2,y}$ are disjoint. First observe that, $G_{1,x_{1j}} \cap G_{2,y_{2j}} = 0_\sim$ implies that $G_{1,x_{1j}} \cap G_{2,y} = 0_\sim$. Thus by distributive law,

$$
\begin{aligned}
G_{1,x} \cap G_{2,y} &= (G_{1,x_{11}} \cup G_{1,x_{12}} \cup \dots \cup G_{1,x_{1n}}) \cap G_{2,y} \\
&= \left(G_{1,x_{11}} \cap G_{2,y}\right) \cup \left(G_{1,x_{12}} \cap G_{2,y}\right) \cup \dots \cup (G_{1,x_{1n}} \cap G_{2,y}) \\
&= 0_\sim \cup 0_\sim \cup \dots \cup 0_\sim \\
&= 0_\sim.
\end{aligned}
$$

Definition 2.5.9. An IFTS (X,τ) is called IF-regular if $x_{(m,n)}$ be an IF-singleton does not belong to a closed set F i.e. $x_{(m,n)} \notin F = (\mu_F, \nu_F)$ then there exists $G = (\mu_G, \nu_G), H = (\mu_H, \nu_H) \in \tau$ such that $x_{(m,n)} \in G, F \subseteq H$ and $G \cap H = (0,1)$.

Theorem 2.5.10. Each IF-compact regular space is IF-normal.

Proof. Let $F_1 = (\mu_{F_1}, \nu_{F_1})$ and $F_2 = (\mu_{F_2}, \nu_{F_2})$ be two disjoint closed set, where $F_1 \cap F_2 = (0,1)$. Consider $x_{(m,n)} \in F_1$then for all $x_{(m,n)} \notin F_2$. Since $x_{(m,n)}$ be an IF-singleton does not belongs to F_2 and X is IF-compact regular there exists $G = (\mu_G, \nu_G), H = (\mu_H, \nu_H) \in \tau$ such that $x_{(m,n)} \in G, F_2 \subseteq H$ and $G \cap H = (0,1)$. Now, we have to show that $F_1 \subset G$. Since, $x_{(m,n)} \in G$, so $\{G: x_{(m,n)} \in F_1\}$ is an open cover of F_1. Since X is IF-compact then there exist finite number of open sets $G_1, G_2, \dots, G_n$ so that $F_1 \subset (G_1 \cup G_2 \cup \dots \cup G_n) = G$, here G is open as G is the union of open sets. So, $F_1 \subset G$.

2.6 IF-Locally Compact Space

Definition 2.6.1. An IFTS is IF-locally compact if every IF-singleton in X belongs in an IF-compact open set.

Theorem 2.6.2. Every IF-compact space (X,τ) is IF-locally compact.

Proof. Let $x_{(m,n)}$ be any fuzzy singleton such that $x_{(m,n)} \in A = (\mu_A, \nu_A) \in \tau$. Again $\{B_i = (\mu_{B_i}, \nu_{B_i})\}$ be an open cover of X i.e. $\cup B_i = (1,0)$. Since X is compact then there exists $j_1, j_2, \dots, j_n$ such that $\bigcup_{k=1}^{n} B_{jk} = (1,0)$. Since $A \subseteq (1,0)$ then $(A \cap B_i)$ is also an open cover of A as A and B_i are both open. Again, we get $A \cap (\bigcup_{k=1}^{n} B_{jk}) = (1,0) \cap A \quad \Rightarrow \bigcup_{k=1}^{n}(A \cap B_{jk}) = A$ i.e. A can be expressed as finite union of open covers $\{A \cap B_{jk}\}$. Hence A is IF-compact. So, (X, τ) is IF-locally compact.

CHAPTER THREE

Q-Compactness in IFTS

In this chapter, we have introduced Q-compactness in intuitionistic fuzzy compact topological spaces. Furthermore, we have established some theorems and examples of Q-compactness in intuitionistic fuzzy topological spaces and discussed different characterizations of Q-compactness.

Also we have defined $\delta - Q$ compactness, $Q - \sigma$ compactness and $\delta - Q - \sigma$ compactness in intuitionistic fuzzy topological spaces and found different properties between Q-compactness and $\delta - Q$ compactness, $Q - \sigma$ compactness and $\delta - Q - \sigma$ compactness in intuitionistic fuzzy topological spaces.

3.1 Definition and Relationship

In this section we have given some definitions and investigated some relations among various definitions.

Definition 3.1.1. Let (X, τ) be an intuitionistic fuzzy topological space (IFTS) and $A = (\mu_A, \nu_A)$ be an IFS in X. Consider $\mathcal{M} = \{B_i : i \in J\}$ be a family of IFS in X, where $B_i = (\mu_{B_i}, \nu_{B_i})$. Then $\mathcal{M}$ is called Q-cover of A if $A \subseteq \cup B_i$, $\mu_A(x) + \mu_{B_i}(x) \geq 1$ for each $\mu_{B_i}$and some $x \in X$. If each B_i is open then $\mathcal{M}$ is called an open Q-cover of A. A subfamily of Q-cover of an IFS A in X which is also a Q-cover of A is called Q-subcover of A.

Definition 3.1.2. An IFS $A = (\mu_A, \nu_A)$ in X is said to be Q-compact if every open Q-cover of A has a finite Q-subcover i.e. $\exists B_{i1}, B_{i2}, \dots, B_{in} \in \mathcal{M}$ such that $A \subseteq \bigcup_{i=1}^{n} B_i$, $\mu_A(x) + \mu_{B_{ij}}(x) \geq 1$ for each $\mu_{B_{ij}}$and some $x \in X$, $j = 1,2, \dots, n$.

Example 3.1.3. Let $X = \{a, b\}$ and $I = [0,1]$. Let $A_1, A_2 \in I^X$ defined by $A_1(a) = (0.5,0.2)$, $A_1(b) = (0.7,0.2)$, $A_2(a) = (0.6,0.3)$ and $A_2(b) = (0.8,0.1)$. Consider $\tau = \{(0,0), A_1, A_2, (1,0)\}$. Then (X, τ) be an intuitionistic fuzzy topological space (IFTS). Again let $A \in I^X$ with $A(a) = (0.5,0.3)$, $A(b) = (0.3,0.2)$. Here $A(a) \subseteq \cup A_i(a)$, $\mu_A(a) + \mu_{A_i}(a) \geq 1$. Again, $A(b) \subseteq \cup A_i(b)$, $\mu_A(b) + \mu_{A_i}(b) \geq 1$. Therefore $\{A_1, A_2\}$ is a Q-cover of A.

Theorem 3.1.5. Let (X, τ)be an IFTS. If $A = (\mu_A, \nu_A)$ and $V = (\mu_V, \nu_V)$ are Q-compact in (X, τ) then $A \cup V$ is also Q-compact in (X, τ).

Proof: Let $\mathcal{M} = \{A_i = (\mu_{A_i}, \nu_{A_i}) : i \in J\}$ be an open Q-cover of $A = (\mu_A, \nu_A)$ and $\aleph = \{B_i = (\mu_{B_i}, \nu_{B_i}) : i \in J\}$ be an open Q-cover of $V = (\mu_V, \nu_V)$ in (X, τ). Now $A \subseteq \bigcup_{i=1}^{n} A_i$ and $V \subseteq \bigcup_{i=1}^{n} B_i$ such that

$A \cup V \subseteq \bigcup_{i=1}^{n} A_i \cup \bigcup_{i=1}^{m} B_i$

$$\Rightarrow A \cup V \subseteq \begin{cases} \bigcup_{i=1}^{m}((A_i \cup B_i) \cup (\bigcup_{i=m+1}^{n} A_i)) & if \quad n > m \\ \bigcup_{i=1}^{n}((A_i \cup B_i) \cup (\bigcup_{i=n+1}^{mn} B_i)) & if \quad m > n \end{cases}$$

$\Rightarrow A \cup V \subseteq \cup (A_i \cup B_i)$

Again, by the definition of Q-compactness, we have $\mu_A(x) + \mu_{A_i}(x) \geq 1$ for each μ_{A_i} and some $x \in X$ and $\mu_V(x) + \mu_{B_i}(x) \geq 1$ for each μ_{B_i} and some $x \in X$

$\Rightarrow \mu_{(A \cup V)}(x) + \mu_{(A_i \cup B_i)}(x) \geq 1.$

Hence $\mathcal{M} \cup \aleph = \{A_i \cup B_i\}$ is an open Q-cover of $A \cup V$.

Again, as A is Q-compact in (X, τ) then A has finite Q-subcover i.e. there exist $A_{ik} \in \{A_i\}, k \in j_n$ such that $A \subseteq \bigcup_{k=1}^{n} A_{ik}$ and $\mu_A(x) + \mu_{A_{ik}}(x) \geq 1$ for each $\mu_{A_{ik}}$ and some $x \in X$. Also as V is Q-compact in (X, τ) then V has finite Q-subcover i.e. there exist $B_{ik} \in \{B_i\}, k \in j_n$ such that $V \subseteq \bigcup_{k=1}^{n} B_{ik}$ and $\mu_V(x) + \mu_{B_{ik}}(x) \geq 1$ for each $\mu_{B_{ik}}$ and some $x \in X$. Now $A \subseteq \bigcup_{k=1}^{n} A_{ik}$ and $V \subseteq \bigcup_{k=1}^{n} B_{ik}$ which gives

$$A \cup V \subseteq \bigcup_{k=1}^{n} A_{ik} \cup \bigcup_{k=1}^{m} B_{ik}$$

$$\Rightarrow A \cup V \subseteq \begin{cases} \bigcup_{k=1}^{m}((A_{ik} \cup B_{ik}) \cup (\bigcup_{i=m+1}^{n} A_{ik})) & if \quad n > m \\ \bigcup_{k=1}^{n}((A_{ik} \cup B_{ik}) \cup (\bigcup_{i=n+1}^{mn} B_{ik})) & if \quad m > n \end{cases}$$

$$\Rightarrow A \cup V \subseteq \cup (A_{ik} \cup B_{ik})$$

Also, $\mu_A(x) + \mu_{A_{ik}}(x) \geq 1$ and $\mu_V(x) + \mu_{B_{ik}}(x) \geq 1$ $\Rightarrow \mu_{(A \cup V)}(x) + \mu_{(A_{ik} \cup B_{ik})}(x) \geq 1$

i.e. $\{A_{ik} \cup B_{ik}\}$ is an open Q-subcover of $A \cup V$.

Hence $A \cup V$ is also Q-compact in (X, τ).

Theorem 3.1.6. Let (X, τ)be an IFTS and $A = (\mu_A, \nu_A)$ be an IFS in X. If every $\{F_i\}$ where $F_i = (\nu_{F_i}, \mu_{F_i})$ of closed subset of X with $\cap F_i = (0, 1)$ implies $\{F_i\}$ contains finite subclass $\{F_{i1}, F_{i2}, \dots, F_{im}\}$ with $F_{i1} \cap F_{i2} \cap \dots \cap F_{im} = (0, 1)$ then A is Q-compact in(X, τ). The converse is not true in general.

Proof: Given $\cap F_i = (0, 1)$ then by De Morgan's law $(\cap F_i)^c = \left((0, 1)\right)^c$

$$\Rightarrow \cup {F_i}^c = (1, 0)$$

$$\Rightarrow \cup \left(\nu_{F_i}, \mu_{F_i}\right)^c = (1, 0)$$

$$\Rightarrow \cup \left(\mu_{F_i}, \nu_{F_i}\right) = (1, 0)$$

$$\Rightarrow \left(\cup \mu_{F_i}, \cap \nu_{F_i}\right) = (1, 0).$$

Let $\mathcal{M} = \{B_i = (\mu_{B_i}, \nu_{B_i}) : i \in J\}$ be an open Q-cover of A in (X,τ), so $A \subseteq \cup B_i$, $\mu_A(x) + \mu_{B_i}(x) \geq 1$ for each μ_{B_i} and some $x \in X$. Since each B_i is open, so $\{{B_i}^c\}$ is a class of closed sets and by given condition $\exists {B_{i1}}^c, {B_{i2}}^c, \dots, {B_{im}}^c \epsilon \{{B_i}^c\}$ such that ${B_{i1}}^c \cap {B_{i2}}^c \cap \dots \cap {B_{im}}^c = (0,1)$. So by De Morgan's law $(1,0) = (0,1)^c = ({B_{i1}}^c \cap {B_{i2}}^c \cap \dots \cap {B_{im}}^c)^c = B_{i1} \cup B_{i2} \cup \dots \cup B_{im}$, hence $A \subseteq \cup_{i=1}^{n} B_{ij}$, $\mu_A(x) + \mu_{B_{ij}}(x) \geq 1, j = 1,2,\dots,n$ for each $\mu_{B_{ij}}$ and some $x \in X$. So, A is Q-compact in (X,τ).

3.2 Mapping in IF-Q-Compact Topological Space

In this section, we have discussed about image and preimage of IF-Q-Compact topological Space.

Theorem 3.2.1. Let (X,τ) and (Y,δ) be two IFTS and $f: X \to Y$ is bijective, open and continuous. If $f(A) = (f(\mu_A), f(\nu_A))$ is Q-compact in (Y,δ) then A is Q-compact in (X,τ).

Proof: Let $A = (\mu_A, \nu_A) \in \tau$. Consider $\mathcal{M} = \{B_i \in \tau\}$ where $B_i = (\mu_{B_i}, \nu_{B_i}), i \in J$ with $A \subseteq \cup B_i$ and $\mu_A(x) + \mu_{B_i}(x) \geq 1$ for each $\mu_{B_i}$and some $x \in X$ i.e. $\mathcal{M}$ is a Q-open cover of A, then $f(A) = (f(\mu_A), f(\nu_A))$ is an IFS of Y. Since $B_i \in \tau$ then $f(B_i) \in \delta$ as f is open. But $f(B_i) = (f(\mu_{B_i)}, f(\nu_{B_i}))$. Now we have $A \subseteq \cup B_i$ $\Longrightarrow$ $f(A) \subseteq f(\cup_i B_i) = \cup_i f(B_i)$ i.e. $f(A) \subseteq \cup_i f(B_i)$. For any $\in Y$, $f(\mu_A)(y) + f(\mu_{B_i})(y) = sup\mu_A(x) + sup\mu_{B_i}(x)$, where $x \in f^{-1}(y)$

$\geq \mu_A(x) + \mu_{B_i}(x) \forall i \in J$, since f is onto and $f(y) \neq \emptyset$

≥ 1

$\Longrightarrow f(\mu_A)(y) + f(\mu_{B_i})(y) \geq 1$

i.e. $\mathcal{H} = \{f(B_i): i \in J\}$ is Q-open cover of $f(A)$. Since $f(A)$ is Q-compact then $\exists f(B_{i1}), f(B_{i2}), \dots, f(B_{in}) \in \delta \ni f(A) \subseteq \bigcup_{k=1}^{n} f(B_{ik})$ and

$f(\mu_A)(y) + f(\mu_{B_{ik}})(y) \geq 1$

$\Longrightarrow f^{-1}\left(f(\mu_A)(y) + f(\mu_{B_{ik}})(y)\right) \geq f^{-1}(1)$

$\Longrightarrow f^{-1}f(\mu_A)(y) + f^{-1}f(\mu_{B_{ik}})(y) \geq 1$

$\Longrightarrow \mu_A(x) + \mu_{B_{ik}}(x) \geq 1$ as f is continuous and so $\forall y \in Y \Longrightarrow \exists$ unique $x \in X$ since $f(x) = y$.

Again, $f(A) \subseteq \bigcup_{k=1}^{n} f(B_{ik})$

$\Longrightarrow f^{-1}f(A) \subseteq f^{-1}(\bigcup_{k=1}^{n} f(B_{ik}))$

$\Longrightarrow A \subseteq \bigcup_{k=1}^{n} f^{-1}f(B_{ik})$

$\Longrightarrow A \subseteq \bigcup_{k=1}^{n} B_{ik}$.

Hence, we clear that $B_{ik} \in \tau \ni A \subseteq \bigcup_{k=1}^{n} B_{ik}$ and $\mu_A(x) + \mu_{B_{ik}}(x) \geq 1$. Now it is clear that A is Q-compact in (X, τ).

Theorem 3.2.2. Let (X, τ) and (Y, δ) be two IFTS and $f: X \to Y$ is bijective, open and continuous. If $A = (\mu_A, \nu_A)$ is Q-compact in (X, τ) then $f(A) = (f(\mu_A), f(\nu_A))$ is Q-compact in (Y, δ).

Proof: Let $\mathcal{M} = \{B_i \in \delta\}$, where $B_i = (\mu_{B_i}, \nu_{B_i}), i \in J$ be an open Q-cover of $f(A)$ with $f(A) \subseteq \bigcup B_i$ and $\mu_{f(A)}(y) + \mu_{B_i}(y) \geq 1$ for each $\mu_{B_i}$and some $y \in Y$. Since $B_i \in \delta$ then $f^{-1}(B_i) \in \tau$ but $f^{-1}(B_i) = \left(f^{-1}(\mu_{B_i}), f^{-1}(\nu_{B_i})\right)$. Now we have, $f(A) \subseteq \bigcup B_i \Longrightarrow A \subseteq f^{-1}(\cup B_i)$ i.e. $A \subseteq \cup f^{-1}(B_i)$. For any $x \in X$, $\mu_A(x) + \mu_{f^{-1}(B_i)}(x) \geq 1$, since A is Q-compact. i.e. $\mathcal{H} = \{f^{-1}(B_i): i \in J\}$ is Q-open cover of A. Since A is Q-compact in (X,τ) then $\exists f^{-1}(B_{i1}), f^{-1}(B_{i2}), \dots, f^{-1}(B_{in}) \in \tau \ni A \subseteq \bigcup_{k=1}^{n} f^{-1}(B_{ik})$ and $\mu_A(x) + \mu_{f^{-1}(B_{ik})}(x) \geq 1$

$\Rightarrow f(\mu_A)(x) + f(\mu_{f^{-1}(B_{ik})})(x) \geq f(1)$

$\Rightarrow \mu_{f(A)}(y) + \mu_{ff^{-1}(B_{ik})}(y) \geq 1$

$\Rightarrow \mu_{f(A)}(y) + \mu_{B_{ik}}(y) \geq 1$ as f is continuous.

Again, $A \subseteq \bigcup_{k=1}^{n} f^{-1}(B_{ik})$

$\Rightarrow f(A) \subseteq f(\bigcup_{k=1}^{n} f^{-1}(B_{ik}))$

$\Rightarrow f(A) \subseteq \bigcup_{k=1}^{n} ff^{-1}(B_{ik})$

$\Rightarrow f(A) \subseteq \bigcup_{k=1}^{n} B_{ik}$.

Hence, we clear that $B_{ik} \in \delta \ni f(A) \subseteq \bigcup_{k=1}^{n} B_{ik}$ and $\mu_{f(A)}(x) + \mu_{B_{ik}}(x) \geq 1$. So, $f(A)$ is Q-compact in (Y,δ).

3.3 Subspace and Product Space of IF-Q-Compact Topological Spaces

Theorem 3.3.1. Let (X,τ) be an IFTS, V is an subset of X and A be an IFS in V, where $A = (\mu_A, \nu_A)$. Then A is Q-compact in (X,τ) iff $A|V$ is Q-compact in (V,τ_V).

Proof: Suppose $A = (\mu_A, \nu_A)$ is Q-compact in (X,τ). Let $\mathcal{M} = \{B_i = (\mu_{B_i}, \nu_{B_i}) : i \in J\}$ be an open Q-cover of A in (V,τ_V).By the definition of subspace topology, $B_i = U_i|V$, where $U_i \in \tau$. Hence $\mu_A(x) + \mu_{B_i}(x) \geq 1$ for each $\mu_{B_i}$and some $x \in V$ and consequently $\mu_A(x) + \mu_{U_i}(x) \geq 1$ for each μ_{U_i} and some $x \in X$ as $V \subseteq X$. Now $A \subseteq \bigcup B_i \Rightarrow A \subseteq \bigcup U_i|V \Rightarrow A|V \subseteq \bigcup U_i|V$. Therefore $\{U_i : i \in J\}$ is an open Q-cover of $A|V$ in (X,τ). As A is Q-compact in (X,τ) then A has finite Q-subcover i.e. there exist

$U_{ik} \in \{U_i\}, k \in j_n$ such that $\mu_A(x) + \mu_{U_{ik}}(x) \geq 1$ for each $\mu_{U_{ik}}$ and some $x \in V$. This implies that $\mu_A(x) + \mu_{(U_{ik}|V)}(x) \geq 1$ for each $x \in V$.

Also $A \subseteq \cup B_{ik} \Rightarrow A \subseteq \cup U_{ik}|V \Rightarrow A|V \subseteq \cup U_{ik}|V$. Thus $\{B_i\}$ contains a finite subcover $\{B_{i1}, B_{i2}, \dots, B_{in}\}$ and hence $A|V$ is Q-compact in (V, τ_V).

Theorem 3.3.2. Let A and V be Q-compact IFS in an IFTS (X, τ). Then $A \times V$ is also Q-compact in $(X \times X, \tau \times \tau)$.

Proof: Let $\mathcal{M} = \{B_i : B_i = (\mu_{B_i}, \nu_{B_i}) \in \tau \times \tau, i \in J\}$ be a Q-cover of $A \times V$ in $(X \times X, \tau \times \tau)$. Then $A \times V \subseteq \cup B_i$ and $\mu_{A \times V}(x, y) + \mu_{B_i}(x, y) \geq 1$ for each $\mu_{B_i}$and some $(x, y) \in X \times X$. Now we can write, $B_i = U_i \times W_i$, where $U_i, W_i \in \tau$. Thus $A \times V \subseteq \cup B_i \Rightarrow A \times V \subseteq \cup(U_i \times W_i) \Rightarrow A \subseteq \cup U_i, V \subseteq \cup W_i$. Also we have $\mu_{A \times V}(x, y) + \mu_{U_i \times W_i}(x, y) \geq 1$ for each $\mu_{U_i \times W_i}$ and some $(x, y) \in X \times X$. Hence it is clear that $\mu_A(x) + \mu_{U_i}(x) \geq 1$ for each $\mu_{U_i}$and some $x \in X$ and $\mu_V(y) + \mu_{W_i}(y) \geq 1$ for each μ_{W_i} and some $y \in X$. Therefore $\{U_i : i \in J\}$ and $\{W_i : i \in J\}$ are open Q-cover of A and V respectively. Since A and V are Q-compacts then $\{U_i : i \in J\}$ and $\{W_i : i \in J\}$ have finite Q-subcovers, say $\{U_{ik} : k \in J_n\}$ and $\{W_{ik} : k \in J_n\}$ respectively such that $A \subseteq \cup U_{ik}, \mu_A(x) + \mu_{U_ik}(x) \geq 1$ for each $\mu_{U_{ik}}$and some $x \in X$ and

$V \subseteq \cup W_{ik}, \mu_V(y) + \mu_{W_{ik}}(y) \geq 1$ for each $\mu_{W_{ik}}$and some y$\in X$. Thus we can write, $A \times V \subseteq \cup(U_{ik} \times W_{ik}) \Rightarrow A \times V \subseteq \cup B_{ik}$ and $\mu_{A \times V}(x, y) + \mu_{B_{ik}}(x, y) \geq 1$ for each $\mu_{B_{ik}}$and some $(x, y) \in X \times X$. Hence $A \times V$ is Q-compact in $(X \times X, \tau \times \tau)$.

3.4 $\delta - Q$ Compactness in IFTS

Definition 3.4.1. Let (X, τ) be an intuitionistic fuzzy topological space and $0 < \delta \leq 1$. A family $\{(\mu_{G_i}, \nu_{G_i}): i \in J\}$ of IFOS in X is called δ-open cover of X if $\cup \mu_{G_i} \geq \delta$ and $\cap \nu_{G_i} = 0$. If every δ-open cover of X has a finite subcover then X is said to be δ-IF-compact.

Definition 3.4.2. Let (X, τ) be an intuitionistic fuzzy topological space and $0 < \delta \leq 1$. An IFS $A = (\mu_A, \nu_A)$ in X is said to be δ-open in X iff $\mu_A(x) \geq \delta$ for all $x \in X$. An IFS is said to be δ-closed iff its complement is δ-open.

Definition 3.4.3. Let $\mathcal{M} = \{B_i: i \in J\}$ where $B_i = (\mu_{B_i}, \nu_{B_i})$ be a family of δ-open IFS in an IFTS (X, τ) and $A = (\mu_A, \nu_A)$ be an IFS in X. Then $\mathcal{M}$ is said to be $\delta -$Q cover of A if $A \subseteq \cup B_i$, $\mu_A(x) + \mu_{B_i}(x) \geq 1$ for each $\mu_{B_i}$and some $x \in X$. If each B_i is open then $\mathcal{M}$ is called an open $\delta - Q$ cover of A. A subfamily of $\delta - Q$ cover of an IFS A in X which is also a $\delta - Q$ cover of A is called $\delta - Q$ subcover of A.

Definition 3.4.4. An IFS $A = (\mu_A, \nu_A)$ in X is said to be $\delta - Q$ compact if every open $\delta - Q$ cover of A has a finite $\delta - Q$ subcover i.e. $\exists B_{i1}, B_{i2}, \ldots, B_{in} \in \mathcal{M}$ such that $A \subseteq \cup_{i=1}^{n} B_i$, $\mu_A(x) + \mu_{B_{ij}}(x) \geq 1$ for each $\mu_{B_{ij}}$and some $x \in X$, $j = 1,2, \ldots, n$.

Example 3.4.5. Let $X = \{a, b\}$ and $I = [0,1]$. Let $A_1, A_2 \in I^X$ defined by $A_1(a) = (0.5,0.2)$, $A_1(b) = (0.7,0.2)$, $A_2(a) = (0.6,0.3)$ and $A_2(b) = (0.8,0.1)$. Consider, $\tau = \{(0,0), A_1, A_2, (1,0)\}$. Then (X, τ) be an intuitionistic fuzzy topological space

(IFTS). Again let $A \in I^X$ with $A(a) = (0.5,0.3)$, $A(b) = (0.3,0.2)$. Here $A(a) \subseteq\cup A_i(a)$, $\mu_A(a) + \mu_{A_i}(a) \geq 1$. Again, $A(b) \subseteq\cup A_i(b)$, $\mu_A(b) + \mu_{A_i}(b) \geq 1$. If we take $\delta = 0.4$ then $\{A_1, A_2\}$ is a $\delta - Q$ cover of A.

3.5 $Q - \sigma$ Compactness and $\delta - Q - \sigma$ Compactness in IFTS

Definition 3.5.1. Let (X, τ) be an intuitionistic fuzzy topological space (IFTS), $A = (\mu_A, \nu_A)$ be an IFS in X and $0 < \sigma \leq 1$. Consider $\mathcal{M} = \{B_i : i \in J\}$ be a family of IFS in X, where $B_i = (\mu_{B_i}, \nu_{B_i})$. Then $\mathcal{M}$ is called $Q - \sigma$ -cover of A if $A \subseteq \cup B_i$, $\mu_A(x) + \mu_{B_i}(x) \geq \sigma$ for each $\mu_{B_i}$and some $x \in X$. If each B_i is open then $\mathcal{M}$ is called an open $Q - \sigma$ cover of A. A subfamily of $Q - \sigma$ cover of an IFS A in X which is also a $Q - \sigma$ cover of A is called $Q - \sigma$ subcover of A.

Definition 3.5.2. An IFS $A = (\mu_A, \nu_A)$ in X is said to be $Q - \sigma$ compact if every open $Q - \sigma$ cover of A has a finite $Q - \sigma$ subcover.

Definition 3.5.3. Let (X, τ) be an intuitionistic fuzzy topological space (IFTS), $A = (\mu_A, \nu_A)$ be an IFS in X, $0 < \delta \leq 1$ and $0 < \sigma \leq 1$. Let $\mathcal{M} = \{B_i : i \in J\}$ where $B_i = (\mu_{B_i}, \nu_{B_i})$ be a family of δ-open IFS. Then $\mathcal{M}$ is said to be $\delta - Q - \sigma$ cover of A if $A \subseteq \cup B_i$, $\mu_A(x) + \mu_{B_i}(x) \geq \sigma$ for each $\mu_{B_i}$and some $x \in X$. If each B_i is open then $\mathcal{M}$ is called an open $\delta - Q - \sigma$ cover of A. A subfamily of $\delta - Q - \sigma$ cover of an IFS A in X which is also a $\delta - Q - \sigma$ cover of A is called $\delta - Q - \sigma$ subcover of A.

Definition 3.5.4. An IFS $A = (\mu_A, \nu_A)$ in X is said to be $\delta - Q - \sigma$ compact if every open $\delta - Q - \sigma$ cover of A has a finite $\delta - Q - \sigma$ subcover.

Chapter Four

Some Type of Compactness in IFTS

In this chapter, we discusses various type of compactness in intuitionistic fuzzy topological spaces. Almost compact fuzzy sets was first constructed by Concilio and Gerla which is local property. Here we give two new possible notions of almost compactness in intuitionistic fuzzy topological spaces are studied and investigated some of their properties. We show that these notions satisfy hereditary and productive property in intuitionistic fuzzy topological spaces. Under some conditions it is shown that image and preimage preserve intuitionistic fuzzy topological spaces. Also we give three new notions of I-compactness, C-compactness and $I-C$-compactness in intuitionistic fuzzy topological spaces and investigate some relationship among our notions.

4.1 Almost Compactness

In this section, we have discussed several characterizations of almost compactness in intuitionistic fuzzy topological spaces and established some of their features.

4.1.1 Definition and Relationship

In this subsection we have given two new possible notions of almost compactness in intuitionistic fuzzy topological spaces and established some relationship among them.

Definition 4.1.1.1. Let (X,τ) be an intuitionistic fuzzy topological space (IFTS) and $A=(\mu_A,\nu_A)$ be an IFS in X. A family $\mathcal{M}=\{G_i=(\mu_{G_i},\nu_{G_i}): i\in J\}$ be a family of IFS is a proximate cover of A, when $\{\bar{G}_i: i\in J\}$ is a cover of A. A subfamily of $\mathcal{M}=\{G_i=(\mu_{G_i},\nu_{G_i}): i\in J\}$ which is also a proximate cover of A is said to be proximate subcover of A.

Definition 4.1.1.2. An IFS $A=(\mu_A,\nu_A)$ in an IFTS (X,τ) is said to be IF-almost compact iff every open cover of A has a finite subfamily whose closures is cover of A or, equivalently, every open cover of A has a finite proximate subcover.

Theorem 4.1.1.3. Let (X,τ)be an IFTS. If $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ are IFS in X. If A and B are IF-almost compact in (X,τ) then $A\cup B$ is also IF-almost compact in (X,τ).

Proof: Let $\mathcal{M}=\{A_i=(\mu_{A_i},\nu_{A_i}): i\in J\}$ be an open cover of $A=(\mu_A,\nu_A)$ and $\aleph=\{B_i=(\mu_{B_i},\nu_{B_i}): i\in J\}$ be an open cover of $B=(\mu_B,\nu_B)$ in (X,τ). So, $\{\bar{A}_i: i\in J\}$ is a cover of A and $\{\bar{B}_i: i\in J\}$ is a cover of B. Hence $A\subseteq\bigcup_{i=1}^{n}\bar{A}_i$ and $B\subseteq\bigcup_{i=1}^{n}\bar{B}_i$.

Now $A\cup B\subseteq\bigcup_{i=1}^{n}\bar{A}_i\cup\bigcup_{i=1}^{m}\bar{B}_i$

$$=\begin{cases}\bigcup_{i=1}^{m}((\bar{A}_i\cup\bar{B}_i)\cup(\bigcup_{i=m+1}^{n}\bar{A}_i)) & if \quad n>m\\ \bigcup_{i=1}^{n}((\bar{A}_i\cup\bar{B}_i)\cup(\bigcup_{i=n+1}^{m}\bar{B}_i)) & if \quad m>n\end{cases}$$

$$\Rightarrow A\cup B\subseteq\cup(\bar{A}_i\cup\bar{B}_i)$$

i.e. $\{\bar{A}_i\cup\bar{B}_i: i\in J\}$ is a cover of $A\cup B$.

Again, as A is IF-almost compact in (X,τ) then A has finite proximate subcover i.e. there exist $\bar{A}_{ik}\in\{\bar{A}_i\}, k\in j_n$ such that $\subseteq\bigcup_{k=1}^{n}\bar{A}_{ik}$. Also as B is IF-almost compact in (X,τ) then B has finite proximate subcover i.e. there exist $\bar{B}_{ik} \in \{\bar{B}_i\}, k \in J_n$ such that $B \subseteq \bigcup_{k=1}^{n} \bar{B}_{ik}$. Now from $A \subseteq \bigcup_{k=1}^{n} \bar{A}_{ik}$ and $B \subseteq \bigcup_{k=1}^{n} \bar{B}_{ik}$ gives

$$A \cup B \subseteq \bigcup_{k=1}^{n} \bar{A}_{ik} \cup \bigcup_{k=1}^{m} \bar{B}_{ik}$$

$$= \begin{cases} \bigcup_{k=1}^{m}((\bar{A}_{ik} \cup \bar{B}_{ik}) \cup (\bigcup_{i=m+1}^{n} \bar{A}_{ik})) & if \quad n > m \\ \bigcup_{k=1}^{n}((\bar{A}_{ik} \cup \bar{B}_{ik}) \cup (\bigcup_{i=n+1}^{m} \bar{B}_{ik})) & if \quad m > n \end{cases}$$

$$\Rightarrow A \cup B \subseteq \cup (\bar{A}_{ik} \cup \bar{B}_{ik})$$

i.e. $\{\bar{A}_{ik} \cup \bar{B}_{ik} : k \in J_n\}$ is a proximate subcover of $A \cup B$.

Hence $A \cup B$ is IF-almost compact in (X,τ).

Theorem 4.1.1.4. Let (X,τ)be an IFTS and $A = (\mu_A, \nu_A)$ be an IFS in X. If every $\{F_i\}$ where $F_i = (\nu_{F_i,}\mu_{F_i})$ of closed subset of X with $\cap F_i = (0,1)$ implies $\{F_i\}$ contains finite subclass $\{F_{i1}, F_{i2}, \dots, F_{im}\}$ with $F_{i1} \cap F_{i2} \cap \dots \cap F_{im} = (0,1)$ then A is IF-almost compact in(X,τ). The converse is not true in general.

Proof: Given $\cap F_i = (0,1)$ then by De Morgan's law $(\cap F_i)^c = \big((0,1)\big)^c$

$$\Rightarrow \cup {F_i}^c = (1,0)$$

$$\Rightarrow \cup \left(\nu_{F_i,}\mu_{F_i}\right)^c = (1,0)$$

$$\Rightarrow \cup \left(\mu_{F_i}, \nu_{F_i}\right) = (1,0)$$

$$\Rightarrow \left(\cup \mu_{F_i}, \cap \nu_{F_i}\right) = (1,0).$$

Let $\mathcal{M} = \left\{G_i = \left(\mu_{G_i}, \nu_{G_i}\right) : i \in J\right\}$ be an open cover of A in (X,τ), so $\{\bar{G}_i : i \in J\}$ is a cover of A. Therefore $A \subseteq \cup \bar{G}_i$. Since each $\bar{G}_i$ is open, so $\{\bar{G}_i{}^c\}$ is a class of closed sets and by given condition $\exists \bar{G}_{i1}{}^c, \bar{G}_{i2}{}^c, \dots, \bar{G}_{im}{}^c \epsilon \{\bar{G}_i{}^c\}$ such that $\bar{G}_{i1}{}^c \cap \bar{G}_{i2}{}^c \cap \dots \cap \bar{G}_{im}{}^c = (0,1)$. So by De Morgan's law $(1,0) = (0,1)^c = (\bar{G}_{i1}{}^c \cap \bar{G}_{i2}{}^c \cap \dots \cap$

$\bar{G}_{im}{}^{c})^{c} = \bar{G}_{i1} \cup \bar{G}_{i2} \cup \dots \cup \bar{G}_{im}$, hence $A \subseteq \bigcup_{i=1}^{n} \bar{G}_{ij}$, i.e. $\{\bar{G}_{ij}: j \in J_n\}$ is a proximate subcover of A. So, A is IF-almost compact in (X, τ).

4.1.2 Mapping of Almost Compactness

In this section, we have discussed about image and pre-image of almost compactness in intuitionistic fuzzy topological spaces.

Theorem 4.1.2.1. Let (X, τ) and (Y, δ) be two IFTS and $f: X \to Y$ is bijective, open and continuous. If $f(A) = (f(\mu_A), f(\nu_A))$ is IF-almost compact in (Y, δ) then $A = (\mu_A, \nu_A)$ is IF-almost compact in (X, τ).

Proof: Let $A = (\mu_A, \nu_A)$ be IFS in X. Consider $\mathcal{M} = \{G_i: i \in J\}$ where $G_i = \left(\mu_{G_i}, \nu_{G_i}\right)$, with $A \subseteq \bigcup G_i$ is an open cover of A, then $\{\bar{G}_i: i \in J\}$ is also a cover of A, i.e. $A \subseteq \bigcup \bar{G}_i$. Since f is open then $f(A) = (f(\mu_A), f(\nu_A))$ is an IFS of Y. Again, since $\bar{G}_i \in \tau$ then $f(\bar{G}_i) \in \delta$. Now we have $A \subseteq \bigcup \bar{G}_i \Rightarrow f(A) \subseteq f(\bigcup_i \bar{G}_i) = \bigcup_i f(\bar{G}_i)$ i.e. $f(A) \subseteq \bigcup_i f(\bar{G}_i)$.

Since $f(A)$ is IF-almost compact then $\exists f(\bar{G}_{i1}), f(\bar{G}_{i2}), \dots, f(\bar{G}_{in}) \in \delta \ni f(A) \subseteq \bigcup_{k=1}^{n} f(\bar{G}_{ik})$

Now $f(A) \subseteq \bigcup_{k=1}^{n} f(\bar{G}_{ik})$

$\Rightarrow f^{-1}f(A) \subseteq f^{-1}(\bigcup_{k=1}^{n} f(\bar{G}_{ik}))$

$\Rightarrow A \subseteq \bigcup_{k=1}^{n} f^{-1}f(\bar{G}_{ik})$

$\Rightarrow A \subseteq \bigcup_{k=1}^{n} \bar{G}_{ik}$.

Hence we clear that $\bar{G}_{ik} \in \tau \ni A \subseteq \bigcup_{k=1}^{n} \bar{G}_{ik}$. So, $\{\bar{G}_{ik}: k \in J_n\}$ is a proximate subcover of A. Hence A is IF-almost compact in (X, τ).

Theorem 4.1.2.2. Let (X,τ) and (Y,δ) be two IFTS and $f: X \to Y$ is bijective, open and continuous. If $A = (\mu_A, \nu_A)$ is IF-almost compact in (X,τ) then $f(A) = (f(\mu_A), f(\nu_A))$ is IF-almost compact in (Y,δ).

Proof: Let $\mathcal{M} = \{G_i \in \delta: i \in J\}$ where $G_i = \left(\mu_{G_i}, \nu_{G_i}\right)$, be an open cover of $f(A)$ then $\{\bar{G}_i: i \in J\}$ is also a cover of $f(A)$ with $f(A) \subseteq \cup\bar{G}_i$. Since f is continuous and $\bar{G}_i \in \delta$ then $f^{-1}(\bar{G}_i) \in \tau$ but $f^{-1}(B_i) = \left(f^{-1}(\mu_{B_i}), f^{-1}(\nu_{B_i})\right)$. Now we have, $f(A) \subseteq \cup\bar{G}_i \Rightarrow A \subseteq f^{-1}(\cup\, \bar{G}_i)$ i.e. $A \subseteq\cup\, f^{-1}(B_i)$. i.e. $\mathcal{H} = \{f^{-1}(\bar{G}_i): i \in J\}$ is a cover of A. Since A is IF-almost compact in (X,τ) then $\exists f^{-1}(\bar{G}_{i1}), f^{-1}(\bar{G}_{i2}), \dots, f^{-1}(\bar{G}_{in}) \in \tau \ni A \subseteq \cup_{k=1}^{n} f^{-1}(\bar{G}_{ik})$. Again $A \subseteq \cup_{k=1}^{n} f^{-1}(\bar{G}_{ik}) \Rightarrow f(A) \subseteq f(\cup_{k=1}^{n} f^{-1}(\bar{G}_{ik})) \Rightarrow f(A) \subseteq \cup_{k=1}^{n} ff^{-1}(\bar{G}_{ik}) \Rightarrow f(A) \subseteq \cup_{k=1}^{n} \bar{G}_{ik}$. Hence we clear that $\bar{G}_{ik} \in \delta \ni f(A) \subseteq \cup_{k=1}^{n} \bar{G}_{ik}$. So, $\{\bar{G}_{ik}: k \in J_n\}$ is a proximate subcover of $f(A)$. Hence $f(A)$ is IF-almost compact in (Y,δ).

4.1.3 Subspace and Product Space of Almost Compactness

In this section, we have discussed about subspace and product space of almost compactness in intuitionistic fuzzy topological spaces.

Theorem 4.1.3.1. Let (X,τ) be an IFTS and (V,τ_V) be a subspace of (X,τ). Let $A = (\mu_A, \nu_A)$ be an IFS in X. If A is IF-almost compact in (X,τ) then $A|V$ is IF-almost compact in (V,τ_V).

Proof: Let $\mathcal{M} = \{G_i = (\mu_{G_i}, \nu_{G_i}): i \in J\}$ be an open cover of A so $\{\bar{G}_i: i \in J\}$ is also a cover of A, i.e. $A \subseteq\cup\, \bar{G}_i$. So, $A|V \subseteq\cup\, \overline{G_i|V}$, hence $\{\overline{G_i|V}: i \in J\}$ is a cover of $A|V$.

Since A is IF-almost compact in (X, τ), then A has finite proximate subcover i.e. there exist $\bar{G}_{ik} \in \{\bar{G}_i\}, k \in J_n$ such that $A \subseteq \bigcup_{k=1}^{n} \bar{G}_{ik} \Rightarrow A|V \subseteq \bigcup_{k=1}^{n} \bar{G}_{ik}|V\, U_{ik}|V$. Thus $\{\overline{G_\iota|V}: i \in J\}$ contains a finite proximate subcover $\{\bar{G}_{ik}|V: k \in J_n\}$ and hence $A|V$ is IF-almost compact in (V, τ_V).

Theorem 4.1.3.2. Let A and V be IF-almost compact IFS in an IFTS (X, τ). Then $A \times V$ is also IF-almost compact in $(X \times X, \tau \times \tau)$.

Proof: Let $\mathcal{M} = \left\{G_i : G_i = \left(\mu_{G_i}, \nu_{G_i}\right) \in \tau \times \tau, i \in J\right\}$ be an open cover of $A \times V$ in $(X \times X, \tau \times \tau)$. So, $\{\bar{G}_i, i \in J\}$ is also a cover of $A \times V$ in $(X \times X, \tau \times \tau)$. Then $A \times V \subseteq \bigcup \bar{G}_i$. Now we can write, $G_i = U_i \times W_i$, where $U_i, W_i \in \tau$.

Thus $A \times V \subseteq \bigcup G_i$

$\Rightarrow A \times V \subseteq \bigcup (U_i \times W_i)$

$\Rightarrow A \subseteq \bigcup U_i, V \subseteq \bigcup W_i$

Therefore $\{U_i : i \in J\}$ and $\{W_i : i \in J\}$ are open cover of A and V respectively. So, $\{\bar{U}_i : i \in J\}$ and $\{\bar{W}_i : i \in J\}$ are also covers of A and V respectively. Since A and V are IF-almost compacts then $\{\bar{U}_i : i \in J\}$ and $\{\bar{W}_i : i \in J\}$ have finite proximate subcovers, say $\{\bar{U}_{ik} : k \in J_n\}$ and $\{\bar{W}_{ik} : k \in J_n\}$ respectively such that $A \subseteq \bigcup \bar{U}_{ik}$ and $V \subseteq \bigcup \bar{W}_{ik}$.

Thus we can write, $A \times V \subseteq \bigcup (\bar{U}_{ik} \times \bar{W}_{ik})$

$\Rightarrow A \times V \subseteq \bigcup \bar{B}_{ik}$.

Hence $A \times V$ is an IF-almost compact in $(X \times X, \tau \times \tau)$.

by definition of τ; $(u, u^c), (v, v^c) \in \tau \; as \; u, v \in t$. Therefore (X, τ) is IF-T2(r-i).

4.2 I-compact, C-compact and $I-C$-compact in IFTS

In this section, we have discussed about the I-compact, C-compact and $I-C$-compact in intuitionistic fuzzy topological spaces.

Definition 4.2.1. (D. Jankovic and T.R. Hamlet, New topologies from old via ideal, Am. Math. Mon., 97 (1990), 295–310)**:** A non-empty collection I of subsets of a non-empty set X is said to be an ideal on X if it satisfies the following two conditions:

i) $A \in I$ and $B \subseteq A \Rightarrow B \in I$ (hereditary)

ii) $A \in I$ and $B \in I \Rightarrow A \cup B \in I$ (finite additivity)

Definition 4.2.2. Let I be an ideal on IFTS (X, τ). A cover $\{(\mu_{G_i}, \nu_{G_i}) : i \in J\}$ of IFOS in X is said to be an I-cover if there exists a finite subset J_0 of J such that $\{(\mu_{G_{i_n}}, \nu_{G_{i_n}}) : i_n \in J_0\}$ covers X excepts, for some IFS which belongs to the ideal I, i.e. $\left(\bigcup_{i_n \in J_0} \left(\mu_{G_{i_n}}, \nu_{G_{i_n}}\right)\right)^C \in I$.

Definition 4.2.3. An IFTS (X, τ) with an ideal I is said to be IF-I-compact if every open cover of X is an I-cover.

Theorem 4.2.4. Let (X, τ) and (Y, δ) be IFTSs and $f: X \to Y$ is bijective, open and continuous. Then (X, τ) is IF-I-compact$\Longrightarrow$ (Y, δ) is IF-I-compact.

Proof: Let I be an ideal on Y and assume $\mathcal{M} = \{G_i = (\mu_{G_i}, \nu_{G_i}) : i \in J\}$ is an open cover of Y. Since f is continuous then $f^{-1}(I)$ is also an ideal on X and $\{f^{-1}(G_i) =$

$\left(f^{-1}(\mu_{G_i}), f^{-1}(\nu_{G_i})\right)\big| i \in J\}$ is an open cover of X. Since (X, τ) is IF-I-compact then there exists $f^{-1}(G_{i1}), f^{-1}(G_{i2}), \dots, f^{-1}(G_{in})$ such that $(f^{-1}(G_{i1}) \cup f^{-1}(G_{i2}) \cup \dots \cup f^{-1}(G_{in}))^C \in f^{-1}(I)$

$\Rightarrow f(f^{-1}(G_{i1}) \cup f^{-1}(G_{i2}) \cup \dots \cup f^{-1}(G_{in}))^C \in f(f^{-1}(I))$

$\Rightarrow (ff^{-1}(G_{i1}) \cup ff^{-1}(G_{i2}) \cup \dots \cup ff^{-1}(G_{in}))^C \in ff^{-1}(I)$

$\Rightarrow (G_{i1} \cup G_{i2} \cup \dots \cup G_{in})^C \in I$

$\Rightarrow \left(\cup_{j=1}^{n}(G_{ij})\right)^C \in I$

So, $\mathcal{M} = \left\{G_i = \left(\mu_{G_i}, \nu_{G_i}\right): i \in J\right\}$ is an I-open cover of Y and hence (Y, δ) is IF-I-compact.

Theorem 4.2.5. Let (X, τ) and (Y, δ) be IFTSs and $f: X \rightarrow Y$ is bijective, open and continuous. Then (Y, δ) is IF-I-compact$\Longrightarrow$ (X, τ) is IF-I-compact.

Proof: Let I be an ideal on X and assume $\mathcal{M} = \left\{G_i = \left(\mu_{G_i}, \nu_{G_i}\right): i \in J\right\}$ is an open cover of X. Since f is open then $f(I)$ is also an ideal on $f(X) = Y$. We know, for $y \in Y$, $f(A_i)$(y) = $(y, f\left(\mu_{A_i}\right)(y), f\left(\nu_{A_i}\right)(y))$, where $f\left(\mu_{A_i}\right)(y) = \sup_{x \in f^{-1}(y)} \mu_{A_i}(x)$ $=\mu_{A_i}(x)$. Similarly we get, $f\left(\nu_{A_i}\right)(y) = \nu_{A_i}(x)$. So, we get $\left\{f(G_i) = \left(f(\mu_{G_i}), f(\nu_{G_i})\right)\big| i \in J\right\}$ is an open cover of Y. Since (Y, δ) is IF-I-compact then there exists $f(G_{i1}), f(G_{i2}), \dots, f(G_{in})$ such that $(f(G_{i1}) \cup f(G_{i2}) \cup \dots \cup f(G_{in}))^C \in f(I)$

$\Rightarrow f^{-1}(f(G_{i1}) \cup f(G_{i2}) \cup \dots \cup f(G_{in}))^C \in f^{-1}(f(I))$

$\Rightarrow (f^{-1}f(G_{i1}) \cup f^{-1}f(G_{i2}) \cup \dots \cup f^{-1}f(G_{in}))^C \in f^{-1}f(I)$

$\Rightarrow (G_{i1} \cup G_{i2} \cup \dots \cup G_{in})^C \in I$ (since from Chang $\mu \subseteq f^{-1}\left(f(\mu)\right)$).

$\Rightarrow \left(\cup_{j=1}^{n}(G_{ij})\right)^C \in I$

So, $\mathcal{M} = \{G_i = (\mu_{G_i}, \nu_{G_i}): i \in J\}$ is an I-open cover of X and hence (X, τ) is IF-I-compact.

Definition 4.2.6. An IFTS (X, τ) is said to be IF- C-compact if for every IFCS $F = (\mu_F, \nu_F)$ and every open cover $\mathcal{M} = \{(\mu_{G_i}, \nu_{G_i}): i \in J\}$ of F, there exists a finite sub-collection $\{G_{i1}, G_{i2}, \dots, G_{in}\}$ of $\mathcal{M}$ such that $F \subseteq \bigcup_{j=1}^{n}(\bar{G}_{ij})$.

Theorem 4.2.7. Let (X, τ) and (Y, δ) be IFTSs and $f: X \to Y$ is bijective, open and continuous. Then (X, τ) is IF-C-compact$\Longrightarrow$ (Y, δ) is IF- C-compact.

Proof: Let $F = (\mu_F, \nu_F)$ be an IFCS of Y and assume $\mathcal{M} = \{G_i = (\mu_{G_i}, \nu_{G_i}): i \in J\}$ is an open cover of Y. Since f is continuous then $f^{-1}(F) = (f^{-1}(\mu_F), f^{-1}(\nu_F))$ is also an IFCS on X and $\{f^{-1}(G_i) = (f^{-1}(\mu_{G_i}), f^{-1}(\nu_{G_i})) | i \in J\}$ is an open cover of X. Since (X, τ) is IF-F-compact then there exists $f^{-1}(G_{i1}), f^{-1}(G_{i2}), \dots, f^{-1}(G_{in})$ such that

$$f^{-1}(F) \subseteq \bigcup_{j=1}^{n} f^{-1}(\bar{G}_{ij}) \Rightarrow f f^{-1}(F) \subseteq f(\bigcup_{j=1}^{n} f^{-1}(\bar{G}_{ij}))$$

$$\Rightarrow F \subseteq \bigcup_{j=1}^{n} f f^{-1}(\bar{G}_{ij})$$

$$\Rightarrow F \subseteq \bigcup_{j=1}^{n}(\bar{G}_{ij})$$

Hence (Y, δ) is IF- F-compact.

Definition 4.2.8. Let I be an ideal on an IFTS (X, τ). Then (X, τ) is said to be IF- I-C-compact if for every IFCS $F = (\mu_F, \nu_F)$ and every open cover $\mathcal{M} = \{(\mu_{G_i}, \nu_{G_i}): i \in J\}$ of F, there exists a finite sub-collection $\{G_{i1}, G_{i2}, \dots, G_{in}\}$ of $\mathcal{M}$ such that $\left(\bigcup_{j=1}^{n}(\bar{G}_{ij})\right)^C \in I$.

Theorem 4.2.9. Let (X,τ) and (Y,δ) be IFTSs and $f: X \to Y$ is bijective, open and continuous. Then (X,τ) is IF- I-C-compact$\Longrightarrow$ (Y,δ) is IF- I- C-compact.

Proof: Let I be an ideal on Y, $F = (\mu_F, \nu_F)$ be an IFCS of Y and assume $\mathcal{M} = \{G_i = (\mu_{G_i}, \nu_{G_i}) : i \in J\}$ is an open cover of Y. Since f is continuous so $f^{-1}(I)$ is also an ideal on X, $f^{-1}(F) = (f^{-1}(\mu_F), f^{-1}(\nu_F))$ is also an IFCS on X and $\{f^{-1}(G_i) = (f^{-1}(\mu_{G_i}), f^{-1}(\nu_{G_i})) | i \in J\}$ is an open cover of X. Since (X,τ) is IF-I-C-compact then there exists $f^{-1}(G_{i1}), f^{-1}(G_{i2}), \ldots, f^{-1}(G_{in})$ such that $(f^{-1}(\bar{G}_{i1}) \cup f^{-1}(\bar{G}_{i2}) \cup \ldots \cup f^{-1}(\bar{G}_{in}))^C \in f^{-1}(I)$

$\Rightarrow f(f^{-1}(\bar{G}_{i1}) \cup f^{-1}(\bar{G}_{i2}) \cup \ldots \cup f^{-1}(\bar{G}_{in}))^C \in f(f^{-1}(I))$

$\Rightarrow (ff^{-1}(\bar{G}_{i1}) \cup ff^{-1}(\bar{G}_{i2}) \cup \ldots \cup ff^{-1}(\bar{G}_{in}))^C \in ff^{-1}(I)$

$\Rightarrow (\bar{G}_{i1} \cup \bar{G}_{i2} \cup \ldots \cup \bar{G}_{in})^C \in I$

$\Rightarrow \left(\bigcup_{j=1}^{n}(\bar{G}_{ij})\right)^C \in I$

So, (Y,δ) is IF-I- C-compact.

4.3 Paracompactness in IFTS

Definition 4.3.1. An open covering $\{U_i\}$ of an IFTS X is locally finite if every IF-singleton $x_{\alpha,\beta}$ admits an intuitionistic fuzzy neighborhood A such that $(A \cap U_i)$ is empty for all finitely many i.

Definition 4.3.2. An IFTS (X,τ) is IF-paracompact if every open covering $\{U_i\}$ of X admits a locally finite refinement.

Definition 4.3.3. An open cover $|A_i|$ of X refines an open cover $|B_i|$ of X if there exists at least one $j_0 \in J$ such that $A_i \supseteq B_{j_0}$, i.e. $\mu_{A_i} \supseteq \mu_{B_{j_0}}$, $\nu_{A_i} \subseteq \nu_{B_{j_0}}$.

Theorem 4.3.4. Every closed intuitionistic fuzzy subset of an IF-paracompact space X is IF-paracompact.

Proof: Let F be a closed IFS of an IF-paracompact space X and $|A_i|$ be an open cover of F. Then $A_i = (F \cap W_i)$ for some open IFS W_i of X. So, the collection W_i, which gives a cover of X, has an open locally finite refinement $|B_j|$. Now $(A \cap B_j)$ is an open locally finite refinement of the cover $|A_i|$ of F. So, F is IF-paracompact.

Theorem 4.3.5. Every IF-paracompact Hausdorff space X is IF-normal.

Proof: Let A and B are two disjoint intuitionistic fuzzy closed subsets of an IF-paracompact space X. As X is IF-paracompact and from the previous theorem we get every closed intuitionistic fuzzy subset of an IF-paracompact space X is IF-paracompact, so we can say that A and B are also IF-paracompact. Let us consider two IF-singletons $x_{m,n}$ and $y_{r,s}$ such that $x_{m,n} \in A$ and $y_{r,s} \in B$ and $x_{m,n} \neq y_{r,s}$. Now for each $x_{m,n} \in A$, $y_{r,s} \in B$, we choose two disjoint neighborhoods $U_{x_{m,n}}$ and $W_{y_{r,s}}$ of $x_{m,n}$ and $y_{r,s}$ respectively. By adding A^C to the collection $|U_{x_{m,n}}|$, we get an open cover of X which has an open locally finite refinement V_i. If we put $J = \{i \in i | V_i \cap A \neq (0,1)\}$ and $W = \bigcup_{j \in J} V_j$, then $A \subset W$. Since each V_i is contained in some $U_{x_{m,n}}$, so we have $y_{r,s} \notin \bigcup_{j \in J} \bar{V}_j = \bar{W}$. Similarly, for $y_{r,s} \in B$, we get T such that $B \subset T$ and $x_{m,n} \notin \bar{T}$. Hence $A \subset W$ and $B \subset T$ and $W \cap T = (0,1)$. Hence X is IF-normal.

Lemma 4.3.6. If an IFTS X is locally IF-compact Hausdorff space that is second countable, then it admits a countable base of opens $\{U_n\}$ with IF-compact closure.

Proof: Let $|V_r|$ be a countable base of opens in a second countable locally IF-compact Hausdorff space X. Now, for each IF-singleton $x_{m,n}$ where $x_{m,n} \in X$ there exists an open covering $U_{x_{m,n}}$ around $x_{m,n}$ with IF-compact closure, yet some $V_{r(x_{m,n})}$ contains $x_{m,n}$ as well as contained in $U_{x_{m,n}}$. Hence the closure of $V_{r(x_{m,n})}$ is a closed intuitionistic fuzzy subset of the IF-compact $\overline{U}_{x_{m,n}}$ and so $\overline{V}_{r(x_{m,n})}$ is also IF-compact. Thus, the V_r's with IF-compact closure are a countable base of opens with IF-compact closure.

Theorem 4.3.7. Any second countable Hausdorff space X that is locally IF-compact is IF-paracompact.

Proof: Let $|V_r|$ be a countable base of opens in a second countable locally IF-compact Hausdorff space X. Assume $|U_i|$ be an open cover of X for which we seek a locally finite refinement. Since each $x_{m,n} \in X$ lies in some U_i and so there exists a open covering $V_{r(x_{m,n})}$ containing intuitionistic fuzzy singleton $x_{m,n}$ with $V_{r(x_{m,n})} \subseteq U_i$. The $V_{r(x_{m,n})}$'s therefore consistute a refinement of U_i that is countable. Since the property of one open covering refines another, is transitive, we therefore lose no generality by seeking locally refinements of countable covers. Hence the locally IF-compact Hausdorff space X is IF-paracompact.

Definition 4.3.8. An IFTS is called IF-σ-compact if it is a countable union of IF-compact IFSs.

Lemma 4.3.9. If $\{A_i\}_{i\in I}$ is a locally finite collection of intuitionistic fuzzy subsets of an IFTS X then $\overline{\bigcup_\iota A_\iota} = \bigcup_i \bar{A}_i$.

Proof: We have to prove that the left hand side is contained in the right hand side, the reverse inclusion being obvious. Suppose $p_{\alpha,\beta} \in X - \bigcup_i \bar{A}_i = \bigcap (X - \bar{A}_i)$. Choose an open neighborhood U of $p_{\alpha,\beta}$ and a finite intuitionistic fuzzy subset $J \subset I$ such that $V \coloneqq \bigcap_{j\in J}(X - \bar{A}_j)$ is a neighborhood of $p_{\alpha,\beta}$ which is disjoint from A_j for all $j \in J$. Hence $U \cap V$ is a neighborhood of $p_{\alpha,\beta}$, which is disjoint from A_i for all $i \in I$, so $p_{\alpha,\beta} \notin \overline{\bigcup_\iota A_\iota}$.

Theorem 4.3.10. Let X be an IFTS in which every singleton $p_{\alpha,\beta}$ has a neighborhood $U_{p_{\alpha,\beta}}$ which is second countable and IF-precompact (i.e. its closure is IF-compact). Then among the following statements the implications $i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv)$ hold:

i) X is second countable.

ii) X is IF- σ -compact.

iii) X is IF-paracompact.

iv) Every components of X is second countable.

Proof: $i) \Rightarrow ii)$: Let X is second countable. Then every open cover of X has a countable subcover. Applying this to the open cover $\left\{U_{p_{\alpha,\beta}}\right\}_{p_{\alpha,\beta}\in X}$, we see that there is a sequence $p_{1_{\alpha,\beta}}, p_{2_{\alpha,\beta}}, p_{3_{\alpha,\beta}}, \dots$ in X such that $X = \bigcup_{k=1}^{\infty} p_{k_{\alpha,\beta}}$. Since $\overline{U}_{p_{\alpha,\beta}}$ is IF-compact, then for each $p_{\alpha,\beta}$, it follows that X is IF- σ -compact.

$ii) \Rightarrow iii)$: Let X be IF- σ-compact, say $X = \bigcup_{n=1}^{\infty} K_n$ where each K_n is IF-compact. We first find a sequence $V_0 \subset V_1 \subset V_2 \subset \cdots$ of IF-paracompact open IFSs of whose union is all of X, such that $\bar{V}_j \subset V_{j+1}$ for all j. Let $V_0 \coloneqq (0,1)$. After, $V_0, V_1, V_2, \dots, V_j$ have been chosen, note that $K_j \cup \bar{V}_j$ is IF-compact, so there are finitely many IFP $p_{\alpha_1,\beta_1}, p_{\alpha_2,\beta_2}, \dots, p_{\alpha_m,\beta_m} \in X$ such that $K_j \cup \bar{V}_j \subset U_{p_{\alpha_1,\beta_1}} \cup U_{p_{\alpha_2,\beta_2}} \cup \dots \cup U_{p_{\alpha_m,\beta_m}}$. Set, $V_{j+1}: U_{p_{\alpha_1,\beta_1}} \cup U_{p_{\alpha_2,\beta_2}} \cup \dots \cup U_{p_{\alpha_m,\beta_m}}$. Now let $\{W_i\}_{i\in I}$ be any open cover of X. Each set $\bar{V}_K - V_{K-1}$ is IF-compact and is therefore contained in $\bigcup_{i\in I_K} W_i$ for some finite IFS $I_K \subset I$. Then $\{W_i - \bar{V}_{K-2}\}_{K\in\mathbb{N}, i\in I_K}$ is a locally finite open refinement of $\{W_i\}_{i\in I}$. So, X is IF-paracompact.

$iii) \Rightarrow iv)$: Suppose X is IF-paracompact and nonempty and let Y be a component of X. We have to show that Y is second countable. Since Y is closed in X then by using lemma 4.3.9 we can say that Y is IF-paracompact.

Chapter Five

Connectedness in IFTS

Intuitionistic fuzzy connectedness first introduced by Ozcag and Coker (S. Ozcag, 1998) in intuitionistic fuzzy topological spaces and mentioned some properties which are global property. In this chapter we give some new notions of separated, connectedness and totally connectedness and one notions of T_1-space in intuitionistic fuzzy topological space and investigate some relationship among them. Also we find a relation about classical topology and intuitionistic fuzzy topology. Further, we show that connectedness in intuitionistic fuzzy topological spaces are productive.

5.1 Definition and Relationship

In this section, we have given five possible new notions of connectedness in intuitionistic fuzzy topological spaces.

Definition 5.1.1. Two disjoint non-empty intuitionistic fuzzy subsets $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ of an IFTS X are said to be separated if A and B neither contains a limit point of the other. i.e. A and B are separated iff $A \cap \bar{B} = (0,1)$ and $\bar{A} \cap B = (0.1)$.

Definition 5.1.2. Two IFS's $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ in X are called Q-separated for an IFTS (X, τ) if and only if there exist closed (open) IFS's $G = (\mu_G, \nu_G)$ and $H = (\mu_H, \nu_H)$ in X such that $A \subseteq G, B \subseteq H$ and $A \cap B = (0,1) = G \cap H$.

Definition 5.1.3. An intuitionistic fuzzy subsets $A = (\mu_A, \nu_A)$ of an IFTS X is disconnected if there exists an open intuitionistic fuzzy subsets $G = (\mu_G, \nu_G)$ and $H = (\mu_H, \nu_H)$ of X such that $(A \cap G) \cup (A \cap H) = (1,0)$ and $(A \cap G) \cap (A \cap H) = (0,1)$. In this case $G \cup H$ is called a disconnection.

Definition 5.1.4. An IFTS X is said to be disconnected if $A \cup B = (1,0)$ and $A \cap B = (0,1)$ where $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are non-empty open intuitionistic fuzzy subsets of X.

Theorem 5.1.5. Union of two non-empty separated intuitionistic fuzzy subsets of an IFTS X is disconnected.

Proof: Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are two non-empty separated intuitionistic fuzzy subsets of an IFTS X, so $A \cap \bar{B} = (0,1)$ and $\bar{A} \cap B = (0.1)$. Let $G = \bar{B}^C$ and $H = \bar{A}^C$. Then G and H are open and $(A \cup B) \cap G = (1_A, 0)$ and $(A \cup B) \cap H = (1_B, 0)$ are non-empty disjoint IFSs whose union is $A \cup B$. Thus G and H form a disconnection of $A \cup B$. Hence $A \cup B$ is disconnected.

Theorem 5.1.6. Consider $\mathcal{M} = \{A_i\}$, where $A_i = (\mu_{A_i}, \nu_{A_i})$ be a class of IF-connected subsets of an IFTS X such that no two members of $\mathcal{M}$ are separated. Then $B = \bigcup_i A_i$ is IF-connected.

Proof: Assume that B is not IF-connected. Let $G = (\mu_G, \nu_G)$ and $H = (\mu_H, \nu_H)$ are two open IFS of X such that $G \cup H$ is a IF-disconnection of B. Now each $A_i \in \mathcal{M}$ is IF-connected and so is contained in either G or H and disjoint from the other. Since any two members of $A_{i_1}, A_{i_2} \in \mathcal{M}$ are not separated and so $A_{i_1} \cup A_{i_2}$ is IF-connected, hence $A_{i_1} \cup A_{i_2}$ is contained in either G or H and disjoint from the other. Accordingly all the members of $\mathcal{M}$ and hence $B = \bigcup_i A_i$ must be contained in either G or H and disjoint from the other. But this contradicts the fact that $G \cup H$ is an IF-disconnection of B, hence B is IF-connected.

Theorem 5.1.7. Let $G \cup H$ be a disconnection of an IFS $A = (\mu_A, \nu_A)$. Show that $A \cap G$ and $A \cap H$ are separated IFSs.

Proof: Here $A \cap G$ and $A \cap H$ are disjoint, hence we need only to show that each IFS contains no limit point of the other. Let $p_{(m,n)}, m, n \in I$ be a limit point of $A \cap G$ and suppose $p_{(m,n)} \in A \cap H$. Then H is an open IFS containing $p_{(m,n)}$ and so H contains a point of $A \cap G$ distinct from $p_{(m,n)}$, i.e. $(A \cap G) \cap H \neq (0,1)$. But $(A \cap G) \cap (A \cap H) = (0,1) = (A \cap G) \cap H$. Accordingly $p_{(m,n)} \notin A \cap H$. Similarly if $p_{(m,n)}$ be a limit point of $A \cap H$ then $p_{(m,n)} \notin A \cap G$. Thus $A \cap G$ and $A \cap H$ are separated IFSs.

Theorem 5.1.8. For an IFTS the following statements are mutually equivalent:

i) X is connected.

ii) The only IFSs of X which are simultaneously open and closed are.

iii) X cannot be expressed as the union of two disjoint nonempty open IFSs.

Proof: $i) \Longrightarrow ii)$: Suppose A is an IFS in X, which is both open and closed. Then $B = A^C$ is both closed and open; further $A \cup B = (1,0)$ and $\bar{A} \cap B = A \cap B = (0,1)$ and $\cap \bar{B} = A \cap B = (0,1)$. Since X is connected, A or B must be $(0,1)$. That is $A = (0,1)$ or $A = (1,0)$.

$ii) \Longrightarrow iii)$: Since $A \cup B = (1,0)$ and $A \cap B = (0,1)$ which implies $A^C = B$ and $B^C = A$. So A and B are simultaneously open and closed. So, between of these IFSs A and B one IFS is $(0,1)$ and the other IFS is $(1,0)$.

$iii) \Longrightarrow i)$: Suppose $iii)$ holds but X is not connected. Then there would exist nonempty open sets A and B such that $A \cup B = (1,0)$ and $\bar{A} \cap B = A \cap \bar{B} = (0,1)$. Then B, being the complement of $\bar{A}$ is open, similarly A is open. Thus X is the union of two disjoint non-empty open IFSs, which contradict the hypothesis, hence X is connected.

Theorem 5.1.9. If an IFTS (X, τ) is IF-disconnected and $\tau^* \supseteq \tau$ then (X, τ^*) is also IF-disconnected.

Proof: Given IFTS (X, τ) is IF-disconnected. Let $A, B \in \tau$ where $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are non-empty open intuitionistic fuzzy subsets of Xthen $A \cup B = (1,0)$ and $A \cap B = (0,1)$. Since $\tau^* \supseteq \tau$ and $A, B \in \tau$ then obviously $A, B \in \tau^*$ hence $A \cup B = (1,0)$ and $A \cap B = (0,1)$, which implies that (X, τ^*) is also IF-disconnected.

Theorem 5.1.10. Let A be an IFS of an IFTS (X, τ) and let τ_A be the relative IFT on A. Then A is τ-IF-connected if and only if A is τ_A-IF-connected.

Proof: Let $A = (\mu_A, \nu_A)$ be IF-disconnected space on an IFTS (X, τ) and $G \cup H$ be a disconnection of (X, τ) where $G = (\mu_G, \nu_G)$ and $H = (\mu_H, \nu_H)$. Then $G \cup H = (1,0), G \cap H = (0,1) \Longrightarrow \mu_G \cup \mu_H = 1, \nu_G \cap \nu_H = 0$ and $\mu_G \cap \mu_H = 1, \nu_G \cup \nu_H = 0$. Now $G, H \in \tau \Longrightarrow A \cap G, A \cap H \in \tau_A$, where $A \cap G = (\mu_A, \nu_A) \cap (\mu_G, \nu_G) = (\mu_A \cap \mu_G, \nu_A \cup \nu_G)$ and $A \cap H = (\mu_A, \nu_A) \cap (\mu_H, \nu_H) = (\mu_A \cap \mu_H, \nu_A \cup \nu_H)$.

We have, $(A \cap G) \cup (A \cap H) = (\mu_A \cap \mu_G, \nu_A \cup \nu_G) \cup (\mu_A \cap \mu_H, \nu_A \cup \nu_H)$

$$= ((\mu_A \cap \mu_G) \cup (\mu_A \cap \mu_H), (\nu_A \cup \nu_G) \cap (\nu_A \cup \nu_H))$$

$= (\mu_A \cap (\mu_G \cup \mu_H), \nu_A \cup (\nu_G \cap \nu_H))$

$= (\mu_A, \nu_A)$

Again, $(A \cap G) \cap (A \cap H) = (\mu_A \cap \mu_G, \nu_A \cup \nu_G) \cap (\mu_A \cap \mu_H, \nu_A \cup \nu_H)$

$= ((\mu_A \cap \mu_G) \cap (\mu_A \cap \mu_H), (\nu_A \cup \nu_G) \cup (\nu_A \cup \nu_H))$

$= (\mu_A \cap (\mu_G \cap \mu_H), \nu_A \cup (\nu_G \cup \nu_H))$

$= (0,1)$

Conversely, let $E = (\mu_E, \nu_E), F = (\mu_F, \nu_F) \in \tau_A$ then $\exists C = (\mu_C, \nu_C), D = (\mu_D, \nu_D) \in \tau \ni C \cap A = E$ and $D \cap A = F$. We have $E \cup F = (1_A, 0_A), E \cap F = (0_A, 1_A)$. We have to show that $C \cup D = (1_X, 0), C \cap D = (0, 1_X)$.

Now $E \cup F = (1_A, 0_A) \Longrightarrow (C \cap A) \cup (D \cap A) = (1_A, 0_A)$

$\Longrightarrow (C \cup D) \cap A = (1_A, 0_A)$

$\Longrightarrow (C \cup D) = (1_X, 0)$

Again, $E \cap F = (0_A, 1_A) \Longrightarrow (C \cap A) \cap (D \cap A) = (0_A, 1_A)$

$\Longrightarrow (C \cap D) \cap A = (0_A, 1_A)$

$\Longrightarrow (C \cap D) = (0, 1_X)$

So, $C \cup D$ form a τ-IF-disconnection of A. Hence A is τ-IF-connected if and only if A is τ_A-IF-connected.

Theorem 5.1.11. Let $\{(X_i, \tau_{X_i}), i \in J\}$ be a family of subspaces of an IFTS (X, τ) such that $\cap X_i \neq \phi$, if (X_i, τ_{X_i}) is IF-connected then $(\cup X_i, \tau_{\cup X_i})$ is IF-connected.

Proof: Suppose $(\cup X_i, \tau_{\cup X_i})$ is not IF-connected, then there exist $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau_{\cup X_i}$ such that $A \cup B = (1,0)$ and $A \cap B = (0,1)$.

Now, $A \cup B = (1,0) \Longrightarrow (A \cup B)|X_i = (1,0),\ \forall X_i \subseteq \cup X_i$

$$\Longrightarrow ((\mu_A, \nu_A) \cup (\mu_B, \nu_B))|X_i = (1,0)$$

$$\Longrightarrow (\mu_A \cup \mu_B, \nu_A \cap \nu_B)|X_i = (1,0)$$

Which gives, $(\mu_A \cup \mu_B)|X_i = 1$ and $(\nu_A \cap \nu_B)|X_i = 0$

From $(\mu_A \cup \mu_B)|X_i = 1$, we get $(\mu_{A_i}|X_i) \cup (\mu_{B_i}|X_i) = 1$ and from $(\nu_A \cap \nu_B)|X_i = 0$ we get $(\nu_{A_i}|X_i) \cap (\nu_{B_i}|X_i) = 0$, where $(\mu_{A_i}|X_i, \nu_{A_i}|X_i), (\mu_{B_i}|X_i, \nu_{B_i}|X_i) \in \tau_{X_i}$.

Again from $A \cap B = (0,1) \Longrightarrow (A \cap B)|X_i = (0,1),\ \forall X_i \subseteq \cup X_i$

$$\Longrightarrow ((\mu_A, \nu_A) \cap (\mu_B, \nu_B))|X_i = (0,1)$$

$\Longrightarrow (\mu_A \cap \mu_B, \nu_A \cup \nu_B)|X_i = (0,1)$, which gives, $(\mu_A \cap \mu_B)|X_i = 0$ and $(\nu_A \cup \nu_B)|X_i = 1$. Therefore, $(\mu_{A_i}|X_i) \cap (\mu_{B_i}|X_i) = 0$ and $(\nu_{A_i}|X_i) \cup (\nu_{B_i}|X_i) = 1$. So, (X_i, τ_{X_i}) is not IF-connected.

Theorem 5.1.12. The continuous image of an IF-connected space X is IF-connected.

Proof: Let$f: (X, \tau) \to (Y, \delta)$ be a continuous function from an IFTS (X, τ) to (Y, δ). Consider (X, τ) is IF-connected, we shall prove that (Y, δ) is also IF-connected. Suppose (Y, δ) is not IF-connected, i.e. (Y, δ) has a disconnection. Let this be $G = (\mu_G, \nu_G)$ and $H = (\mu_H, \nu_H)$ be two IFS on X then $G \cup H = (1,0)$ i.e. $\mu_G \cup \mu_H = 1$ and $\nu_G \cap \nu_H = 0$.

Now $f^{-1}(G) = (f^{-1}(\mu_G), f^{-1}(\nu_G))$ and $f^{-1}(H) = (f^{-1}(\mu_H), f^{-1}(\nu_H))$.

So, $f^{-1}(G) \cup f^{-1}(H) = (\max(f^{-1}(\mu_G), f^{-1}(\mu_H))(x), \min(f^{-1}(\nu_G), f^{-1}(\nu_H))(x))$

$$= (max(f^{-1}(\mu_G)(x), f^{-1}(\mu_H)(x)), min(f^{-1}(\nu_G)(x), f^{-1}(\nu_H)(x)))$$

$$= (\max(\mu_G(f(x), \mu_H(f(x))), min(\nu_G(f(x), \nu_H(f(x)))$$

$$= ((\mu_G \cup \mu_H)f(x), (\nu_G \cap \nu_H)f(x))$$

$= (f^{-1}(\mu_G \cup \mu_H)(x), f^{-1}(\nu_G \cap \nu_H)(x))$

$= (1_X, 0)$

Again, $f^{-1}(G) \cap f^{-1}(H) = (\min\big(f^{-1}(\mu_G), f^{-1}(\mu_H)\big)(x),$

$\max(f^{-1}(\nu_G), f^{-1}(\nu_H))(x))$

$= \big(min(f^{-1}(\mu_G)(x), f^{-1}(\mu_H)(x)\big),$

$max(f^{-1}(\nu_G)(x), f^{-1}(\nu_H)(x)))$

$= (\min(\mu_G(f(x), \mu_H\big(f(x)\big)), max(\nu_G(f(x), \nu_H(f(x)))$

$= ((\mu_G \cap \mu_H)f(x), (\nu_G \cup \nu_H)f(x))$

$= (f^{-1}(\mu_G \cap \mu_H)(x), f^{-1}(\nu_G \cup \nu_H)(x))$

$= (0,1_X)$

Hence , $f^{-1}(G)$ and $f^{-1}(H)$ give a disconnection for X, which gives the prove.

Definition 5.1.13. An IFTS (X, τ) is called

a) Intuitionistic fuzzy connected (IFC) (i) if (X, τ) has no proper clopen (clopen means closed-open) IFS.

b) IFC (ii) if there do not exist non-empty IFSs A, B in X which are separated and $A \cup B = (1, 0)$.

c) IFC (iii) if there is no clopen IFS $A >> (0, 1)$ which is C1 separated.

d) IFC (iv) if there do not exist $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau \backslash \{(0, 1), (1.0)\}$ such that $A \cup B = (r, 0)$ with $0 < r \leq 1$ and $A \cap B = (0, 1)$.

e) IFC (v) iff for any $\alpha \in I_0$, there exist no non-empty proper subset $H \subseteq X$ such that $\alpha 1_H = \alpha(1_H, 1_{X-H}), \alpha 1_{X-H} = \alpha(1_{X-H}, 1_H) \in \tau$.

f) IFC (vi) iff there exist no non-zero Q-separated IFSs $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ in X with $A \cup B = (1, 0)$.

Theorem 5.1.14. The following statements are equivalent:

a) IFTS (X, τ) is IFC (vi)

b) There do not exist two non-zero disjoint closed IFSs $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ where $\max(\mu_A, \mu_B) = (1, 0)$.

c) There do not exist two non-zero disjoint open IFSs $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ where $\max(\mu_A, \mu_B) = (1, 0)$.

d) IFTS (X, τ) is IFC (ii)

Proof: a) $\Rightarrow b)$: Let there exist IFSs $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau^C$ such that $A \neq B, A \cup B = (1, 0)$ and $A \cap B = (0, 1)$ then clearly A and B are Q-separated. So that, (X, τ) is not IFC (vi), which contradicts $a)$.

b) $\Rightarrow c)$: If $A, B \in \tau$ where $A = (\mu_A, \nu_A), B = (\mu_B, \nu_B), A \cup B = (1, 0)$ and $A \cap B = (0, 1)$ then A and B closed which contradicts $b)$.

c) $\Rightarrow d)$: If (X, τ) is not IFC (ii) then there exist $A, B \in I^X - \{(1, 0)\}$ such that A, B are separated and $A \cup B = (1, 0)$. Now $\exists G, H \in \tau$ such that $A \subseteq G, B \subseteq H$ and $G \cap B = (1, 0) = H \cap A$. But then G and H satisfying $G \cap H = (0,1)$ and $G \cup H = (1,0)$ which contradicting $c)$.

d) $\Rightarrow a)$: If there exist some IFS $A = \left(1_A, 1_{A^c}\right) \in \tau \cap \tau^C - \{(0, 1), (1, 0)\}$ then $A = \left(1_A, 1_{A^c}\right), A^C = (1_{A^c}, 1_A)$ are two non-zero separated sets with $\max\left(1_A, 1_{A^c}\right) = (1, 0)$. This contradicts d).

Theorem 5.1.15. An IFTS (X,τ) is IF-connected if and only if there exists no non-empty IFOS A and B in X such that $A = B^C$.

Proof: Let A and B are two IFOSs in X such that $A \neq (0,1) \neq B$ and $A = B^C$, since B is an IFOS which implies that $B^C = A$ is an IFCS and $B \neq (0,1)$ implies that $B^C \neq (1,0)$ i.e. $A \neq (1,0)$. Hence there exists a proper IFS A as $A \neq (0,1)$ and $A \neq (1,0)$, such that A is both IFOS and IFCS. But this is a contradiction that (X,τ) is IF-connected.

Conversely, suppose (X,τ) is an IFTS and A is both IFOS and IFCS in X such that $(0,1) \neq A \neq (1,0)$. Here $A = B^C$. In this case B is an IFOS and $A \neq (1,0)$. This implies that, $B = A^C \neq (0,1)$, which is a contradiction. Hence there exist no proper IFS in X which is both IFO and IFC. So, (X,τ) is IF-connected.

5.2 Good Extension of Connectedness

Theorem 5.2.1. Let (X,T) be a topological space and (X,τ) be its corresponding IFTS, where $\tau = \{(1_A, 1_{A^c}) : A \in T\}$. Then (X,T) is connected if and only if (X,τ) is IF-connected.

Proof: Suppose (X,T) is disconnected, so there exist two nonempty subsets A, B of X such that $A \cup B = X, A \cap B = \emptyset$. Since $A, B \in T$ then $1_A = (1_A, 1_{A^c}) \in \tau$ and $1_B = (1_B, 1_{B^c}) \in \tau$. Now, $1_A \cup 1_B = (1_A, 1_{A^c}) \cup (1_B, 1_{B^c})$

$$= (1_A \cup 1_B, 1_{A^c} \cap 1_{B^c})$$

$$= (1_{A\cup B}, 1_{A^c \cap B^c})$$

$$= (1_{A\cup B}, 1_{(A\cup B)^c})$$

$= (1_X, 1_\emptyset)$

$= (1,0)$

Again, $1_A \cap 1_B = (1_A, 1_{A^c}) \cap (1_B, 1_{B^c})$

$= (1_A \cap 1_B, 1_{A^c} \cup 1_{B^c})$

$= (1_{A\cap B}, 1_{A^c \cup B^c})$

$= (1_{A\cap B}, 1_{(A\cap B)^c})$

$= (1_\emptyset, 1_X)$

$= (0, 1)$

So, (X,τ) is IF-disconnected. Hence (X,T) is connected if (X,τ) is IF-connected.

Conversely, suppose (X,τ) is IF-disconnected. Since $1_A, 1_B \in \tau$ so $1_A \cup 1_B = (1, 0)$ and $1_A \cap 1_B = (0,1)$, then we can write

$$1_A \cup 1_B = (1,0)$$

$$\Rightarrow \left(1_A, 1_{A^c}\right) \cup \left(1_B, 1_{B^c}\right) = (1, 0)$$

$$\Rightarrow \left(1_A \cup 1_B, 1_{A^c} \cap 1_{B^c}\right) = (1, 0)$$

$$\Rightarrow \left(1_{A\cup B}, 1_{A^c \cap B^c}\right) = (1, 0)$$

So, $1_{A\cup B} = 1 = 1_X \Rightarrow A \cup B = X$

Again, $1_A \cap 1_B = (0, 1)$

$$\Rightarrow \left(1_A, 1_{A^c}\right) \cap \left(1_B, 1_{B^c}\right) = (0, 1)$$

$$\Rightarrow \left(1_A \cap 1_B, 1_{A^c} \cup 1_{B^c}\right) = (0, 1)$$

$$\Rightarrow \left(1_{A\cap B}, 1_{A^c \cup B^c}\right) = (0, 1)$$

So, $1_{A\cap B} = 0 = 1_\emptyset \Rightarrow A \cap B = \emptyset$. Hence (X,T) is disconnected.

So, (X,T) is connected if and only if (X,τ) is IF-connected.

Theorem 5.2.2. Let $(X,\mathcal{T})$ be an intuitionistic topological space and (X,τ) be its corresponding IFTS, where $\tau = \{(1_{A_1}, 1_{A_2}): A = (A_1, A_2) \in \mathcal{T}\}$. Then $(X,\mathcal{T})$ is connected if and only if (X,τ) is IF-connected.

Proof: Suppose $(X,\mathcal{T})$ is disconnected, so there exist two nonempty subsets A, B of X such that $A \cup B = (X, \emptyset), A \cap B = (\emptyset, X)$. Since $A, B \in \mathcal{T}$ then $1_A = (1_{A_1}, 1_{A_2}) \in \tau$ and $1_B = (1_{B_1}, 1_{B_2}) \in \tau$.

Here, $A \cup B = (X,\emptyset) \Rightarrow (A_1, A_2) \cup (B_1, B_2) = (X,\emptyset)$

$$\Rightarrow (A_1 \cup B_1, A_2 \cap B_2) = (X,\emptyset)$$

$$\Rightarrow A_1 \cup B_1 = X, A_2 \cap B_2 = \emptyset$$

Again, $A \cap B = (\emptyset, X) \Rightarrow (A_1, A_2) \cap (B_1, B_2) = (\emptyset, X)$

$$\Rightarrow (A_1 \cap B_1, A_2 \cup B_2) = (\emptyset, X)$$

$$\Rightarrow A_1 \cap B_1 = \emptyset, A_2 \cup B_2 = X$$

Now, $1_A \cup 1_B = (1_{A_1}, 1_{A_2}) \cup (1_{B_1}, 1_{B_2})$

$$= (1_{A_1} \cup 1_{B_1}, 1_{A_2} \cap 1_{B_2})$$

$$= (1_{A_1 \cup B_1}, 1_{A_2 \cap B_2})$$

$$= (1_X, 1_\emptyset)$$

$$= (1, 0)$$

Again, $1_A \cap 1_B = (1_{A_1}, 1_{A_2}) \cap (1_{B_1}, 1_{B_2})$

$$= (1_{A_1} \cap 1_{B_1}, 1_{A_2} \cup 1_{B_2})$$

$$= (1_{A_1 \cap B_1}, 1_{A_2 \cup B_2})$$

$$= (1_\emptyset, 1_X)$$

$$= (0, 1)$$

So, (X,τ) is IF-disconnected. Hence $(X,\mathcal{T})$ is connected if (X,τ) is IF-connected.

Conversely, suppose (X,τ) is IF-disconnected. Since $1_A, 1_B \in \tau$ so $1_A \cup 1_B = (1,0)$ and $1_A \cap 1_B = (0,1)$, then we can write

$$1_A \cup 1_B = (1,0)$$

$$\Rightarrow (1_{A_1}, 1_{A_2}) \cup (1_{B_1}, 1_{B_2}) = (1,0)$$

$$\Rightarrow (1_{A_1} \cup 1_{B_1}, 1_{A_2} \cap 1_{B_2}) = (1,0)$$

$$\Rightarrow \left(1_{A_1 \cup B_1}, 1_{A_2 \cap B_2}\right) = (1,0)$$

So, $1_{A_1 \cup B_1} = 1 = 1_X \Rightarrow A_1 \cup B_1 = X$

and $1_{A_2 \cap B_2} = 0 = 1_\emptyset \Rightarrow A_2 \cap B_2 = \emptyset$

Again, $1_A \cap 1_B = (0,1)$

$$\Rightarrow (1_{A_1}, 1_{A_2}) \cap (1_{B_1}, 1_{B_2}) = (0,1)$$

$$\Rightarrow (1_{A_1} \cap 1_{B_1}, 1_{A_2} \cup 1_{B_2}) = (0,1)$$

$$\Rightarrow \left(1_{A_1 \cap B_1}, 1_{A_2 \cup B_2}\right) = (0,1)$$

So, $1_{A_1 \cap B_1} = 0 = 1_\emptyset \Rightarrow A_1 \cap B_1 = \emptyset$

and $1_{A_2 \cup B_2} = 1 = 1_X \Rightarrow A_2 \cup B_2 = X$.

Here, $A_1 \cup B_1 = X, A_2 \cap B_2 = \emptyset \Rightarrow A \cup B = (X, \emptyset)$ and $A_1 \cap B_1 = \emptyset, A_2 \cup B_2 = X \Rightarrow A \cap B = (\emptyset, X)$. Hence $(X, \mathcal{T})$ is disconnected.

So, $(X, \mathcal{T})$ is connected if and only if (X, τ) is IF-connected.

Theorem 5.2.3. Let (X, t) be a intuitionistic topological space and (X, τ) be its corresponding IFTS, where $\tau = \{(\lambda, \lambda^C): \lambda \in t\}$. Then (X, t) is connected if and only if (X, τ) is IF-connected.

Proof: Suppose (X, t) is disconnected, so there exist two nonempty subsets α, β of X such that $\alpha \cup \beta = 1, \alpha \cap \beta = 0$ where $\alpha \neq 0 \neq \beta, \alpha \neq 1 \neq \beta$. Since $\alpha, \beta \in t$ then $(\alpha, \alpha^C), (\beta, \beta^C) \in \tau$.

Now, $(\alpha, \alpha^C) \cup (\beta, \beta^C) = (\alpha \cup \beta, \alpha^C \cap \beta^C)$

$= (\alpha \cup \beta, (\alpha \cup \beta)^C)$

$= (1, 0)$

Again, $(\alpha, \alpha^C) \cap (\beta, \beta^C) = (\alpha \cap \beta, \alpha^C \cup \beta^C)$

$= (\alpha \cap \beta, (\alpha \cap \beta)^C)$

$= (0, 1)$

So, (X, τ) is IF-disconnected.

Conversely, suppose (X, τ) is IF-disconnected. Since $(\alpha, \alpha^C), (\beta, \beta^C) \in \tau$, then we can write $(\alpha, \alpha^C) \cup (\beta, \beta^C) = (1, 0)$ and $(\alpha, \alpha^C) \cap (\beta, \beta^C) = (0, 1)$.

Now, $(\alpha, \alpha^C) \cup (\beta, \beta^C) = (1, 0)$

$\Rightarrow (\alpha \cup \beta, \alpha^C \cap \beta^C) = (1, 0)$

$\Rightarrow (\alpha \cup \beta, (\alpha \cup \beta)^C) = (1, 0)$

$\Rightarrow \alpha \cup \beta = 1$

Again, $(\alpha, \alpha^C) \cap (\beta, \beta^C) = (0, 1)$

$\Rightarrow (\alpha \cap \beta, \alpha^C \cup \beta^C) = (0, 1)$

$\Rightarrow (\alpha \cap \beta, (\alpha \cap \beta)^C) = (0, 1)$

$\Rightarrow \alpha \cap \beta = 0$

So, (X, t) is disconnected. Hence (X, t) is connected if and only if (X, τ) is IF-connected.

5.3 Product space of Connectedness

In this section, we have discuss about productivity of connectedness in intuitionistic fuzzy topological space.

Theorem 5.3.1. If (X, τ) and (Y, δ) are IF-connected space then $(X \times Y, \tau \times \delta)$ is also IF-connected.

Proof: Consider $(X \times Y, \tau \times \delta)$ is not IF-connected then $\exists A, B \in \tau \times \delta$ such that $A \cup B = (1, 0)$ and $A \cap B = (0, 1)$. Since $A, B \in \tau \times \delta$ then $A = C \times D$ and $B = E \times F$ where $C = (\mu_C, \nu_C), E = (\mu_E, \nu_E) \in \tau$, and $D = (\mu_D, \nu_D), F = (\mu_F, \nu_F) \in \delta$. Now $C \times D = (\mu_C \dot{\times} \mu_D, \nu_C \dot{\times} \nu_D)$, where $\left(\mu_C \dot{\times} \mu_D\right)(x, y) = \min(\mu_C(x), \mu_D(y))$ and $(\nu_C \dot{\times} \nu_D)(x, y) = \max\big(\nu_C(x), \nu_D(y)\big), \forall (x, y) \in \tau \times \delta.$

Similarly, $E \times F = (\mu_E \dot{\times} \mu_F, \nu_E \dot{\times} \nu_F)$.

Now $A \cup B = (1,0) \Rightarrow (C \times D) \cup (E \times F) = (1, 0)$

$\Rightarrow \left(\mu_C \dot{\times} \mu_D, \nu_C \dot{\times} \nu_D\right) \cup \left(\mu_E \dot{\times} \mu_F, \nu_E \dot{\times} \nu_F\right) = (1, 0)$

$\Rightarrow (\min\big(\mu_C(x), \mu_D(y)\big) \cup \min\big(\mu_E(x), \mu_F(y)\big), \max\big(\nu_C(x), \nu_D(y)\big)$

$\cap \max\big(\nu_E(x), \nu_F(y)\big)) = (1, 0)$

i.e., $\min\big(\mu_C(x), \mu_D(y)\big) \cup \min\big(\mu_E(x), \mu_F(y)\big) = 1$

$\Rightarrow$ Either, $\min\big(\mu_C(x), \mu_D(y)\big) = 1$ or, $\min\big(\mu_E(x), \mu_F(y)\big) = 1$

$\Rightarrow$ Either $\mu_C(x) = 1, \mu_D(y) = 1$ or, $\mu_E(x) = 1, \mu_F(y) = 1$

For, $\max\big(\nu_C(x), \nu_D(y)\big) \cap \max\big(\nu_E(x), \nu_F(y)\big) = 0$

$\Rightarrow \max\big(\nu_C(x), \nu_D(y)\big) = 0$ and $\max\big(\nu_E(x), \nu_F(y)\big) = 0$

$\Rightarrow \nu_C(x) = 0, \nu_D(y) = 0, \nu_E(x) = 0, \nu_F(y) = 0$

Case I: Suppose $\mu_C(x) = 1, \mu_D(y) = 1$

Then $C \cup E = (\mu_C, \nu_C) \cup (\mu_E, \nu_E) = (\mu_C \cup \mu_E, \nu_C \cap \nu_E) = (1,0)$ as $\mu_C(x) = 1$

Case II: Suppose $\mu_E(x) = 1, \mu_F(y) = 1$

Then $D \cup F = (\mu_D, \nu_D) \cup (\mu_F, \nu_F) = (\mu_D \cup \mu_F, \nu_D \cap \nu_F) = (1,0)$ as $\mu_F(y) = 1$

Again, $A \cap B = (0,1) \Rightarrow (C \times D) \cap (E \times F) = (0,1)$

$\Rightarrow \left(\mu_C \underset{\cdot}{\overset{\times}{}} \mu_D, \nu_C \underset{\times}{\overset{\cdot}{}} \nu_D\right) \cap \left(\mu_E \underset{\cdot}{\overset{\times}{}} \mu_F, \nu_E \underset{\times}{\overset{\cdot}{}} \nu_F\right) = (1, 0)$

$\Rightarrow (\min\big(\mu_C(x), \mu_D(y)\big) \cap \min\big(\mu_E(x), \mu_F(y)\big), \max\big(\nu_C(x), \nu_D(y)\big)$

$\cup \max\big(\nu_E(x), \nu_F(y)\big)) = (0, 1)$

i.e. $\min\big(\mu_C(x), \mu_D(y)\big) \cap \min\big(\mu_E(x), \mu_F(y)\big) = 0$

$\Rightarrow \min\big(\mu_C(x), \mu_D(y)\big) = 0$ and $\min\big(\mu_E(x), \mu_F(y)\big) = 0$

$\Rightarrow$ Either $\mu_C(x) = 0$, or $\mu_D(y) = 0$ and either $\mu_E(x) = 0$ or $\mu_F(y) = 0$

Again, for, $\max\big(\nu_C(x), \nu_D(y)\big) \cup \max\big(\nu_E(x), \nu_F(y)\big) = 1$

$\Rightarrow$Either $\max\big(\nu_C(x), \nu_D(y)\big) = 1$ or, $\max\big(\nu_E(x), \nu_F(y)\big) = 1$

$\Rightarrow$ Either $\nu_C(x) = 1$ or $\nu_D(y) = 1$, or, either $\nu_E(x) = 1$ or $\nu_F(y) = 1$

Case III: Suppose $\mu_C(x) = 0$, or $\mu_D(y) = 0$ and $\nu_C(x) = 1$

Then $C \cap E = (\mu_C, \nu_C) \cap (\mu_E, \nu_E) = (\mu_C \cap \mu_E, \nu_C \cup \nu_E) = (0, 1)$

Case IV: Suppose $\mu_E(x) = 0$ or $\mu_F(y) = 0$ and $\nu_F(y) = 1$

Then $D \cap F = (\mu_D, \nu_D) \cap (\mu_F, \nu_F) = (\mu_D \cap \mu_F, \nu_D \cup \nu_F) = (0, 1)$

So, (X, τ) and (Y, δ) are not connected, hence if (X, τ) and (Y, δ) are IF-connected then $(X \times Y, \tau \times \delta)$ is IF-connected.

Theorem 5.3.2. The product of IF-connected space is IF-connected.

Proof: Let (X_i, τ_i) be a collection of IF-connected space. Also let $(X, \tau) = (\Pi_i X_i, \Pi_i \tau_i)$ be the product space. Consider $(\Pi_i X_i, \Pi_i, \tau_i)$ are not IF-connected then there exists $A, B \in \tau_1 \times \tau_2 \times \tau_3 \times ....$ such that $A \cup B = (1,0)$ and $A \cap B = (0,1)$. Since $A, B \in \tau_1 \times \tau_2 \times \tau_3 \times ....$ then $A = A_1 \times A_2 \times A_3 \times ...$ and $B = B_1 \times B_2 \times B_3 \times ...$, where $A_i = \left(\mu_{A_i}, \nu_{A_i}\right) \in \tau$ and $B_i = \left(\mu_{B_i}, \nu_{B_i}\right) \in \tau$.

Now, $A \cup B = (1,0) \Rightarrow (A_1 \times A_2 \times A_3 \times ...) \cup (B_1 \times B_2 \times B_3 \times ...) = (1,0)$

$$\Rightarrow \left(\left(\mu_{A_1}, \nu_{A_1}\right) \times \left(\mu_{A_2}, \nu_{A_2}\right) \times \left(\mu_{A_3}, \nu_{A_3}\right) \times ...\right) \cup$$

$$\left(\left(\mu_{B_1}, \nu_{B_1}\right) \times \left(\mu_{B_2}, \nu_{B_2}\right) \times \left(\mu_{B_3}, \nu_{B_3}\right) \times ...\right) = (1,0)$$

$\Rightarrow$ (inf$\left(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \mu_{A_3}(x_3), ...\right) \cup$ inf$\left(\mu_{B_1}(x_1), \mu_{B_2}(x_2), \mu_{B_3}(x_3), ...\right)$,

sup$\left(\nu_{A_1}(x_1), \nu_{A_2}(x_2), \nu_{A_3}(x_3), ...\right) \cap$

sup$\left(\nu_{B_1}(x_1), \nu_{B_2}(x_2), \nu_{B_3}(x_3), ...\right)) = (1,0)$, where $(x_1, x_2, x_3, ...) \in \Pi_i X_i$

i.e. inf$\left(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \mu_{A_3}(x_3), ...\right) \cup$ inf$\left(\mu_{B_1}(x_1), \mu_{B_2}(x_2), \mu_{B_3}(x_3), ...\right) = 1$

$\Rightarrow$Either, inf$\left(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \mu_{A_3}(x_3), ...\right) = 1$

or, inf$\left(\mu_{B_1}(x_1), \mu_{B_2}(x_2), \mu_{B_3}(x_3), ...\right) = 1$

$\Rightarrow$ Either $\mu_{A_1}(x_1) = 1, \mu_{A_2}(x_2) = 1, \mu_{A_3}(x_3) = 1, ...$

or, $\mu_{B_1}(x_1) = 1, \mu_{B_2}(x_2) = 1, \mu_{B_3}(x_3) = 1, ...$

Again, sup$\left(\nu_{A_1}(x_1), \nu_{A_2}(x_2), \nu_{A_3}(x_3), ...\right) \cap$ sup$\left(\nu_{B_1}(x_1), \nu_{B_2}(x_2), \nu_{B_3}(x_3), ...\right) = 0$

$\Rightarrow$ sup$\left(\nu_{A_1}(x_1), \nu_{A_2}(x_2), \nu_{A_3}(x_3), ...\right) = 0$ and sup$\left(\nu_{B_1}(x_1), \nu_{B_2}(x_2), \nu_{B_3}(x_3), ...\right) = 0$

$\Rightarrow \nu_{A_1}(x_1) = 0, \nu_{A_2}(x_2) = 0, \nu_{A_3}(x_3) = 0, ..., \nu_{B_1}(x_1) = 0, \nu_{B_2}(x_2) = 0, \nu_{B_3}(x_3) = 0, ...$

Case I: Suppose $\mu_{A_1}(x_1) = 1, \mu_{B_i}(x_i) = 1, \nu_{A_1}(x_1) = 0, \nu_{B_i}(x_i) = 0$

Then $A_1 \cup B_i = (\mu_{A_1}, \nu_{A_1}) \cup (\mu_{B_i}, \nu_{B_i}) = (\mu_{A_1} \cup \mu_{B_i}, \nu_{A_1} \cap \nu_{B_i}) = (1, 0)$,

for any $(\mu_{B_i}, \nu_{B_i}) \in \tau_i$

Again, $A \cap B = (0, 1) \quad \Rightarrow (A_1 \times A_2 \times A_3 \times \dots) \cap (B_1 \times B_2 \times B_3 \times \dots) = (0, 1)$

$\Rightarrow \left((\mu_{A_1}, \nu_{A_1}) \times (\mu_{A_2}, \nu_{A_2}) \times (\mu_{A_3}, \nu_{A_3}) \times \dots\right) \cap$

$$\left((\mu_{B_1}, \nu_{B_1}) \times (\mu_{B_2}, \nu_{B_2}) \times (\mu_{B_3}, \nu_{B_3}) \times \dots\right) = (0,1)$$

$\Rightarrow (\inf(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \mu_{A_3}(x_3), \dots) \cap \inf(\mu_{B_1}(x_1), \mu_{B_2}(x_2), \mu_{B_3}(x_3), \dots),$

$\sup(\nu_{A_1}(x_1), \nu_{A_2}(x_2), \nu_{A_3}(x_3), \dots) \cup$

$\sup(\nu_{B_1}(x_1), \nu_{B_2}(x_2), \nu_{B_3}(x_3), \dots)) = (0, 1)$

i.e. $\inf(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \mu_{A_3}(x_3), \dots) \cap \inf(\mu_{B_1}(x_1), \mu_{B_2}(x_2), \mu_{B_3}(x_3), \dots) = 0$

$\Rightarrow \inf(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \mu_{A_3}(x_3), \dots) = 0$ and $\inf(\mu_{B_1}(x_1), \mu_{B_2}(x_2), \mu_{B_3}(x_3), \dots) = 0$

For, $\sup(\nu_{A_1}(x_1), \nu_{A_2}(x_2), \nu_{A_3}(x_3), \dots) \cup \sup(\nu_{B_1}(x_1), \nu_{B_2}(x_2), \nu_{B_3}(x_3), \dots) = 1$

$\Rightarrow$Either $\sup(\nu_{A_1}(x_1), \nu_{A_2}(x_2), \nu_{A_3}(x_3), \dots) = 1$

or, $\sup(\nu_{B_1}(x_1), \nu_{B_2}(x_2), \nu_{B_3}(x_3), \dots) = 1$

Case II: Suppose $\inf(\mu_{B_1}(x_1), \mu_{B_2}(x_2), \mu_{B_3}(x_3), \dots) = 0$,

and $\sup(\nu_{B_1}(x_1), \nu_{B_2}(x_2), \nu_{B_3}(x_3), \dots) = 1$

Then $A_1 \cap B_i = (\mu_{A_1}, \nu_{A_1}) \cap (\mu_{B_i}, \nu_{B_i}) = (\mu_{A_1} \cap \mu_{B_i}, \nu_{A_1} \cup \nu_{B_i}) = (0, 1)$

Since $A_1 \in \tau_1$ and $B_i \in \tau_i$gives $A_1 \cup B_i = (1,0)$ and $A_1 \cap B_i = (0,1)$, then $A_1 \cup B_1$ is a disconnection of τ_1. Thus every coordinate space of τ_i are IF-disconnected. Hence, (X_i, τ_i) be a collection of IF-disconnected space, which is a contradiction. So, the product of IF-connected space is IF-connected.

5.4 Totally Connectedness

In this section, we have studied about totally connectedness in intuitionistic fuzzy topological space.

Definition 5.4.1. An IFTS (X, τ) is said to be totally IF-connected if for each pair of IFP $p_{\alpha,\beta}, q_{\rho,\theta} \in X$, there exists a disconnection $G \cup H$ of X with $p_{\alpha,\beta} \in G$ and $q_{\rho,\theta} \in H$.

Theorem 5.4.2. The maximal of a totally IF-disconnected space is the IF-singleton.

Proof: Let E be the maximal of a totally IF-disconnected space X and suppose $x_{\alpha,\beta}, y_{m,n} \in E$ with $x_{\alpha,\beta} \neq y_{m,n}$. Since X is totally IF-disconnected there exists a disconnection $G \cup H$ of X such that $x_{\alpha,\beta} \in G = (\mu_G, \nu_G)$ and $y_{m,n} \in H = (\mu_H, \nu_H) \Rightarrow G \cup H = (1, 0), G \cap H = (0, 1)$. Consequently $E \cap G$ and $E \cap H$ are nonempty and $(E \cap G) \cup (E \cap H) = E \cap (G \cup H) = E$ and $(E \cap G) \cap (E \cap H) = E \cap (G \cap H) = (0, 1)$, so $(E \cap G) \cup (E \cap H)$ forms a disconnection of E. But this contradicts the fact that E is a maximal and so is IF-connected. So, we conclude that E consists of exactly one intuitionistic fuzzy point, hence E is the IF-singleton of X.

Theorem 5.4.3. The continuous image of a totally IF-disconnected space is totally IF-disconnected.

Proof: Let $f: (X, \tau) \to (Y, \delta)$ be a continuous function from an IFTS (X, τ) to (Y, δ). Consider $x_{\alpha,\beta}, y_{r,s}$ be two IFP in $Y = f(X)$. Since f is continuous $f^{-1}(x_{\alpha,\beta})$ and $f^{-1}(y_{r,s})$ are IFP in X. If (X, τ) is totally IF-disconnected then there exists a disconnection $G \cup H$ of X where $f^{-1}(x_{\alpha,\beta}) \in G = (\mu_G, \nu_G)$ and $f^{-1}(y_{r,s}) \in H = (\mu_H, \nu_H)$. Since $f^{-1}(x_{\alpha,\beta}) \in G \Rightarrow x_{\alpha,\beta} \in f(G)$ and $f^{-1}(y_{r,s}) \in H \Rightarrow y_{r,s} \in f(H)$. Again $G \cup H$ is a disconnection of X such that $G \cup H = (1,0)$ and $\cap H = (0,1)$.

Here, $G \cup H = (1,0) \Rightarrow (\mu_G, \nu_G) \cup (\mu_H, \nu_H) = (1,0)$

$$\Rightarrow (\mu_G \cup \mu_H, \nu_G \cap \nu_H) = (1,0)$$

And $G \cap H = (0,1) \Rightarrow (\mu_G, \nu_G) \cap (\mu_H, \nu_H) = (1,0)$

$$\Rightarrow (\mu_G \cap \mu_H, \nu_G \cup \nu_H) = (1,0)$$

So, $f(G) = (f(\mu_G), f(\nu_G))$ and $f(H) = (f(\mu_H), f(\nu_H))$ gives

$$\begin{aligned} f(G) \cup f(H) &= (f(\mu_G), f(\nu_G)) \cup (f(\mu_H), f(\nu_H)) \\ &= (f(\mu_G) \cup f(\mu_H), f(\nu_G) \cap f(\nu_H)) \\ &= ((\mu_G \cup \mu_H)(f^{-1}(x)), (\nu_G \cap \nu_H)(f^{-1}(x))) \\ &= (1,0) \end{aligned}$$

And
$$\begin{aligned} f(G) \cap f(H) &= (f(\mu_G), f(\nu_G)) \cap (f(\mu_H), f(\nu_H)) \\ &= (f(\mu_G) \cap f(\mu_H), f(\nu_G) \cup f(\nu_H)) \\ &= ((\mu_G \cap \mu_H)(f^{-1}(x)), (\nu_G \cup \nu_H)(f^{-1}(x))) \\ &= (0,1) \end{aligned}$$

So, $Y = f(X)$ is totally IF-disconnected.

Definition 5.4.4. An IFTS (X, τ) is T_1-space if $\forall$ IF-singleton $x_{\alpha,\beta}, y_{m,n} \in X$ with $x_{\alpha,\beta} \neq y_{m,n}$ then $\exists A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ such that $x_{\alpha,\beta} \in A, y_{m,n} \notin A$ and $x_{\alpha,\beta} \notin B, y_{m,n} \in B$.

Theorem 5.4.5. Every IF- T_1 space is totally IF-disconnected space.

Proof: Let (X,τ) be an IFTS and also IF- T_1 space. consider $x_{\alpha,\beta}, y_{m,n} \in X$ with $x_{\alpha,\beta} \neq y_{m,n}$ then $\exists A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ such that $x_{\alpha,\beta} \in A, y_{m,n} \notin A$ and $x_{\alpha,\beta} \notin B, y_{m,n} \in B$.

Now $x_{\alpha,\beta} \in A = (\mu_A, \nu_A) \Rightarrow \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta$

$x_{\alpha,\beta} \notin B = (\mu_B, \nu_B) \Rightarrow \mu_B(x) < \alpha, \nu_B(x) > \beta$

$y_{m,n} \notin A = (\mu_A, \nu_A) \Rightarrow \mu_A(y) < m, \nu_A(y) > n$

$y_{m,n} \in B = (\mu_B, \nu_B) \Rightarrow \mu_B(y) \geq m, \nu_B(y) \leq n$

So, $(A \cup B)(x) = (\mu_A \cup \mu_B, \nu_A \cap \nu_B) > (\alpha,\beta), (A \cap B)(x) = (\mu_A \cap \mu_B, \nu_A \cup \nu_B) < (\alpha,\beta)$

and $(A \cup B)(y) = (\mu_A \cup \mu_B, \nu_A \cap \nu_B) > (m,n), (A \cap B)(y) = (\mu_A \cap \mu_B, \nu_A \cup \nu_B) < (m,n)$.

This result is true for any $x_{\alpha,\beta}, y_{m,n} \in X$ with $x_{\alpha,\beta} \neq y_{m,n}$. Hence it is clear that $A \cup B$ is a disconnection of X, so (X,τ) is totally IF-disconnected.

Theorem 5.4.6. Every IF- T_2 space is totally IF-disconnected space.

Proof: Let (X,τ) be an IFTS and also IF- T_2 space. Consider $x_{\alpha,\beta}, y_{m,n} \in X$ with $x_{\alpha,\beta} \neq y_{m,n}$ then $\exists A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ with $\mu_A(x_{\alpha,\beta}) = 1, \nu_A(x_{\alpha,\beta}) = 0, \mu_B(y_{m,n}) = 1, \nu_B(y_{m,n}) = 0$ and $A \cap B = (0,1)$.

Now $x_{\alpha,\beta} \in A = (\mu_A, \nu_A) \Rightarrow \mu_A(x_{\alpha,\beta}) = 1, \nu_A(x_{\alpha,\beta}) = 0$

$x_{\alpha,\beta} \notin B = (\mu_B, \nu_B) \Rightarrow \mu_B(x_{\alpha,\beta}) = 0, \nu_B(x_{\alpha,\beta}) = 1$

$y_{m,n} \notin A = (\mu_A, \nu_A) \Rightarrow \mu_A(y_{m,n}) = 0m, \nu_A(y_{m,n}) = 1$

$y_{m,n} \in B = (\mu_B, \nu_B) \Rightarrow \mu_B(y_{m,n}) = 1, \nu_B(y_{m,n}) = 0$

So, $(A \cup B)(x_{\alpha,\beta}) = (\mu_A \cup \mu_B, \nu_A \cap \nu_B) = (1,0), (A \cap B)(x_{\alpha,\beta}) = (\mu_A \cap \mu_B, \nu_A \cup \nu_B) = (0,1)$

and $(A \cup B)(y_{m,n}) = (\mu_A \cup \mu_B, \nu_A \cap \nu_B) = (1,0), (A \cap B)(y_{m,n}) = (\mu_A \cap \mu_B, \nu_A \cup \nu_B) = (0,1)$.

Hence it is clear that $A \cup B$ is a disconnection of X, so (X, τ) is totally IF-disconnected.

5.5 Super Connectedness and Strong Connectedness

Definition 5.5.1. An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ in X is proper if $\mu_A \neq 0\ or\ 1$ and $\nu_A \neq 0\ or\ 1$.

Definition 5.5.2. An IFTS X is said to be IF-super connected if X does not have non-zero intuitionistic fuzzy open subsets $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ such that $\mu_A + \mu_B \leq 1$.

Definition 5.5.3. An IFTS X is said to be IF-strong connected if X does not have non-zero intuitionistic fuzzy closed subsets $F = (\nu_F, \mu_F)$ and $K = (\nu_K, \mu_K)$ such that $\nu_F + \nu_K \leq 1$.

Theorem 5.5.4. If $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are intuitionistic fuzzy subsets of an IFTS (X, τ) and $\mu_A \subseteq \mu_B \subseteq \overline{\mu_A}$ i.e. $A \subseteq B \subseteq \bar{A}$, if A is IF-strong connected of X then B is also IF-strong connected.

Proof: Let $B = (\mu_B, \nu_B)$ is not IF-strong connected, then there exist two non-zero intuitionistic fuzzy closed subsets $F = (\nu_F, \mu_F)$ and $K = (\nu_K, \mu_K)$ such that $\nu_F|B + \nu_K|B \subseteq 1 \dots (i)$. If $\nu_F|A = 0$ then $\nu_F + \mu_A \subseteq 1$ and this implies $\nu_F + \mu_A \subseteq \nu_F + \mu_B \subseteq \nu_F + \overline{\mu_A} \dots (ii)$. So, $\nu_F + \mu_B \subseteq 1$ and thus $\nu_F|B = 0$, a contradiction and therefore $\nu_F|A \neq 0$. Similarly we can show that $\nu_K|A \neq 0$. By (i) with the relation $\mu_A \subseteq \mu_B$ imply $\nu_F|A + \nu_K|A \subseteq 1$, so A is not IF-strong connected which is a contradiction also.

Theorem 5.5.5. Let $f: (x, \tau) \to (Y, \delta)$ be an intuitionistic fuzzy continuous mapping. If X is an IF-strongly connected then so is Y.

Proof: Suppose that Y is not IF-strongly connected then there exists intuitionistic fuzzy closed set $F = (\mu_F, \nu_F)$ and $K = (\mu_K, \nu_K)$ in Y such that $F \neq (0,1), K \neq (0,1)$ and $\nu_F + \nu_K \subseteq 1$. Since f is intuitionistic fuzzy continuous $f^{-1}(F), f^{-1}(K)$ are intuitionistic fuzzy closed sets in X and $f^{-1}(F) \cap f^{-1}(K) = (0,1), f^{-1}(F) \neq (0,1), f^{-1}(K) \neq (0,1)$. If $f^{-1}(F) = (0,1)$ then $f(f^{-1}(F)) = F$ which implies $f(0,1) = F$, so $F = (0,1)$ a contradiction. Hence X is IF-strongly disconnected, a contradiction. Thus (Y, δ) is IF-strongly connected.

Definition 5.5.6. An IFTS X is said to be IF-locally connected at an IFP $p_{\alpha,\beta}$ in X if for every IFOS $A = (\mu_A, \nu_A)$ in X containing $p_{\alpha,\beta}$, there exists an IF-connected open set $B = (\mu_B, \nu_B)$ in X such that $p_{\alpha,\beta} \in B \subseteq A$.

Theorem 5.5.7. An IFTS (X,τ) is IF-locally connected if every IFOS of X is IF-locally connected.

Proof: Let V be an intuitionistic fuzzy open subspace of X and let $A=(\mu_A,\nu_A)$ be an IFOS in X. Let $p_{\alpha,\beta}$ be an IFP in V and let A|V be an IFOS in V containing $p_{\alpha,\beta}$. We must prove that there exists an IF-connected open set $B=(\mu_B,\nu_B)$ in V such that $p_{\alpha,\beta}\in B|V\subseteq A|V$. Clearly the IFP $p_{\alpha,\beta}$ in X lies in A. Since X is IF-locally connected, there exists an IF-connected open set $B=(\mu_B,\nu_B)$ such that $p_{\alpha,\beta}\in B\subseteq A$. It is easy to prove that $p_{\alpha,\beta}\in B|V\subseteq A|V$. If B|V is not IF-connected then there exists a proper IF-clopen C|V in B|V (where $C=(\mu_C,\nu_C)$ is proper IF-clopen in B). This is a contradiction with the fact that B is IF-connected and hence V is IF-locally connected.

Definition 5.5.8. An IFTS (X,τ) is said to be IF-locally super connected at an IFP $p_{\alpha,\beta}$ in X if for every IFOS $A=(\mu_A,\nu_A)$ in X containing $p_{\alpha,\beta}$, there exists an IF-super connected open set $B=(\mu_B,\nu_B)$ in X such that $p_{\alpha,\beta}\in B\subseteq A$.

Theorem 5.5.9. Let an IFTS (X,τ) is IF-locally super connected space and (Y,δ) be an IFTS. Suppose f be a continuous function from X onto Y, then Y is also IF-locally super connected.

Proof: Let $p_{\alpha,\beta}$ be an intuitionistic fuzzy point of Y. To prove Y is IF-locally super connected, then we have to show that for every IF-open set $A=(\mu_A,\nu_A)$ be an IFOS in Y containing $p_{\alpha,\beta}$ there exists an IF-super connected open set $B=(\mu_B,\nu_B)$ in Y such that $p_{\alpha,\beta}\in B\subseteq A$. Since $f:(X,\tau)\rightarrow(Y,\delta)$ is IF-continuous then there exist an

IFP $q_{\rho,\theta}$ of X such that $f(q_{\rho,\theta}) = p_{\alpha,\beta}$ and $f^{-1}(A)$ is IF-open set in X then $f^{-1}(A)(q_{\rho,\theta}) = A\left(f(q_{\rho,\theta})\right) = A(p_{\alpha,\beta})$, so $f(q_{\rho,\theta}) \subseteq A$ and thus $q_{\rho,\theta} \subseteq f^{-1}(A)$. Since X is IF- locally super connected then there exists an IF-super connected open set $C = (\mu_C, \nu_C)$ such that $q_{\rho,\theta} \in C \subseteq f^{-1}(A)$, which gives $f(q_{\rho,\theta}) \in f(C) \subseteq$ A i.e. $p_{\alpha,\beta} \in B \subseteq A$, where $B = f(C)$ is IF-super connected. Hence Y is IF-locally super connected.

Chapter Six

(r, s)-Connectedness in IFTS

In this chapter, we have introduced (r, s)-connectedness in intuitionistic fuzzy topological spaces. Furthermore, we have established some theorems and examples of (r, s)-connectedness in intuitionistic fuzzy topological spaces and discussed different characterizations of (r, s)-connectedness.

6.1 Definition and Relationship

In this section, we have introduced the notions of (r, s)-connectedness in intuitionistic fuzzy topological spaces and discussed its properties.

Definition 6.1.1. An IFTS (X, τ) is said to be (r, s)-disconnected for $r \in I_0, s \in I_1$ if there exist non-empty open IFSs $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ in X such that $(A \cup B)(x) > (s, r)$ and $(A \cap B)(x) < (r, s), \forall x \in X$.

Theorem 6.1.2. Let (X, τ) is an IFTS. If (X, τ) is IF-connected then (X, τ) is IF-(r, s)-connected. But converse of the above theorem is not true in general.

Proof: Let (X, τ) is not IF-connected then there exist non-empty open IFSs $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ in X such that $A \cup B = (1, 0)$ and $A \cap B = (0, 1)$.

Now, $A \cup B = (1, 0)$

i.e. $(\mu_A, \nu_A) \cup (\mu_B, \nu_B) = (1, 0)$

$\Rightarrow \mu_A \cup \mu_B = 1, \nu_A \cap \nu_B = 0$

$\Rightarrow \mu_A \cup \mu_B > s, \nu_A \cap \nu_B < r$, as $r \in I_0 = (0,1], s \in I_1 = [0,1)$

$\Rightarrow A \cup B > (s,r)$

Again, $A \cap B = (0,1)$

i.e. $(\mu_A, \nu_A) \cap (\mu_B, \nu_B) = (0,1)$

$\Rightarrow \mu_A \cap \mu_B = 0, \nu_A \cup \nu_B = 1$

$\Rightarrow \mu_A \cap \mu_B < r, \nu_A \cup \nu_B > s$, as $r \in I_0 = (0,1], s \in I_1 = [0,1)$

$\Rightarrow A \cap B < (r,s)$

So, (X,τ) is IF-(r,s)-disconnected. Hence if (X,τ) is IF-connected then (X,τ) is IF-(r,s)-connected.

The second part of the theorem can be prove by an example.

Let τ be an IFT and A and B are two IFS on X, where $A = \{\langle x, (0.8,0.5), (0.5,0.3\rangle; x \in X\}$ and $B = \{\langle x, (0.4,0.2), (0.3,0.5)\rangle; x \in X\}$, then $A \cup B = \{\langle x, (0.8,0.2), (0.5,0.3)\rangle; x \in X\} > (s,r)$ and $A \cap B = \{\langle x, (0.4,0.5), (0.3,0.5)\rangle; x \in X\} < (r,s)$ where $r = 0.8, s = 0.3$. So, (X,τ) is (r,s)-disconnected. But $A \cup B = \{\langle x, (0.8,0.2), (0.5,0.3)\rangle; x \in X\} \neq (1,0)$ and $A \cap B = \{\langle x, (0.4,0.5), (0.3,0.5)\rangle; x \in X\} \neq (0,1)$, so (X,τ) is not IF-disconnected.

Theorem 6.1.3. An IFTS (X,τ) is IF-(r,s)-connected if and only if there exists no non-empty IFOS A and B in X such that $A = B^C$.

Proof of the above theorem is obvious.

Theorem 6.1.4. The continuous image of an IF-(r, s)-connected space X is IF-(r, s)-connected.

Proof: Let$f: (X, \tau) \to (Y, \delta)$ be a continuous function from an IFTS (X, τ) to (Y, δ). Consider (X, τ) is IF-(r, s)-connected, we shall prove that (Y, δ) is also IF-(r, s)-connected. Suppose (Y, δ) is not IF-(r, s)-connected, i.e. (Y, δ) has a (r, s)-disconnection. Let this be $G = (\mu_G, \nu_G)$ and $H = (\mu_H, \nu_H)$ be two IFS on X then $G \cup H > (s, r)$ i.e. $\mu_G \cup \mu_H > s$ and $\nu_G \cap \nu_H < r$. Again, $G \cap H < (r, s)$ i.e. $\mu_G \cap \mu_H < r$ and $\nu_G \cup \nu_H > s$.

Now $f^{-1}(G) = (f^{-1}(\mu_G), f^{-1}(\nu_G))$ and $f^{-1}(H) = (f^{-1}(\mu_H), f^{-1}(\nu_H))$.

So, $f^{-1}(G) \cup f^{-1}(H)$

$= (\max\big(f^{-1}(\mu_G), f^{-1}(\mu_H)\big)(x), \min(f^{-1}(\nu_G), f^{-1}(\nu_H))(x))$

$= \big(max(f^{-1}(\mu_G)(x), f^{-1}(\mu_H)(x)\big), min(f^{-1}(\nu_G)(x), f^{-1}(\nu_H)(x)))$

$= (\max(\mu_G(f(x), \mu_H\big(f(x)\big)), min(\nu_G(f(x), \nu_H(f(x)))$

$= ((\mu_G \cup \mu_H)(f(x)), (\nu_G \cap \nu_H)(f(x)))$

$= (f^{-1}(\mu_G \cup \mu_H)(x), f^{-1}(\nu_G \cap \nu_H)(x))$

$> (s, r)$

Again, $f^{-1}(G) \cap f^{-1}(H)$

$= (\min\big(f^{-1}(\mu_G), f^{-1}(\mu_H)\big)(x), \max(f^{-1}(\nu_G), f^{-1}(\nu_H))(x))$

$= \big(min(f^{-1}(\mu_G)(x), f^{-1}(\mu_H)(x)\big), max(f^{-1}(\nu_G)(x), f^{-1}(\nu_H)(x)))$

$= (\min(\mu_G(f(x), \mu_H\big(f(x)\big)), max(\nu_G(f(x), \nu_H(f(x)))$

$= ((\mu_G \cap \mu_H)(f(x)), (\nu_G \cup \nu_H)(f(x)))$

$= (f^{-1}(\mu_G \cap \mu_H)(x), f^{-1}(\nu_G \cup \nu_H)(x))$

$< (r, s)$

Hence , $f^{-1}(G)$ and $f^{-1}(H)$ give a (r,s)-disconnection for X, which gives the prove.

Theorem 6.1.5. Let $\{(X_i,\tau_{X_i}), i \in J\}$ be a family of subspaces of an IFTS (X,τ) such that $\cap X_i \neq \phi$, if (X_i,τ_{X_i}) is IF-(r,s)-connected then $(\cup X_i,\tau_{\cup X_i})$ is also IF-(r,s)-connected.

Proof: Suppose $(\cup X_i,\tau_{\cup X_i})$ is not IF-(r,s)-connected, there exist $A=(\mu_A,\nu_A), B=(\mu_B,\nu_B) \in \tau_{\cup X_i}$ such that $A\cup B > (s,r)$ and $A\cap B<(r,s)$.

Now, $A\cup B>(s,r) \Rightarrow (A\cup B)|X_i>(s,r),\ \forall X_i \subseteq \cup X_i$

$$\Rightarrow ((\mu_A,\nu_A)\cup(\mu_B,\nu_B))|X_i>(s,r)$$

$$\Rightarrow (\mu_A\cup\mu_B,\nu_A\cap\nu_B)|X_i>(s,r)$$

Which gives, $(\mu_A\cup\mu_B)|X_i>s$ and $(\nu_A\cap\nu_B)|X_i<r$

From $(\mu_A\cup\mu_B)|X_i<s$, we get $(\mu_{A_i}|X_i)\cup(\mu_{B_i}|X_i)>s$ and from $(\nu_A\cap\nu_B)|X_i<r$ we get $(\nu_{A_i}|X_i)\cap(\nu_{B_i}|X_i)<r$, where $(\mu_{A_i}|X_i,\nu_{A_i}|X_i),(\mu_{B_i}|X_i,\nu_{B_i}|X_i)\in\tau_{X_i}$.

Again from $A\cap B<(r,s)\Rightarrow(A\cap B)|X_i<(r,s),\ \forall X_i\subseteq\cup X_i$

$$\Rightarrow ((\mu_A,\nu_A)\cap(\mu_B,\nu_B))|X_i<(r,s)$$

$\Rightarrow (\mu_A\cap\mu_B,\nu_A\cup\nu_B)|X_i<(r,s)$, which gives, $(\mu_A\cap\mu_B)|X_i<r$ and $(\nu_A\cup\nu_B)|X_i>s$. Therefore, $(\mu_{A_i}|X_i)\cap(\mu_{B_i}|X_i)<r$ and $(\nu_{A_i}|X_i)\cup(\nu_{B_i}|X_i)>s$.

Hence, (X_i,τ_{X_i}) is not IF-(r,s)-connected.

6.2 Good Extension of (r,s)-Connectedness

Theorem 6.2.1. Let (X,T) be a topological space and (X,τ) be its corresponding IFTS, where $\tau = \{(1_A, 1_{A^c}) : A \in T\}$. If (X,T) is connected then (X,τ) is IF-(r,s)-connected.

Proof: Suppose (X,T) is disconnected, so there exist two nonempty subsets A, B of X such that $A \cup B = X, A \cap B = \emptyset$. Since $A, B \in T$ then $1_A = (1_A, 1_{A^c}) \in \tau$ and $1_B = (1_B, 1_{B^c}) \in \tau$. Now, $1_A \cup 1_B = (1_A, 1_{A^c}) \cup (1_B, 1_{B^c})$

$= (1_A \cup 1_B, 1_{A^c} \cap 1_{B^c})$

$= (1_{A\cup B}, 1_{A^c \cap B^c})$

$= (1_{A\cup B}, 1_{(A\cup B)^c})$

$= (1_X, 1_\emptyset)$

$= (1,0)$

$> (s,r)$, as $r \in I_0 = (0,1], s \in I_1 = [0,1)$

Again, $1_A \cap 1_B = (1_A, 1_{A^c}) \cap (1_B, 1_{B^c})$

$= (1_A \cap 1_B, 1_{A^c} \cup 1_{B^c})$

$= (1_{A\cap B}, 1_{A^c \cup B^c})$

$= (1_{A\cap B}, 1_{(A\cap B)^c})$

$= (1_\emptyset, 1_X)$

$= (0,1)$

$< (r,s)$, as $r \in I_0 = (0,1], s \in I_1 = [0,1)$

So, (X,τ) is IF-(r,s)-disconnected.

Hence if (X,T) is connected then (X,τ) is IF-(r,s)-connected.

Theorem 6.2.2. Let $(X,\mathcal{T})$ be an intuitionistic topological space and (X,τ) be its corresponding IFTS, where $\tau=\{(1_{A_1},1_{A_2}):A=(A_1,A_2)\in\mathcal{T}\}$. If $(X,\mathcal{T})$ is connected then (X,τ) is IF-(r,s)-connected.

Proof: Suppose $(X,\mathcal{T})$ is disconnected, so there exist two nonempty subsets A,B of X such that $A\cup B=(X,\emptyset), A\cap B=(\emptyset,X)$. Since $A,B\in\mathcal{T}$ then $1_A=(1_{A_1},1_{A_2})\in\tau$ and $1_B=(1_{B_1},1_{B_2})\in\tau$.

Here, $A\cup B=(X,\emptyset)\Rightarrow(A_1,A_2)\cup(B_1,B_2)=(X,\emptyset)$

$$\Rightarrow(A_1\cup B_1,A_2\cap B_2)=(X,\emptyset)$$

$$\Rightarrow A_1\cup B_1=X, A_2\cap B_2=\emptyset$$

Again, $A\cap B=(\emptyset,X)\Rightarrow(A_1,A_2)\cap(B_1,B_2)=(\emptyset,X)$

$$\Rightarrow(A_1\cap B_1,A_2\cup B_2)=(\emptyset,X)$$

$$\Rightarrow A_1\cap B_1=\emptyset, A_2\cup B_2=X$$

Now, $1_A\cup 1_B=(1_{A_1},1_{A_2})\cup(1_{B_1},1_{B_2})$

$$=(1_{A_1}\cup 1_{B_1},1_{A_2}\cap 1_{B_2})$$

$$=\left(1_{A_1\cup B_1},1_{A_2\cap B_2}\right)$$

$$=(1_X,1_\emptyset)$$

$$=(1,0)$$

$$>(s,r),\text{ as } r\in I_0=(0,1], s\in I_1=[0,1)$$

Again, $1_A\cap 1_B=(1_{A_1},1_{A_2})\cap(1_{B_1},1_{B_2})$

$$=(1_{A_1}\cap 1_{B_1},1_{A_2}\cup 1_{B_2})$$

$$=\left(1_{A_1\cap B_1},1_{A_2\cup B_2}\right)$$

$$=(1_\emptyset,1_X)$$

$$=(0,1)$$

$$> (r,s), \text{ as } r \in I_0 = (0,1], s \in I_1 = [0,1)$$

So, (X,τ) is IF-(r,s)-disconnected.

Hence if $(X,\mathcal{T})$ is connected then (X,τ) is IF-(r,s)-connected.

Theorem 6.2.3. Let (X,t) be a intuitionistic topological space and (X,τ) be its corresponding IFTS, where $\tau = \{(\lambda,\lambda^C): \lambda \in t\}$. Then if (X,t) is connected then (X,τ) is IF-(r,s)-connected.

Proof: Suppose (X,t) is disconnected, so there exist two nonempty subsets α,β of X such that $\alpha \cup \beta = 1, \alpha \cap \beta = 0$ where $\alpha \neq 0 \neq \beta, \alpha \neq 1 \neq \beta$. Since $\alpha,\beta \in t$ then $(\alpha,\alpha^C),(\beta,\beta^C) \in \tau$.

Now, $(\alpha,\alpha^C) \cup (\beta,\beta^C) = (\alpha \cup \beta, \alpha^C \cap \beta^C)$

$$= (\alpha \cup \beta, (\alpha \cup \beta)^C)$$

$$= (1,0)$$

$$> (s,r), \text{ as } r \in I_0 = (0,1], s \in I_1 = [0,1)$$

Again, $(\alpha,\alpha^C) \cap (\beta,\beta^C) = (\alpha \cap \beta, \alpha^C \cup \beta^C)$

$$= (\alpha \cap \beta, (\alpha \cap \beta)^C)$$

$$= (0,1)$$

$$< (r,s), \text{ as } r \in I_0 = (0,1], s \in I_1 = [0,1)$$

So, (X,τ) is IF-(r,s)-disconnected.

Hence if (X,t) is connected then (X,τ) is IF-(r,s)-connected.

6.3 Productivity of $(\boldsymbol{r}, \boldsymbol{s})$-Connectedness

In this section, we discuss about productive property of (r, s)-connectedness in intuitionistic fuzzy topological space.

Theorem 6.3.1. If (X, τ) and (Y, δ) are IF-(r, s)-connected space then $(X \times Y, \tau \times \delta)$ is also IF-(r, s)-connected.

Proof: Consider $(X \times Y, \tau \times \delta)$ is not IF-(r, s)-connected then $\exists A, B \in \tau \times \delta$ such that $A \cup B > (s, r)$ and $A \cap B < (r, s)$. Since $A, B \in \tau \times \delta$ then $A = C \times D$ and $B = E \times F$ where $C = (\mu_C, \nu_C), E = (\mu_E, \nu_E) \in \tau$, and $D = (\mu_D, \nu_D), F = (\mu_F, \nu_F) \in \delta$. Now $C \times D = (\mu_C \underset{\cdot}{\overset{\times}{\ }} \mu_D, \nu_C \underset{\times}{\overset{\cdot}{\ }} \nu_D)$, where $\left(\mu_C \underset{\cdot}{\overset{\times}{\ }} \mu_D\right)(x, y) = \min(\mu_C(x), \mu_D(y))$ and $(\nu_C \underset{\times}{\overset{\cdot}{\ }} \nu_D)(x, y) = \max\big(\nu_C(x), \nu_D(y)\big)$, $\forall (x, y) \in \tau \times \delta$. Similarly, $E \times F = (\mu_E \underset{\cdot}{\overset{\times}{\ }} \mu_F, \nu_E \underset{\times}{\overset{\cdot}{\ }} \nu_F)$.

Now $A \cup B > (s, r)$

$\Rightarrow (C \times D) \cup (E \times F) > (s, r)$

$\Rightarrow \left(\mu_C \underset{\cdot}{\overset{\times}{\ }} \mu_D, \nu_C \underset{\times}{\overset{\cdot}{\ }} \nu_D\right) \cup \left(\mu_E \underset{\cdot}{\overset{\times}{\ }} \mu_F, \nu_E \underset{\times}{\overset{\cdot}{\ }} \nu_F\right) > (s, r)$

$\Rightarrow (\min\big(\mu_C(x), \mu_D(y)\big) \cup \min\big(\mu_E(x), \mu_F(y)\big), \max\big(\nu_C(x), \nu_D(y)\big)$

$\quad \cap \max\big(\nu_E(x), \nu_F(y)\big)) > (s, r)$

i.e. $\min\big(\mu_C(x), \mu_D(y)\big) \cup \min\big(\mu_E(x), \mu_F(y)\big) > s$

$\Rightarrow$ Either, $\min\big(\mu_C(x), \mu_D(y)\big) > s$ or, $\min\big(\mu_E(x), \mu_F(y)\big) > s$

$\Rightarrow$ Either $\mu_C(x) > s, \mu_D(y) > s$ or, $\mu_E(x) > s, \mu_F(y) > s$

For, $\max\big(\nu_C(x), \nu_D(y)\big) \cap \max\big(\nu_E(x), \nu_F(y)\big) < r$

$\Rightarrow \max\big(\nu_C(x), \nu_D(y)\big) < r$ and $\max\big(\nu_E(x), \nu_F(y)\big) < r$

$\Rightarrow \nu_C(x) < r, \nu_D(y) < r, \nu_E(x) < r, \nu_F(y) < r$

Case I: Suppose $\mu_C(x) > s, \mu_D(y) > s$

Then $C \cup E = (\mu_C, \nu_C) \cup (\mu_E, \nu_E) = (\mu_C \cup \mu_E, \nu_C \cap \nu_E) > (s, r)$ as $\mu_C(x) > s$

Case II: Suppose $\mu_E(x) > s, \mu_F(y) > s$

Then $D \cup F = (\mu_D, \nu_D) \cup (\mu_F, \nu_F) = (\mu_D \cup \mu_F, \nu_D \cap \nu_F) > (s, r)$ as $\mu_F(y) > s$

Again, $A \cap B < (r, s)$

$\Rightarrow (C \times D) \cap (E \times F) < (r, s)$

$\Rightarrow \left(\mu_C \underset{\cdot}{\times} \mu_D, \nu_C \overset{\cdot}{\times} \nu_D\right) \cap \left(\mu_E \underset{\cdot}{\times} \mu_F, \nu_E \overset{\cdot}{\times} \nu_F\right) < (r, s)$

$\Rightarrow (\min\big(\mu_C(x), \mu_D(y)\big) \cap \min\big(\mu_E(x), \mu_F(y)\big), \max\big(\nu_C(x), \nu_D(y)\big)$

$\cup \max\big(\nu_E(x), \nu_F(y)\big)) < (r, s)$

i.e., $\min\big(\mu_C(x), \mu_D(y)\big) \cap \min\big(\mu_E(x), \mu_F(y)\big) < r$

$\Rightarrow \min\big(\mu_C(x), \mu_D(y)\big) < r$ and $\min\big(\mu_E(x), \mu_F(y)\big) < r$

$\Rightarrow$ Either $\mu_C(x) < r$, or $\mu_D(y) < r$ and either $\mu_E(x) < r$ or $\mu_F(y) < r$

Again, for, $\max\big(\nu_C(x), \nu_D(y)\big) \cup \max\big(\nu_E(x), \nu_F(y)\big) > s$

$\Rightarrow$Either $\max\big(\nu_C(x), \nu_D(y)\big) > s$ or, $\max\big(\nu_E(x), \nu_F(y)\big) > s$

$\Rightarrow$ Either $\nu_C(x) > s$ or $\nu_D(y) > s$, or, either $\nu_E(x) > s$ or $\nu_F(y) > s$

Case III: Suppose $\mu_C(x) < r$, or $\mu_D(y) < r$ and $\nu_C(x) > s$

Then $C \cap E = (\mu_C, \nu_C) \cap (\mu_E, \nu_E) = (\mu_C \cap \mu_E, \nu_C \cup \nu_E) < (r, s)$

Case IV: Suppose $\mu_E(x) < r$ or $\mu_F(y) < r$ and $\nu_F(y) > s$

Then $D \cap F = (\mu_D, \nu_D) \cap (\mu_F, \nu_F) = (\mu_D \cap \mu_F, \nu_D \cup \nu_F) < (r, s)$

So, (X, τ) and (Y, δ) are not (r, s)-connected, hence if (X, τ) and (Y, δ) are IF-(r, s)-connected then $(X \times Y, \tau \times \delta)$ is IF-(r, s)-connected.

Theorem 6.3.2. The product of IF-(r,s)-connected space is IF-(r,s)-connected.

Proof: Let (X_i,τ_i) be a collection of IF-(r,s)-connected space. Also let $(X,\tau) = (\Pi_i X_i, \Pi_i \tau_i)$ be the product space. Consider $(\Pi_i X_i, \Pi_i, \tau_i)$ are not IF-(r,s)-connected then there exists $A,B \in \tau_1 \times \tau_2 \times \tau_3 \times ....$ such that $A \cup B > (s,r)$ and $A \cap B < (r,s)$. Since $A,B \in \tau_1 \times \tau_2 \times \tau_3 \times ....$ then $A = A_1 \times A_2 \times A_3 \times ...$ and $B = B_1 \times B_2 \times B_3 \times ...$, where $A_i = \left(\mu_{A_i}, \nu_{A_i}\right) \in \tau$ and $B_i = \left(\mu_{B_i}, \nu_{B_i}\right) \in \tau$.

Now, $A \cup B > (s,r)$

$\Rightarrow (A_1 \times A_2 \times A_3 \times ...) \cup (B_1 \times B_2 \times B_3 \times ...) > (s,r)$

$\Rightarrow \left(\left(\mu_{A_1}, \nu_{A_1}\right) \times \left(\mu_{A_2}, \nu_{A_2}\right) \times \left(\mu_{A_3}, \nu_{A_3}\right) \times ...\right) \cup$

$$\left(\left(\mu_{B_1}, \nu_{B_1}\right) \times \left(\mu_{B_2}, \nu_{B_2}\right) \times \left(\mu_{B_3}, \nu_{B_3}\right) \times ...\right) > (s,r)$$

$\Rightarrow (\inf\left(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \mu_{A_3}(x_3), ...\right) \cup \inf\left(\mu_{B_1}(x_1), \mu_{B_2}(x_2), \mu_{B_3}(x_3), ...\right),$

$\sup\left(\nu_{A_1}(x_1), \nu_{A_2}(x_2), \nu_{A_3}(x_3), ...\right) \cap$

$\sup\left(\nu_{B_1}(x_1), \nu_{B_2}(x_2), \nu_{B_3}(x_3), ...\right)) > (s,r)$ where $(x_1, x_2, x_3, ...) \in \Pi_i X_i$

i.e., $\inf\left(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \mu_{A_3}(x_3), ...\right) \cup \inf\left(\mu_{B_1}(x_1), \mu_{B_2}(x_2), \mu_{B_3}(x_3), ...\right) > s$

$\Rightarrow$Either, $\inf\left(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \mu_{A_3}(x_3), ...\right) > s$

or, $\inf\left(\mu_{B_1}(x_1), \mu_{B_2}(x_2), \mu_{B_3}(x_3), ...\right) > s$

$\Rightarrow$ Either $\mu_{A_1}(x_1) > s, \mu_{A_2}(x_2) > s, \mu_{A_3}(x_3) > s, ...$

or, $\mu_{B_1}(x_1) > s, \mu_{B_2}(x_2) > s, \mu_{B_3}(x_3) > s, ...$

Again, $\sup\left(\nu_{A_1}(x_1), \nu_{A_2}(x_2), \nu_{A_3}(x_3), ...\right) \cap \sup\left(\nu_{B_1}(x_1), \nu_{B_2}(x_2), \nu_{B_3}(x_3), ...\right) < r$

$\Rightarrow \sup\left(\nu_{A_1}(x_1), \nu_{A_2}(x_2), \nu_{A_3}(x_3), ...\right) < r$ and $\sup\left(\nu_{\square_1}(\square_1), \nu_{\square_2}(\square_2), \nu_{\square_3}(\square_3), ...\right) < r$

$\Rightarrow \nu_{A_1}(x_1) < r, \nu_{A_2}(x_2) < r, \nu_{A_3}(x_3) < r, \dots, \nu_{B_1}(x_1) < r, \nu_{B_2}(x_2) < r, \nu_{B_3}(x_3) <$

$r, \dots$

Case I: Suppose $\mu_{A_1}(x_1) > s, \mu_{B_i}(x_i) > s, \nu_{A_1}(x_1) < r, \nu_{B_i}(x_i) < r$

Then $A_1 \cup B_i = (\mu_{A_1}, \nu_{A_1}) \cup (\mu_{B_i}, \nu_{B_i}) = (\mu_{A_1} \cup \mu_{B_i}, \nu_{A_1} \cap \nu_{B_i}) > (s, r)$, for any $(\mu_{B_i}, \nu_{B_i}) \in \tau_i$

Again, $A \cap B < (r, s)$

$\Rightarrow (A_1 \times A_2 \times A_3 \times \dots) \cap (B_1 \times B_2 \times B_3 \times \dots) < (r, s)$

$$\Rightarrow \left((\mu_{A_1}, \nu_{A_1}) \times (\mu_{A_2}, \nu_{A_2}) \times (\mu_{A_3}, \nu_{A_3}) \times \dots\right) \cap$$

$$\left((\mu_{B_1}, \nu_{B_1}) \times (\mu_{B_2}, \nu_{B_2}) \times (\mu_{B_3}, \nu_{B_3}) \times \dots\right) < (r, s)$$

$\Rightarrow (\inf(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \mu_{A_3}(x_3), \dots) \cap \inf(\mu_{B_1}(x_1), \mu_{B_2}(x_2), \mu_{B_3}(x_3), \dots),$

$\sup(\nu_{A_1}(x_1), \nu_{A_2}(x_2), \nu_{A_3}(x_3), \dots) \cup$

$\sup(\nu_{B_1}(x_1), \nu_{B_2}(x_2), \nu_{B_3}(x_3), \dots)) < (r, s)$

i.e. $\inf(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \mu_{A_3}(x_3), \dots) \cap \inf(\mu_{B_1}(x_1), \mu_{B_2}(x_2), \mu_{B_3}(x_3), \dots) < r$

$\Rightarrow \inf(\mu_{A_1}(x_1), \mu_{A_2}(x_2), \mu_{A_3}(x_3), \dots) < r$ and $\inf(\mu_{B_1}(x_1), \mu_{B_2}(x_2), \mu_{B_3}(x_3), \dots) < r$

For, $\sup(\nu_{A_1}(x_1), \nu_{A_2}(x_2), \nu_{A_3}(x_3), \dots) \cup \sup(\nu_{B_1}(x_1), \nu_{B_2}(x_2), \nu_{B_3}(x_3), \dots) > s$

$\Rightarrow$Either $\sup(\nu_{A_1}(x_1), \nu_{A_2}(x_2), \nu_{A_3}(x_3), \dots) > s$

or, $\sup(\nu_{B_1}(x_1), \nu_{B_2}(x_2), \nu_{B_3}(x_3), \dots) > s$

Case II: Suppose $\inf(\mu_{B_1}(x_1), \mu_{B_2}(x_2), \mu_{B_3}(x_3), \dots) < r$, $\sup(\nu_{B_1}(x_1), \nu_{B_2}(x_2), \nu_{B_3}(x_3), \dots) > s$

Then $A_1 \cap B_i = (\mu_{A_1}, \nu_{A_1}) \cap (\mu_{B_i}, \nu_{B_i}) = (\mu_{A_1} \cap \mu_{B_i}, \nu_{A_1} \cup \nu_{B_i}) < (r, s)$

Since $A_1 \in \tau_1$ and $B_i \in \tau_i$gives $A_1 \cup B_i > (s, r)$ and $A_1 \cap B_i < (r, s)$, then $A_1 \cup B_1$ is a (r, s)-disconnection of τ_1. Thus every coordinate space of τ_i are IF-(r, s)-

disconnected. Hence, (X_i, τ_i) be a collection of IF-(r, s)-disconnected space, which is a contradiction. So, the product of IF-(r, s)-connected space is IF-(r, s)-connected.

6.4 Totally (r, s)-Connectedness

Definition 6.4.1. An IFTS (X, τ) is said to be totally IF-(r, s)-disconnected for $r \in I_0, s \in I_1$ if for each pair of IFP $p_{\alpha,\beta}, q_{\rho,\theta} \in X$, there exists a (r, s)-disconnection $G \cup H$ of X with $p_{\alpha,\beta} \in G$ and $q_{\rho,\theta} \in H$ i.e. $G \cup H > (s, r)$ and $G \cap H < (r, s)$.

Theorem 6.4.2. The continuous image of a totally IF-(r, s)-disconnected space is totally IF-(r, s)-disconnected.

Proof: Let $f: (X, \tau) \to (Y, \delta)$ be a continuous function from an IFTS (X, τ) to $(\square, \square)$. Consider $x_{\alpha,\beta}, y_{r,s}$ be two IFP in $Y = f(X)$. Since f is continuous $f^{-1}(x_{\alpha,\beta})$ and $f^{-1}(y_{r,s})$ are IFP in X. If (X, τ) is totally IF-(r, s)-disconnected then there exists a (r, s)-disconnection $G \cup H$ of X where $f^{-1}\left(x_{\alpha,\beta}\right) \in G = (\mu_G, \nu_G)$ and $f^{-1}\left(y_{r,s}\right) \in H = (\mu_H, \nu_H)$. Since $f^{-1}(x_{\alpha,\beta}) \in G \Rightarrow x_{\alpha,\beta} \in f(G)$ and $f^{-1}(y_{r,s}) \in H \Rightarrow y_{r,s} \in f(H)$. Again $G \cup H$ is a (r, s)-disconnection of X such that $G \cup H > (s, r)$ and $G \cap H < (r, s)$.

Here, $G \cup H > (s, r) \Rightarrow (\mu_G, \nu_G) \cup (\mu_H, \nu_H) > (s, r)$

$$\Rightarrow (\mu_G \cup \mu_H, \nu_G \cap \nu_H) > (s, r)$$

And $G \cap H < (r, s) \Rightarrow (\mu_G, \nu_G) \cap (\mu_H, \nu_H) < (r, s)$

$$\Rightarrow (\mu_G \cap \mu_H, \nu_G \cup \nu_H) < (r, s)$$

So, $f(G) = (f(\mu_G), f(\nu_G))$ and $f(H) = (f(\mu_H), f(\nu_H))$ gives

$f(G) \cup f(H) = (f(\mu_G), f(\nu_G)) \cup (f(\mu_H), f(\nu_H))$

$= (f(\mu_G) \cup f(\mu_H), f(\nu_G) \cap f(\nu_H))$

$= ((\mu_G \cup \mu_H)(f^{-1}(x)), (\nu_G \cap \nu_H)(f^{-1}(x)))$

$> (s, r)$

And $f(G) \cap f(H) = (f(\mu_G), f(\nu_G)) \cap (f(\mu_H), f(\nu_H))$

$= (f(\mu_G) \cap f(\mu_H), f(\nu_G) \cup f(\nu_H))$

$= ((\mu_G \cap \mu_H)(f^{-1}(x)), (\nu_G \cup \nu_H)(f^{-1}(x)))$

$< (r, s)$

So, $Y = f(X)$ is totally IF-(r,s)-disconnected.

Theorem 6.4.3. Every IF- T_1 space is totally IF-(r,s)-disconnected space.

Proof: Let (X, τ) be an IFTS and also IF- T_1 space. consider $x_{\alpha,\beta}, y_{m,n} \in X$ with $x_{\alpha,\beta} \neq y_{m,n}$ then $\exists A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ such that $x_{\alpha,\beta} \in A, y_{m,n} \notin A$ and $x_{\alpha,\beta} \notin B, y_{m,n} \in B$.

Now $x_{\alpha,\beta} \in A = (\mu_A, \nu_A) \Rightarrow \mu_A(x) \geq \alpha, \nu_A(x) \leq \beta$

$x_{\alpha,\beta} \notin B = (\mu_B, \nu_B) \Rightarrow \mu_B(x) < \alpha, \nu_B(x) > \beta$

$y_{m,n} \notin A = (\mu_A, \nu_A) \Rightarrow \mu_A(y) < m, \nu_A(y) > n$

$y_{m,n} \in B = (\mu_B, \nu_B) \Rightarrow \mu_B(y) \geq m, \nu_B(y) \leq n$

So, $(A \cup B)(x) = \big((\mu_A \cup \mu_B)(x), (\nu_A \cap \nu_B)(x)\big) > (\alpha, \beta)$,

$(A \cap B)(x) = ((\mu_A \cap \mu_B)(x), (\nu_A \cup \nu_B)(x)) < (\alpha, \beta)$

and $(A \cup B)(y) = \big((\mu_A \cup \mu_B)(y), (\nu_A \cap \nu_B)(y)\big) > (m, n)$,

$(A \cap B)(y) = ((\mu_A \cap \mu_B)(y), (\nu_A \cup \nu_B)(y)) < (m, n)$

This result is true for any $x_{\alpha,\beta}, y_{m,n} \in X$ with $x_{\alpha,\beta} \neq y_{m,n}$. Hence it is clear that $A \cup B$ is a IF-(r,s)-disconnection of X, so (X, τ) is totally IF-(r,s)-disconnected.

Theorem 6.4.4. Every IF- T_2 space is totally IF-(r,s)-disconnected space.

Proof: Let (X,τ) be an IFTS and also IF- T_2 space. Consider $x_{\alpha,\beta}, y_{m,n} \in X$ with $x_{\alpha,\beta} \neq y_{m,n}$ then $\exists A = (\mu_A, \nu_A), B = (\mu_B, \nu_B) \in \tau$ with $\mu_A(x_{\alpha,\beta}) = 1, \nu_A(x_{\alpha,\beta}) = 0, \mu_B(y_{m,n}) = 1, \nu_B(y_{m,n}) = 0$ and $A \cap B = (0,1)$.

Now $x_{\alpha,\beta} \in A = (\mu_A, \nu_A) \Rightarrow \mu_A(x_{\alpha,\beta}) = 1, \nu_A(x_{\alpha,\beta}) = 0$

$x_{\alpha,\beta} \notin B = (\mu_B, \nu_B) \Rightarrow \mu_B(x_{\alpha,\beta}) = 0, \nu_B(x_{\alpha,\beta}) = 1$

$y_{m,n} \notin A = (\mu_A, \nu_A) \Rightarrow \mu_A(y_{m,n}) = 0, \nu_A(y_{m,n}) = 1$

$y_{m,n} \in B = (\mu_B, \nu_B) \Rightarrow \mu_B(y_{m,n}) = 1, \nu_B(y_{m,n}) = 0$

So, $(A \cup B)(x_{\alpha,\beta}) = \left((\mu_A \cup \mu_B)(x_{\alpha,\beta}), (\nu_A \cap \nu_B)(x_{\alpha,\beta})\right) = (1,0) > (s,r)$,

$(A \cap B)(x_{\alpha,\beta}) = \left((\mu_A \cap \mu_B)(x_{\alpha,\beta}), (\nu_A \cup \nu_B)(x_{\alpha,\beta})\right) = (0,1) < (r,s)$

as $r \in I_0 = (0,1], s \in I_1 = [0,1)$

and $(A \cup B)(y_{m,n}) = \left((\mu_A \cup \mu_B)(y_{m,n}), (\nu_A \cap \nu_B)(y_{m,n})\right) = (1,0) > (s,r)$,

$(A \cap B)(y_{m,n}) = \left((\mu_A \cap \mu_B)(y_{m,n}), (\nu_A \cup \nu_B)(y_{m,n})\right) = (0,1) < (r,s)$

as $r \in I_0 = (0,1], s \in I_1 = [0,1)$.

6.5 $\boldsymbol{\beta}$-level connectedness

Definition 6.5.1. Two disjoint non-empty IFSs $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ of an IFTS X are said to be β-level separated for $\beta \in I_0$ if there exist $G, H \in \tau$ such that $A \subseteq G, B \subseteq H$ and $A \cap B = (0,r) = G \cap H$, where $\beta < r \leq 1$.

Definition 6.5.2. An IFTS X is said to be β-level disconnected for $\beta \in I_0$ if $A \cup B = (1,0)$ and $A \cap B = (0,r)$, where $\beta < r \leq 1$ and $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are non-empty open IFSs of X.

Theorem 6.5.3. Union of two non-empty □-level separated intuitionistic fuzzy subsets of an IFTS X is β-level IF-disconnected.

Proof: Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are two non-empty β-level separated intuitionistic fuzzy subsets of an IFTS X, so $A \cap \bar{B} = (0,r)$ and $\bar{A} \cap B = (0,r)$. Let $G = \bar{B}^C$ and $H = \bar{A}^C$. Then G and H are open and $(A \cup B) \cap G = (1_A, 0)$ and $(A \cup B) \cap H = (1_B, 0)$ are non-empty disjoint IFSs whose union is $A \cup B$. Again, as A and B are two non-empty β-level separated intuitionistic fuzzy subsets of X, so $A \cap B = (0,r) = G \cap H$. Thus G and H form a β-level disconnection of $A \cup B$. Hence $A \cup B$ is β-level disconnected.

Theorem 6.5.4. Let (X, τ) be an IFTS. If X is IF-disconnected then it is also β-level IF-disconnected. The converse is not true in general.

Proof: Since X is IF-disconnected then there exist two non-empty IFSs $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ in X such that $A \cup B = (1,0)$ and $A \cap B = (0,1)$. Now for $\beta \in I_0$, $A \cup B = (1,0)$ and $(A \cap B)(x) = (0,r), \forall x \in X$ where $\beta < r = 1$, so X is β-level disconnected. Now for $0 < r < 1$, $A \cup B = (1,0)$ and $A \cap B \neq (0,1)$, so for X is β-level disconnected, X is not IF-disconnected.

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