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Study of Radicals and Semisimple Classes of Rings

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Study of Radicals and Semisimple Classes of Rings

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Certified that the thesis entitle " Study of Radicals and Semisimple Classes of Rings " submitted by Kalyan Kumar Dey in fulfillment of the requirements for the degree of Master of Philosophy in Mathematics, University of Rajshahi, Rajshahi, has been completed under my supervision. I believe that this research work contains exposition of others' work and some original work, and it has not been submitted elsewhere for any degree.

Subrata Majumdar
(Subrata Majumdar)

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Statement of Originality

This thesis does not incorporate without acknowledgement any material previously submitted for a degree or diploma in any University, and to the best of my knowledge and belief, does not contain any material previously published or written by another person except where due reference is made in the text.

Kalyan Kumar Dey

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SYNOPSIS

The concept of the radical of a ring was introduced by Artin for rings with the descending chain condition with a view to obtaining a nice structure theorem for the ring. The idea was to single out the troublesome part, of a ring, called the radical of the ring, and factor out the original ring with respect to the radical. The resulting ring, termed, semisimple has a nice description. Radicals for rings without chain conditions were proposed by Koethe, Jacobson, Brown, McCoy, Levitzki and others for a similar purpose in an attempt to generalize Artin's radical. All these attempts were later further generalized by Kurosh and Amitsur to define the concept of a general radical of a ring and the corresponding semisimple ring and study these in their generality. Andrunakievic advanced these studies further.

The class of rings which are radicals of themselves with respect to some radical is called a *radical class*, or simply, a *radical*, and the corresponding class of the semisimple rings is called a *semisimple class*. A class of rings may be simultaneously a radical ring with respect to some radical and a semisimple ring with respect to another radical. Such a class of rings is called a *semisimple radical class*. In this thesis we have studied radical classes, semisimple classes and semisimple radical classes of rings. We have given an account of works of various mathematicians regarding these classes and have incorporated some of our own contribution in this context.

The first chapter introduces the fundamental concepts in ring theory and those in the theory of radicals. Various definitions, constructions and results have been stated, and sometimes, proved.

Different important radicals, viz., the nil radical, Baer lower radical, Levitzki radical, Jacobson radical, Brown-McCoy radical, etc. have been described in the second chapter. The concept of special radicals and their semisimple classes due to Andrunakievic, too, has been introduced and described. All of the above mentioned radicals are special.

In the third chapter the lower radical and the upper radical constructions of Kurosh have been described. Leavitt's, and independently Majumdar's, ideas of the join and the meet of two radicals have been described with the use of the lower radical construction. Also radical pairs due to Sulinsky and Divinsky and their generalized constructions due to Majumdar and Paul, have been described. An account has been given of our own study of a number of radicals arising from the join and the meet.

The fourth chapter discusses semisimple classes. Different properties of this class have been proved and various characterizations have been given.

The final chapter deals with semisimple radical classes. Characterizations of these classes due to Stewart and Majumdar have been given. Majumdar's example of a family of such classes and Stewart's complete description of such classes have been described in details.

Contents

Chapter-1
Radicals

Introduction 1.1

In this chapter we introduce the concept of radicals of rings. We start with a few basic definitions and examples in ring theory which will be essential for our development of the theory of radicals. We have given the specific definitions in order to avoid any confusion as to the significance of the terms used in the thesis, as some of these terms have also been used in the literature with meanings different form ours. Many characterizations of radicals have been stated and proved , and basic constructions connected with radicals and fundamental results in the theory of rings needed to develop the theory of radicals have been given in details so as to make the exposition self-sufficient. We mostly follow the terminology of Divinsky [11].

2. Definitions and Examples of Rings and Related concepts

Definition 1.2.1

A nonempty set R is said to be **an associative ring**, or simply, **a ring** if on R there are two binary operations, called addition (+) and multiplication (.), defined such that for each a, b and c in R

(i) $a+b = b+a$, (ii) $a+(b+c) = (a+b)+c$, (iii) $a.(b.c) = (a.b).c$, (iv) $a.(b+c)=a.b+a.c$, $(a+b).c = a.c+ b.c$, (v) there exists an element $0\in R$ such that $a+0=a$, for every $a\in R$, (vi) for every a, there exists an element $(-a)\in R$ such that $a+(-a) =0$.

We shall write ab for a.b.

Definition 1.2.2

A ring R is said to be **commutative** if $ab = ba$ for each $a,b\in R$.

Definition 1.2.3

If there exists an element 1 in a ring R such that $1.a = a$ for each $a \in R$ then R is called **a ring with a unity element**. Such an element 1 when it exists is unique and is called the **unity element** of R.

Examples 1.2.4

1. The set **Z** of all integers, Q, the set of all rational numbers, R, the set of all real numbers, C, the set of all complex numbers are commutative ring with unity element under usual addition (+) and multiplication (.) of numbers.
2. The set R[x] of all polynomials in x with real coefficients is a commutative ring with unity under the usual addition and multiplication of polynomials.
3. The set 2Z of all even integers is a commutative ring without a unity element under the usual addition and multiplication of integers.
4. The set $M^{n\times n}(R)$ of all $n\times n$ matrices with real entries is a non-commutative ring with unity, for each $n\geq 2$ under the usual matrix addition and multiplication.
5. The set $M^{n\times n}(2Z)$ of all $n\times n$ matrices with entries in 2Z i.e. with even integers entries is a non-commutative ring without unity element under the usual matrix addition and multiplication.
6. The set Z_n of residue classes of the integers modulo n, is a ring under the usual addition and the multiplication of residue classes.

Definition 1.2.5

A non empty subset S of a ring R called a **subring** of R if S is itself a ring under the operations of R.

Examples 1.2.6

1. The set 2Z of all even integers is a subring of Z.
2. Z[x] is a subring of R[x]. Here Z[x] is the set of all polynomials in x with integer coefficients.

Definition 1.2.7

A non-empty subset I of a ring R is said to be a **left (resp. right) ideal** of R if (i) I is a subgroup of R under addition and (ii) for every $i \in I$ and $r \in R$, ri (resp. ir) $\in I$.

A non-empty subset I of a ring R which is both a left and a right ideal of R is called a two sided ideal or simply, an **ideal** of R.

Examples 1.2.8

1. The set $M^{n \times n}(R)$ of all $n \times n$ matrices (a_{ij}) with real entries for which $a_{ij} = 0$, $j \geq 2$ is a left ideal but not a right ideal of $M^{n \times n}(R)$.
2. 2Z is an ideal of Z.
3. The set $P_0[x]$ of all polynomials in $R[x]$ with zero constant term is an ideal of $R[x]$.

Definition 1.2.9

Let I and J be two ideals of a ring R. The **union** or the **sum** I+J is the set of all $i+j$ where i is in I and j is in J and the product I.J or IJ is the set of all finite sums $\sum i_m j_m$ where i_m is in I and j_m is in J. In particular $I^2 = I.I$. Both I+J and IJ are ideals of R. If C is a non-empty collection of ideals of a ring A, then the union or the sum of the ideals in C written by $\sum I$ is defined as $\sum I = \{i_1+i_2+ \dots +i_r \mid i_1 \in I_1, i_2 \in I_r, i_n \in I_n, I_1, \dots I_r \in C, r \in N\}$. The above sum is again an ideal of A.

Definition 1.2.10

Let R be a ring and A, B two subrings of R. When R=A+B with $A \cap B = 0$ or equivalently, 0 is uniquely expressible as $0=0+0$, then R is the supplementary sum of subrings A and B. Similarly R is the supplementary sum of the subrings $A_1, A_2 \dots A_m$ if $R=A_1+A_2+ \dots +A_m$ and 0 is uniquely expressible as $0=0+0+ \dots +0$.

A ring R is a direct sum of $A_1, A_2, A_3, \dots A_m$ if it is a supplementary sum and if each A_i is an ideal of R and we write this as $R=A_1 \oplus A_2 \oplus \dots \oplus A_m$.

Let $\{A_i\}$ $i \in I$ be a class of rings. Consider $S = \{\{a_i\}_{i \in I} \mid a_i \in A_i\}$. These are infinite vectors. Addition and multiplication are defined coordinatewise as

$$(a_1, a_2, \ldots)+(b_1, b_2, \ldots) = (a_1+b_1, a_2+b_2, \ldots\ldots)$$

$$(a_1, a_2, \ldots)(b_1, b_2, \ldots) = (a_1b_1, a_2b_2, \ldots).$$

Then S is a ring and called the **complete direct sum** of A_i's.

Now, consider

$A^*_i = \{(0, 0, \ldots, a_j, 0\ldots)\}$.It is a subring, which is isomorphic to A_i and the mapping.

$(a_1, a_2, \ldots a_j \ldots) \to (0,0, \ldots a_i, \ldots)$ is a homomorphism of S onto A^*_i.

Definition 1.2.11

The subring which consists of all those elements of S which have only a finite number of nonzero entries is called **the weak direct sum** or **discrete direct sum** of the A_i's.

Definition 1.2.12

A subring S* of the complete direct sum S is a **subdirect sum** of the rings A_i if the natural homomorphism ϕ_i of S* to A^*_i, $(a_1, a_2, \ldots, a_i, \ldots) \to (0,0, \ldots, a_i, 0 \ldots)$ is an onto mapping for every i.

Moreover, if T is a subdirect sum of the rings A_i then for every i, $A^*_i \cong T/T_i$ where T_i is the kernel of the natural homomorphisms of T to A_i. Also $\cap T_i = 0$;for if $x=(a_1, a_2 \ldots)$ has the property $a_i = 0$, the zero element of A_i, for every i, then $x = (0, 0\ldots)$ the zero element of T. Thus if T has a set of ideals $\{T_i\}$ such that $\cap T_i = 0$ and $T/T_i \cong A_i$.

Let A be any ring and let $\{A_i\}$ $i \in I$ be the class of ideals of A such that $\cap Ai = 0$. Define $B_i = A/A_i$ and consider the complete direct sum S of A_i. Now for every element $x \in A$, we may associate the element $(b_1, b_2 \ldots)$ of S where b_i is the element $x+A_i$ of B_i that is determined by x. Then A is isomorphic to a subring $\overline{A}$ of S and A/A_i is isomorphic to B_i. Thus the natural homomorphism from A to B_i is onto, and so, A is isomorphic to a subdirect sum of the rings B_i.

Thus we have

Theorem 1.2.13

A ring A is isomorphic to a subdirect sum of a collection of rings $\{B_i\}$ if there is a collection $\{A_i\}$ of ideals of A such that (i)$\bigcap_i A_i = \{0\}$, (ii)$A/A_i \cong B_i$, for each i.

Definition 1.2.14

A ring R is called **simple** if $R \neq 0$, its only ideals are itself and zero.

Definition 1.2.15

A ring R is said to be **without zero-divisor** if the product of no two nonzero elements of R is zero i.e. if ab=0 implies a=0 or b=0.

Definition 1.2.16

A commutative ring R with unity is called an **integral domain** if the cancellation laws hold i.e.ab=ac implies b=c for each $a,b,c \in R$ .In other words ,a commutative ring R with unity is called an integral domain if R is without zero divisors.

Definition 1.2.17

A ring R with at least two elements is called a **field** if its non-zero elements form an abelian group under multiplication.

Examples 1.2.18

1. Z is an integral domain .
2. Z_6 is not an integral domain since it has zero divisors i.e. $\bar{2} \neq \bar{0}$, $\bar{3} \neq \bar{0}$ but $\bar{2}.\bar{3} = \bar{6} = \bar{0}$.
3. Q , R, C are fields and also Z_p is a field for each prime p. Z_n is not a field where n is not a prime.

Definition 1.2.19

A ring R with unity element and with at least two elements is called a **division ring** or **skew field** if its nonzero elements form a multiplicative group.

Definition 1.2.20

A mapping ϕ from a ring R into a ring R′ is called a **ring homomorphism** if (i) $\phi(a+b)=\phi(a)+\phi(b)$

(ii) $\phi(ab)=\phi(a)\phi(b)$ for each $a,b\in R$.

A homomorphism ϕ from a ring R into a ring R′ is called an **ring isomorphism** if it is an one-one and onto mapping.

Two rings are said to be isomorphic if there exists an isomorphism of one onto the other.

Examples 1.2.21

1. Z_p is a simple ring where p is a prime integer.
2. $M^{n\times n}(R)$ is a simple ring.

   In particular, every field is a simple ring.

3. Radicals

Definition 1.3.1

Let R be a nonempty class of rings. Any ring A in R is called an R-ring. An ideal I of A which is in R, is called an **R-ideal**. (A ring which does not contain any nonzero R-ideals, will be called a **R-semisimple ring**).

Definition 1.3.2

A nonempty class R of rings is called a **radical** if

(i) R is homomorphically closed i.e. if A is in R and I is an ideal of A, then A/I is in R,

(ii) Every ring A in R contains an R-ideal R(A) of A which contains every other R-ideals of A.

(iii) For every ring A in R, the factor ring $\frac{A}{R(A)}$ contains no R-ideal i.e. $R(\frac{A}{R(A)})=0$.

Here R(A) is called the **R-radical** of A. If R(A)=A, A is called an **R-radical ring** and if R(A) =0, A is called an **R-semisimple ring**.

A radical property or a radical class or simply, a radical in the category of rings was first defined independently by A.G. Kurosh [33] and S. A. Amitsur [15] .The above definition is due to Kurosh [33] . Later radical has also been described and characterized by Divinisky [11], Weigandts [13], Leavitt [29], Majumdar (unpublished)). We shall give these characterizations below and prove their equivalence. We shall thus establish that radical is the common concept described by them.

The following theorem gives a characterization of radicals due to Kurosh ([33]).

Theorem 1.3.3 (Divinsky [11])

Let C be a non empty class of rings in W then C is a radical if and only if the following conditions are satisfied ;

(i) C is closed under homomorphism

and (iv) if every non zero homomorphic image of a ring A contains a non-zero C-ideal, then $A \in C$.

Proof ;

First suppose that C is a radical. We have only to prove the condition (iv).Let A be a ring such that every non zero homomorphic image of A contains a non zero C-ideal .If A=C(A),then $A \in C$. If $A \neq C(A)$ then $\frac{A}{C(A)}$ is a non zero homomorphic image of A and $\frac{A}{C(A)}$ contains a nonzero C-ideal. But this contradicts the condition (ii) of definition 1.3.2. Hence A=C(A) and A is in C.

Conversely suppose that C satisfies the conditions (i) and (iv). Let A be a ring and let B be the union of all C-ideals of A.

Let B be a non zero homomorphic image of B. Then B =B/ I where I is an ideal of B and B≠I. If B=0. B is in C. So let B≠0. Then there exists of least one C-ideal, say I_0 such that $I_0 \not\subset I$. Then $\frac{I_0+I}{I}$ is a non zero ideal of B/I. Since $\frac{Io+I}{I} \cong \frac{Io}{I \cap Io}$ The left hand side of this isomorphism is a non- zero ideal of B/I while the right hand side is a homomorphic image of the C-ideal I_0 and is therefore an C-ring by condition (i). Hence $\frac{Io+I}{I}$ is in C. Therefore every homomorphic image of B contains a nonzero C-ideal. Hence by the condition (iv), B is in C. Thus the condition (ii) of definition 1.3.2 holds with C(A) = B.

If A/B has a nonzero C-ideal let $\bar{I}$ be an ideal such that an C-ideal I of A such that I⊇B and I≠B. Let K be an ideal of with K≠I. If B⊇K then I/K≅$\frac{I/B}{K/B}$. Since I/B is in C, then by the condition (iv) I/K is in C. If B⊇K. then $\frac{K+B}{K}$ is a non-zero ideal of I/K, and $\frac{B+K}{K} \cong \frac{B}{B \cap K}$. Since B is in C, by the condition (i) $\frac{B+K}{K}$ is in C. Thus every non zero homomorphic image of I contains a non-zero C-ideal. Hence by the condition (iv), I in C. This contradicts the fact I≠B. Hence $\frac{A}{B}$ has no nonzero C-ideal. Thus the condition (iii) of definition 1.3.2 holds.

We next state and prove a characterization of radicals due to Amitsur([15]). The proof given here is due to Majumdar ([24]).

Theorem 1.3.4 (Amitsur [15])

Let C be a nonempty class of rings in W, then C is a radical if and only if the following conditions satisfy

(i) C is closed under homomorphisms

(v) C is closed under extensions i.e. for a ring A and an ideal Iof A, both I and A/I are in C, then A is in C.

(vi) If $I_1 \subseteq I_2 \subseteq I_3 \subseteq$ ....is an ascending chain of C-ideals of a ring A, then $\bigcup_{\alpha} I_{\alpha}$ is in C.

Proof (Majumdar [21])

First suppose that C is a radical . Then we need only to show that the conditions (v) and (vi) hold. Now let A be a ring and I be an ideal of A such that both I and A/I are in C. Let C(A) be the C-ideal of A. Then I $\subseteq$ (A). Hence $\frac{A}{C(A)}$ is a homomorphic image of $\frac{A}{I}$, for $\frac{A/I}{C(A)/I}$ ≅A/C(A) . Hence A/C(A) is in C. By the condition (iii) of definition ( ) $\frac{A}{C(A)}$ =0 i.e. A=C(A). Hence A is in C.

By the condition (iii) of definition ( ) A/C(A)=0 i.e. A=C(A). Hence A is in C. Thus condition (v) holds.

Now Let A be a ring and let $I_1 \subseteq I_2 \subseteq I_3 \subseteq$ ... be an ascending chain of C-ideals of A. Let I = UI_{α}. Let C(I) be the C-radical of I. If I ≠C(I), there exists an ideal I_{α} such that $I_{\alpha} \subseteq$ C (I). Then I_{α}+C(I)/C(I) is a nonzero ideal of I/C(I) and since Iα+C(I)/C(I) ≅ $I_{\alpha}/I_{\alpha} \cap$C(I). The right hand side is a homomorphic image of I_{α} and is in C. Therefore I_{α}+C(I)/C(I) is a nonzero C-ideal of I/C(I). This contradicts the condition (iii) of definition 1.3.2. Hence I = C(I), and so I is in C. Thus the condition (vi) holds.

We now prove the converse .The first condition is same. Now let R be the class of all C-ideals of a ring A. Then R is nonempty since 0∈R. R is partially ordered with respect to inclusion ($\subseteq$) and by the condition (iv) every totally ordered subset of R has a least upper bound in R. Hence by Zorn's lemma R has a maximal member, say M then M is in C. Let I be any C-ideal of

A. Then I+M/M $\cong$ I/I$\cap$M. Since M is in C and I/I$\cap$M is in C. Then the maximality of M implies that I+M$\subseteq$M i.e. I$\subseteq$M. Then the condition (ii) of definition 1.2.1 holds Let I is a C-ideal of A/M. Then I be an C-ideal of A/M. Then I = I/M for some ideal I of A such that I$\geq$M. Since I is in C and M is in C, the condition (v) implies that I is in C. Hence I$\subseteq$M. Then I=0. Hence the condition (iii) of definition 1.3.2 holds.

Theorem 1.3.5 (Majumdar)

Let C be a non empty class of rings in W, then C is a radical if and only if the following conditions are satisfied

(i) C is closed under homomorphism (ii) for each ring A, there exists a C-ideal C(A) of A such that for every C-ideal I of A then I$\subseteq$C(A). (v) if a ring A has an C-ideal I such that A/I is a C-ideal, then A is in C.

Proof:

Let C be a radical. Then the condition (i) are same. Let A be a ring which has a C-ideal I such that A/I is a C-ring. We consider C to be the lower radical L_c determined by C. If I = A, then A is a C-ring. So Let I$\neq$A, Let J be any ideal of A. If J $\geq$ 1 then A/J is a homomorphic image of A/I., and so is a C-ring. If I$\not\subset$J then I+J / J is a nonzero ideal of A/J, and the isomorphism I+J/J $\cong$ I/I$\cap$J shows that I+J / J is a C-ring , since I is a C-ring. Thus A is of degree 2 over C. Hence A$\in L_c$=C. This implies A$\in$C so that the condition (v) holds.

Conversely, Let the conditions (i), (ii) and (v) hold. To only prove the condition (iii), Let I be a C-ideal of A/C(A). Then I= I/C(A) for some ideal I of A such that I$\supseteq$C(A). Since I and C(A) are C-rings, the condition (v) implies that I is a C-ring. Therefore I $\subseteq$ C(A). Hence I = C(A) and so I=0 thus A/C(A) has no nonzero C-ideals . Hence C is a radical.

Theorem 1.3.6 (Weigandt [13])

Let C be a nonempty class of rings in W, then C is a radical if and only if the following condition satisfy (i) C is closed under homomorphism (iii) the factor ring $A/C(A)$ has no nonzero C-ideal. (viii) the sum $C(A)$ of all C-ideals of a ring A is an C-ideal of A.

Proof

Let C be a radical. Then $C(A)$ the C-ideal of A containing each C-ideal of A, is the sum of all C-ideals of A and $C(A)$.

Hence the condition (viii) holds. Clearly the conditions (i) and (iii) are identical.

Next, let C satisfy (i), (iii) and (viii). Since $C(A)$, the sum of all C-ideals of A, is a C-ideals of A, $C(A)$ contains every C-ideal of A. Thus the condition (ii) of definition ( ) holds.

Theorem 1.3.7 (Leavitt [29])

Let C be a nonempty class of rings in W, then C is a radical if and only if the following conditions are satisfied.

(i) C is closed under homomorphism and (ix) for a ring $A \notin C$ implies some $0 \neq A/I \in S_c$ where I is an ideal of A and $S_c \setminus \{B \in W|$ of $I\ (\neq 0)$ is an ideal of B then $I \notin C$ and $C \in W\}$.

Proof

Let C be a radical. By Theorem it satisfy the condition (i) and (iv) .Let $A \notin C$. Then by the condition (iv) then there exist an ideal I of A such that $A/I \neq 0$ has no nonzero C-ideals i.e. $0 \neq A/I \in S_c$ where $S_c = \{B \notin W \mid$ if $0 \neq J$ be an ideal of B, then $J \notin C\}$.

Then the condition (ix) holds.

Next let C satisfy the condition (i) and (ix). Let $A \in W$ such that every nonzero homomorphic image of A contains a nonzero C-ideal. If $A \notin C$, then by the condition (ix), there exist an ideals I of A such that $0 \neq A/I \in S_c$ where $S_c = \{B \in W| \ 0 \neq I$ be an ideal of B, then $I \notin C\}$. this is a condition to the condition (i). Hence $A \in C$. Thus C satisfies the condition (iv). The condition (i) is the same as the condition of definition 1.3.2 . Hence the theorem proved.

We need to define a construction, called the lower radical defined by Kurosh ([33]). We have another construction due to Kurosh called the **upper radical** . We shall describe this later.

Lower Radical

Let C be a nonempty class of rings. Write C_0=C. Let C_1 to be the homomorphic closure of C i.e. C_1 to be the homomorphic images of the rings in C. For any ordinal $\beta>1$, suppose C_α has been defined for each $\alpha<\beta$. If β is not a limit ordinal, define C_β to be the class of all rings A such that every non-zero homomorphic image of A contains a non zero ideal in $C_{\beta-1}$. If β is a limit ordinal, define $C_\beta = \bigcup_\alpha C_\alpha$ .Thus C_β is defined $\alpha < \beta$ for each ordinal β. The lower radical determined by C written L_c is defined by $L_c = \bigcup_\gamma C\gamma$. The ring in C_γ are said to be of degree γ over C.

Definition 1.3.8

L_c is called the lower radical determined by the class C.

Theorem 1.3.9

L_C is a radical.

Proof

The construction of L_C clearly shows that L_C satisfy the condition (i) of Theorem 1.3.4.

Let A be a ring such that every nonzero homomorphic image of A has a nonzero ideal in L_C .

Let B be a homomorphic image of A. Then B has a non zero ideal I in L_c. So I belongs to C_α for some ordinal α not a limit ordinal. By proposition 8, (Divinsky [ ], p.10) there exists an ordinal β which is not less than any ordinal α , thus obtained. Then every nonzero ideal I which belongs to C_β. Then A belongs to $C_{\beta+1}$., and hence, to L_C. Thus L_C satisfy the condition (iv) of Theorem 1. Hence by theorem 1, L_C is a radical.

Definition 1.3.10

A subring B of a ring R is said to be accessible in R if there exists a finite set of subrings $A_1, A_2, \ldots A_{n-1}$, in R such that $B = A_0 \subseteq A_1 \subseteq A_2 \subseteq \quad \subseteq A_{n-1} \subseteq A_n = R$ where A_i is an ideal of A_{i+1} for every $i = 0, 1.2 \ldots n-1$.

Lemma 1.3.11

Let S be the lower radical determined by a homomorphically closed class M of rings. Then if $R \neq 0$ is in S, R contains a nonzero accessible M-subring B.

Proof :

Since R is in $S = \cup M_\alpha$, R is in M_α for some α. Take α to be minimal. We proceed by induction on α, If $\alpha=1$, R is in $M_1 = M$. Then B=R.

Assume that the lemma is true for rings T in M_γ for every $\gamma<\alpha$ and take R in M_α, α minimal (i.e. R is not in if $\gamma<\alpha$). By definition, every nonzero homomorphic image of R and in particular of itself (note that $R\neq 0$ also R is in M_1) has a nonzero ideal I with I in M_γ for some $\gamma<\alpha$. By induction I has a nonzero accessible subring B with B in M. Since I is an ideal of R, it is clear that B is also an accessible subring of R. This ends the induction and the proof.

We now have the following characterization of a radical due to Yu_Lee Lee([28]) .

Theorem 1.3.12

Let C be a non empty class of rings in W then C is a radical if and only if the following conditions are satisfied (i) C is closed under homomorphism (vii) if every nonzero homomorphic image of a ring A contains a nonzero accessible subring which has nonzero C-ideal then A is in C.

Proof

We shall first prove that if B is an accessible subring of A. Then C(B) is an accessible subring of C(A). Since B is an accessible subring of A, therefore there exists a finite set of subrings $A_1, A_2, A_3 \ldots A_{n-1}$ of A such that $B = A_0 \subseteq A_1$

$\subseteq \dots \subseteq A_{n-1} \subseteq A_n = A$, where A_i is an ideal of A_{i+1} for every i=0,1, 2. n-1. Now (Yu-lee-lee [28], Theorem 2) A_{n-1} is an ideal of A_n implies that C (A_{n-1}) is an C-ring, therefore $C(A_{n-1}) \subseteq C(A_n)$ and $C(A_{n-1})$ is an ideal of A_n implies that $C(A_{n-1})$ is an ideal of $C(A_n)$. Similarly $C(A_{n-2})$ is an ideal of $C(A_{n-1})$ and so on. Hence the chain $C(B) = C(A_0) \subseteq \dots \subseteq C(A_{n-1}) \subseteq C(A_n) = C(A)$ shows that C(B) is an accessible subring of C(A). Now let C be a radical. Let A be a nonzero homomorphic image of A and let A has an accessible subring K which has a nonzero C-ideal. Then there exists a finite set of subring $A_1, A_2, \dots A_{n-1}$ in A such that $K= A_0 \subseteq A_1 \subseteq \dots \subseteq A_{n-1} \subseteq A_n=A$ with $C(K) \neq 0$, Therefore $C(A) \neq 0$ and A_n contains a nonzero C-ideal .Hence by the condition (iv) of Theorem 1.3.3. A is in C. Thus C satisfy the condition (vii).

Conversely, let the condition (i) and (vii) hold. Then C is a radical.

Examples

1. The class of Boolean rings is a radical.

(i) Let R be a Boolean ring, and let I be an ideal of R. Then clearly, R/I is Boolean.

(ii) Let I_1 and I_2 be two Boolean ideals of a ring R. Let $a \in I_1$, $b \in I_2$. Then, $(ab+ba)a = (ab)a+ba^2=a(ab) + ba=a^2b + ba=ab+ba$. Therefore, $(ab+ba)ab=(ab+ba)b = ab+ba$. Similarly, $(ab+ba)ba=ab+ba$. Therefore, $(ab+ba)^2=2(ab+ba) = 0$, since $ab+ba \in A \cap B$,which is a Boolean ring .But $(ab+ba)^2=(ab+ba)$ .Hence $(ab+ba) = 0$. Therefore$(a+b)^2 =a^2+ab+ba+b^2 =a+b$ .Thus, I_1+I_2 is Boolean. It therefore follows that the sum of a finite number of Boolean ideals of R is a Boolean ideal.

Let S be the sum of all Boolean ideals of R. Let $x \in S$. Then x can be written in the form $x=a_1+a_2+\dots\dots + a_r$, where a_i, i=1, ... r, belongs to J_i, a Boolean ideal of R. Therefore, $x \in J_1+ \dots J_r$. Thus, $x^2=x$. Hence S is a Boolean ideal. Clearly, S contains every Boolean ideal of R.

(iii) Let J^* be a Boolean ideal of $S(R)$. Then, $J^*=S(J)$, where J is an ideal of R such that $J \supseteq S$. Let $x^* \in J^*$. Then $x^*=x+S$, for some $x \in J$, Since $x^{*2} = x^*$, we have $x^2+S = x+S$, i.e., $x^2-x \in S$.

Therefore $x^2-x =(x^2-x)^2 =x^4-2x^3+x^2 =2(x^2-x)x^2 =x^4+x^2=x^2-x^4$ .Hence $x= x^4$.

Since $x^3 - x^2 = (x^2-x)\, x \in S$, $x^3 - x^4 = x^3-x$. Therefore, $x^2 = x$, i.e., $x \in S$. Hence, $J=S$, i.e., $J^*=0$.

2. The class of rings with $R^2=R$ is a radical.

This is proved by the following.

(i) Let R be a ring with $R^2=R$ and let I be an ideal of R. Then

$$\left(\frac{R}{I}\right)^2 = \frac{R^2+I}{I} = \frac{R+I}{I} = \frac{R}{I}$$

(ii) Let R be a ring and let S be the sum of all ideals A of R such that $A^2=A$. We shall have to show that $S^2=S$.

Let $r \in S$. Then, $r \in A_1 + A_2 + \dots + A_t$, for some ideals $A_1, A_2, \dots, A_t$ such that $A^2_i = A_i$ $(i=1, 2, \dots, t)$. We shall show that $(A_1 + A_2 + \dots, + A_t)^2 = A_1 + A_2 + \dots A_t$, That will imply that $r \in S^2$, i.e., $S=S^2$. It will be sufficient to show that if A and B are two ideals of R with $A^2=A$ and B^2 B, then $(A+B)^2=A+B$. So, let A and B be two ideals of R with $A^2=A$ and $B^2=B$. Then, $A(A+B) \supseteq A^2+AB=A+AB=A$. Similarly, $B(A+B) \supseteq B$. Therefore, $(A+B)^2 \supseteq A\,(A+B) +B\,(A+B) \supseteq A+B$. Thus, $(A+B)^2 = A+B$.

(iii) Let A^* be an ideal of $S(R)$ such that $A^{*2} = A^*$. Then, $A^*=S(A)$, for some ideal A of R such that $A \supseteq S$. Since $A^{*2} = A^*$, we have

$$\frac{A}{S} = \frac{A^2+S}{S} \cong \xrightarrow{\Phi} \frac{A^2}{A^2 \cap S} \xrightarrow{i} \frac{A}{A^2 \cap S} \xrightarrow{j} \frac{A}{A \cap S} = \frac{A}{S}$$

where i is the inclusion map, j the canonical homomorphism and Φ the usual isomorphism. The composition map $\frac{A}{S} \xrightarrow{i} \frac{A}{S}$ given by the above sequence is the identity map. Hence i is onto and j is 1-1. Thus, $A=A^2$ i.e., $A \subseteq S$, i.e., $A=S$. Hence $A^*=0$. The proof is complete.

Chapter-2

A few important Radicals

Introduction 2.1

The most important among the radicals of rings are the Baer Lower radical, the nil radical, the Jacobson radical the Levitzki radical and the Brown-McCoy radical. In this chapter we shall describe those and other interesting radicals which are important because of their close relation with the structure theory of rings. These radicals developed by Baer, Koethe , Levitzki, and Jacobson, Brown and McCoy in the 1940's and 1950's form pillars on which the general theory of radicals due to Kurosh and Amitsur stands. Besides the above-mentioned radicals we shall describe the special radicals of Andrunakievic which generalize the above famous radicals and still retain importance by being able to shed significant light on the structures of the corresponding semisimple rings. Behrens radical and Thierrin's radical are also described. We first define a few terms and provide examples. These will be needed for description of the radicals.

Definition 2.2.1

An element e of a ring is idempotent if $e \neq 0$ and $e^2 = e$.

Example:

Let $M_n(R)$ denote the ring of all $n \times n$ matrices with real entries. If $A = (a_{ij})$ with $a_{11}=1$, $a_{ij} = 0$, $i, j \neq 1$, then $A^2 = A$, $A \neq 0$. Thus A is an idempotent element. Also I, the identity matrix is an idempotent element.

Definition : 2.2.2

An element x is said to be **nilpotent** if there exists a positive integer n such that $x^n = 0$.

Example:

If $x=\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, $a\in R$ then $x^2 = 0$. Thus x is a nilpotent element of $M_2(R)$.

0 is obviously a nilpotent element.

Definition 2.2.3

A ring R is said to be **nil** of every element if R is nilpotent i.e. $x^n = 0$, where n depends on the particular element x in R.

Examples

1. $R=\{\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}, a\in R\}$ is a nil ring.

2. Let $R=Z_{16} = \{\bar{0},\bar{1},\bar{2}....\overline{15}\}$ and $R'=\{\bar{0},\bar{4},\bar{8},\overline{12}\}$ then R′ is a nil ring , since $\bar{4}^2 = \bar{8}^2 = \overline{12}^2 = \bar{0}^2 = \bar{0}$.

Definition 2.2.4

A ring R is called **nilpotent** if there exists a positive integer n such that $R^n =0$.

Example:

Let $R =Z_8 =\{\bar{0},\bar{1},\bar{2}....\bar{7}\}$, then $R'=\{\bar{0},\bar{2},\bar{4},\bar{6}\}$ is nilpotent .

In general if $R = Z_{n_r}$,then , $R'==\{\bar{0},\bar{n},\overline{2n},\overline{n^r - n}\}$ is a nilpotent subring of R ,for each $n \geq 2$ .Since $R^n =0$ implies $x^n =0$ for each $x\in R$, every nilpotent ring is nil. However, every nil ring is not nilpotent as the following example shows :

Example :

Consider the set of symbols x_α where α is any real number, $0 < \alpha <1$. Let *F* be some field, and let *A* be the commutative algebra over *F* with these x_α as a basis. Multiplication of the basal elements is defined as

$$x_\alpha x_\beta = x_{\alpha+\beta} \quad \text{if } \alpha+\beta < 1$$
$$= 0 \quad \text{if } \alpha+\beta \geq 1$$

Thought of as a ring, *A* is the set of all finite sums $\sum a_\alpha x_\alpha$ where the a_α are elements in a field *F*. Addition is defined artificially as $a_\alpha x_\alpha + a_\beta x_\beta$ just written together, if $\alpha \neq \beta$, and if $\alpha = \beta$, then, of course, $a_\alpha x_\alpha + a'_\alpha x_\alpha = (a_\alpha + a'_\alpha)x_\alpha$. Multiplication is distributive and defined as above. The ring A is then commutative.

It is clear that every element in A is nilpotent, for if we let n be any integer $>1/\alpha$, $(x_\alpha)^n = x_{n\alpha} = 0$ and for any finite sum , the term with the smallest x-subscript will yield an integer which as a power will yield zero. Thus, A is a nil ring.

However, A is not nilpotent, for $x_{1/2}.x_{1/4}.x_{1/8}...x_{1/2^n}... \neq 0$. In fact $A^2 = A$, for given any α, there exists a β such that $x_\beta x_\beta = x_\alpha$.

Take any basal element x_α and consider the ideal (x_α) generated by it . this is a nilpotent ideal , for $(x_\alpha)^n = 0$ for any integer $n > 1/\alpha$. The union of all of the ideas (x_α) fills out all of A and , therefore , the union of the nilpotent ideal s of A is not a nilpotent ideal.

Definition 2.2.5

A ring R is **locally nilpotent** if any finite set of elements of R generates a nilpotent subring i.e., a ring is locally nilpotent if any finite set of element generates a subring which is nilpotent. Thus every nilpotent ring is locally nilpotent ring and every locally nilpotent ring is nil.

Definition 2.2.6

A right ideal V of a ring R is **regular** if there exists an element e in R such that er-r is in V for every r in R.

An element x in a ring R is right quasi-regular if there exists an element y in R such that x+y+xy =0. Thus every nilpotent element is right quasi-regular.

A ring R is **right quasi-regular** if every element in R is right quasi-regular. The union or sum of two right quasi-regular ideals is a right quasi-regular ideal.

Definition 2.2.7

An element a is **G-regular** if a is in G(a), where $G(a)=\{ar+r+\sum(x_i a y_i + x_i y_i)\}$ for each element a of a ring R, and r, x_i, y_i range over R, the summation is finite.

If G (a) = R for some a, then a is in G (a) and a is G-regular. An ideal I is G-regular if every element in I is G-regular.

Definition 2.2.8

An ideal P of a ring R is a **prime ideal** if $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ where A and B are ideals of R. We shall call a ring R a prime ring if 0 is a prime ideal of R i.e. if A and B are ideals of R such that AB = 0 then either A = 0 or B = 0.

Definition 2.2.9

Let $(M : A) = \{a \in A : Aa \subseteq M\}$ where M be a regular maximal ideal

A ring A is **right primitive** if A contains a maximal right ideal M such that (M : A) =0.

An ideal K of A is right primitive ideal if A/K is right primitive. (M:A) is a right primitive ideal of A if M is a maximal regular right ideal of A.

3. Important radicals

We are now ready to describe the most important radicals in the theory of rings. Their importance lies in the fact that these were obtained by Koethe , Jacobson , Levitzki and Brown-McCoy as generalizations to arbitrary rings of Artin's radical for rings with d.c.c. with a view to obtaining semisimple rings with nicely describable structures in terms of well-behaved classes of rings. Among these the Jacobson radical is the best in the sense that the descriptions of both the radical of a ring and that of the structure of a semi simple ring are very good, and the best balance is maintained. Artin's results on rings with d.c.c were generalizations of Wedderburn's corresponding results on finite dimensional algebras.

Let N = The class of all nil rings,

J = The class of all right quasi-regular rings,

L = The class of all locally nilpotent rings ,

G = The class of all G-regular rings,

β = The lower radical determined by all nilpotent ring,

Then each of β, N, J, L and G is a radical (for proof see [ ]). β is called the **Baer lower radical** , N the **nil radical** , J the **Jacobson radical** , L the **Levitzki radical** and G the **Brown-McCoy** radical .

As a particular example, we verify here that

Theorem 2.3.1

N is a radical.

Proof:

Let A be a nil ring in N. Therefore every element in A is nilpotent. Let I be an ideal of A. Now we have to show that A/I is nil. Let $\bar{a} \in A/I$ then $\bar{a}$ = a+I, for some $a \in A$. since A is nil, $a^n = 0$, for some positive integer n. Hence $(a+I)^n = a^n+I = 0+I = I$ .Thus $\bar{a}^n = 0$.

Let W be the sum of all nil ideals of A. If $x \in W$, then x belongs to a finite sum of nil ideals of A. Since the sum of any finite number of nilpotent elements of a ring is nilpotent , x is nilpotent . Therefore W is a nil ideal of A which contains every other nil ideal of A. Write W=N(A). Finally if possible, let $\frac{I}{N(A)}$ have a nonzero nil ideal $\bar{I}$, then $\bar{I} = \frac{I+N(A)}{N(A)}$ for an ideal I of A, $I \not\subset N(A)$. Let $x \in I$, then x = a+N (A), $a \in I$ and there exists a positive integer n such that $x^n = 0$,i.e.,$a^n \in N(A)$, Since N (A) is a nil ring a is nilpotent. Thus I is a nil ring. Thus $I \subseteq N(A)$ and so, I =N(A)/N(A) is the zero ideal of A/N(A). a contradiction to the assumption on I. Hence A/N (A) has no non zero nil ideal. Thus the conditions (i), (ii) and (iii) of the definition 1.3.2 are satisfied. Hence N is a radical.

Definition 2.3.2

Let a class M of rings is called a **special class** if it satisfies the following three conditions:

(x) Every ring in the class M is a prime ring .

(y) Every nonzero ideal of a ring in M is itself a ring in M .

(z) If A is a ring in M, and A is an ideal of a ring K, then K/A^* is in M, where A^* is the annihilator of A, i.e. , $A^*=\{x\in K: xA = Ax = 0\}$.

Definition 2.3.3

A special class M satisfies the condition (E) of Divinisky [11] so that M determines an upper radical S_M . S_M is called a **special radical** .

Andrunakievic introduced and studied special radicals .His work is an important and significant contribution to the extension of the general radical theory.

Examples of Special radicals

β , **N** , **L** , **J** , **G** are special radicals .

1. β is the upper radical determined by the class of prime rings .

2. **N** is the upper radical determined by the class of all nil semisimple prime rings.

3. **L** is the upper radical determined by the class of all L-semisimple prime rings.

4. **J** is the upper radical determined by the class of all primitive rings.

5. **G** is the upper radical determined by the class of all simple rings with unity .

There are a few more important special radicals which too are very useful and historically significant. These are briefly described here.

(i) N_g, the **generalized nil radical**, is the upper radical determined by the class of rings without zero divisors . N_g properly contains N (see [11]).The radical N_g was introduced simultaneously by Andrunakievic and Thierrin [6] .The latter called it the **compressive radical** .

(ii) **F** is the upper radical determined by the class of all fields, it is the largest special radical which reduces to the nil radical for the rings with d.c.c (see [11])

(iii) For any supernilpotent radical R, R_φ the upper radical determined by the class of all subdirectly irreducible R-semisimple rings with idempotent hearts is a special radical. If R = R_φ, them R is called *dual radical.* R_φ is a dual radical since $(R_\varphi)_\varphi$ = R_φ. However, all special radicals are not dual, nor is every dual radical special (see [11]). For example, β is not a dual radical. β_φ is called the antisimple radical.

(vii) The class of all subdirectly irreducible rings with idempotent hearts such that the heart contains an idempotent elements is a special class. This special class determined by this special class is called Beheren's radical and is denoted by J_β.

We prove here a new result: **N**, too is not dual. Thus, N_φ is a distinct special radical.

Theorem 2.3.4

N is not a dual radical .

Proof:

Let A be the ring of all rational numbers of the form $\frac{2x}{2y+1}$ where x and y are integers with (x ,2y+1) =1. A is obviously N-semisimple.

Let I be any nonzero ideal A and let n_0 be the smallest positive integer such that $2^{n_0}x/(2y+1) \in I$, for some odd integer x . Let $i = 2^{n_0}(2z+1)/(2y+1) \in I$, for some integers y and z such that (2z+1, 2y+1) =1.

We note that $(2z+1)2^{n_0} = (2y+1)i \in I$ and so, $2.2^{n_0} + 2/(2z+1).(2z+1)2^{n_0} \in I$. Since (2 , 2z+1) =1,there exist integers λ and μ such that $2\lambda+(2z+1)\mu=1$ and so $2^{n_0}=\lambda.2.\ 2^{n_0}\ +(2z+1)\ \mu.2^{n_0}\ \in I$. Therefore $2^{n_0}A\subseteq I$

Also for each $i\in I$,

$i=2^{n_0}\ u/(2v+1)$, for some integers u and v with $(u,2v+1)=1$,

$=2^{n_0-1}.2u/(2v+1)\in 2^{n_0-1}A$.

Thus $2^{n_0}A\subseteq I\subseteq 2^{n_0-1}A$.

Also for each non -negative integer n, 2^nA is an ideal of A. Therefore it follows that A is subdirectly irreducible and that every non-trivial homomorphic image of A is nil. Thus A can not be mapped homomorphically onto a subdirectly irreducible ring with a nil semisimple heart. So A is N_φ radical. Hence $N\neq N_\varphi$ and the proof is complete .

We only verify that N_g is a special radical through the theorem below

Theorem (Divinisky [11]) **2.3.5**

The class of all non-zero rings without zero divisors is a special class of rings.

Proof

Let M be he class of all non-zero rings without is also divisors. Clearly, every ring in M is prime and every ideal of a ring in M is also zero-divisor-free and thus in M. To show that M is a special class of rings, it is therefore sufficient to establish property (z) of the definition of special class .

Suppose, then, that A is a ring without zero divisors, and that A is an ideal of a ring K. Consider K/A^*, where , $A^*=\{x\in K : Ax=xA=0\}$. We wish to show that K/A^* has no zero divisors. Suppose, then, that $x.y$ is in A^* and that x is not in A^*.

If $w.A=0$, then $aw.\ aw=0$ and since aw is in A and A is zero-divisor-free, $aw=0$ for every a. Thus $A.w=0$. Similarly, if $Aw=0$, then $wA=0$. Thus, an element is in A^* if it annihilates A on any one side.

Since x is not in A^* there must exist an element b in A such that $bx \neq 0$. Then for any a in A we have $bxya = 0$, for xy is in A^*. Then $bx.ya = 0$ and since both bx and ya are in A and A is zero divisor free one of them is 0.Since $bx \neq 0$ we must have ya = 0 .This holds for any a in A, and thus $ya = 0$ and y is in A^*. Therefore, K/A^* has no zero divisors and (z) is established. The proof is complete.

4. Alternative Description of the Baer Lower Radical

We know that the sum N_o of all nilpotent ideals of ring A may not be nilpotent but it must be nil. Also $\frac{A}{N_0}$ may have nilpotent ideal. Let N_1 be the ideal of A such that $\frac{N_1}{N_0}$ is the sum of all nilpotent ideals of $\frac{A}{N_0}$. Let α be an ordinal >1. If α is not a limit ordinal, define Nα to be the ideal of A such that $\frac{N_\alpha}{N_{\alpha-1}}$ is the sum of all nilpotent ideals of $\frac{A}{N_{\alpha-1}}$. If α is a limit ordinal, define $N_\alpha = \sum_{\beta<\alpha} N_\beta$. Thus we obtain an ascending chain of ideals $N_0 \subseteq N_1 \subseteq N_2 \subseteq N_3$ .... Consider the smallest ordinal τ such that $N_\tau = N_{\tau+1}$. We denote this by B(A). Here $\frac{A}{B(A)}$ has no nonzero nilpotent ideal and B(A) is the smallest ideal which gives such a factor ring.

B(A) has the following important property

Theorem 2.4.1

B(A) is the intersection of all ideal Q_i of A such that $\frac{A}{Q_i}$ has no nonzero nilpotent ideals.

Proof:

Let $W = \bigcap Q_i$. Then, since $\frac{A}{B(A)}$ has no nonzero nilpotent ideals, $W \subseteq$ B(A) conversely, take any Q_i such $\frac{A}{Q_i}$ has on nonzero nilpotent ideals. Then

$N_0 \subseteq Q_i$, By transfinite induction assume that $N_\alpha \subseteq Q_i$ for every $\alpha<\beta$. If β is limit ordinal, then if β is not a limit ordinal then β-1 exists and $N_{\beta-1}$ exist and $N_{\beta-1} \subseteq Q_i$. If N_β is not in Q_i then some nilpotent ideal $\frac{I}{N_{\beta-1}}$ of $\frac{A}{B(A)}$ is not in $\frac{Q_i}{N_{\beta-1}}$.

Then consider some power $I_r \subseteq N_{\beta-1} \subseteq Q_i$, and therefore $\frac{A}{B(A)}$ is a nilpotent ideal. Thus $\frac{I}{B(A)}$ is a nilpotent ideal of $\frac{A}{B(A)}$. Since $I \subseteq Q_i$, this is a nonzero nilpotent ideal and this contradicts the fact that $\frac{A}{B(A)}$ has no such ideals. Therefore $N_\beta \subseteq Q_i$ and thus $B(A) \subseteq W$ and $B(A) = W$, which proves the theorem.

Definition 2.4.2

A ring A is called a **B-ring** if $A = B(A')$ for some ring A'.

Theorem 2.4.3

The class of B-rings is a radical.

We first proof the following lemma

Lemma 2.4.4

If a ring R has no nonzero nilpotent ideals and C is an ideal of r, then C has no nonzero nilpotent ideals.

Proof

Let J be a nilpotent ideal of C, with $J^n=0$. Then the ideal RJR of R is nilpotent and $(RJR)^{2n-1} = 0$. We can prove this by induction on n. If n=2, then $(RJR)^3 = RJ(RRJRR)JR$ and $RRJRR \subseteq C$; therefore $(RJR)^3 \subseteq RJCJR \subseteq RJJR = 0$. Assume than that $(RJR)^{2n-1} \subseteq RJ^nR$ and consider

$(RJR)^{2(n+1)-1} = (RJR)^{2n+1} \subseteq RJ^nR\,(RJRRJR) \subseteq RJ^nCJR \subseteq RJ^nRJR \subseteq RJ^{n+1}R$.

Thus for all n, we have $(RJR)^{2n-1} \subseteq RJ^nR$, and therefore, if $J^n=0$, RJR is a nilpotent ideal of R. Since R has no nonzero nilpotent ideals, RJR=0. Thus for any $x \in J$, RxR=0. Let X be the ideal of R generated by x. Then $X=\{ix+r_1x+xr_2 + \Sigma r_3xr_4\}$, where i is an integer and the terms Σr_3xr_4 must be 0. Then $X^3 \subseteq$

$R \times R = 0$, as a straight forward computation shows. Thus X is a nilpotent ideal of R, and therefore X=0. Thus x=0 and J=0. Therefore, C has no nonzero nilpotent ideals.

Proof of the theorem

Let A be a B-ring and assume that A/J is not a B-ring. Clearly 0 is a B-ring from the definition of B-ring. Thus we may take $A/J \neq 0$. Since A/J is not a B-ring, it is not equal to B(A′) for any ring A′. Let M/J be the B-ring of A/J. Then $M/J \neq A/J$ and $\cong$ has no nonzero nilpotent ideals, and it is not equal to 0. Thus M contains a B-ring B(R) for a ring A by the lemma [ ] and in particular, A is not equal to its own B-ring though it is the equal to B-ring for some ring A′. As such, $R = \cup N_\alpha$. Let B be the Baer lower radical of R, $B \neq R$. Then not all the N_α are in B and in particular, there must exist an ordinal α, not a limit ordinal, such that N_α B but $N_{\alpha-1} \subseteq B$. Then $B/N_{\alpha-1}$ does not contain some nilpotent ideal $I/N_{\alpha-1}$ of $R'/N_{\alpha-1}$. Consequently I B, but some power $I^r \subseteq N_{\alpha-1} \subseteq B$. Then if we consider the right hand side is nilpotent since $I^r \subseteq B \cap I$. Therefore the left hand side is a nilpotent ideal in R/B. It is nonzero for $I \subseteq B$. But R/B has not nonzero nilpotent ideals and therefore I must be in B; thus $N\alpha \subseteq B$ and therefore B=R. Thus no such M can exist; in particular, R/J must be an B-ring and the condition (i) of definition [ ] is established. Take any ring R and consider its Baer lower radical B. If is clear that every ring contains an B-ideal.

Chapter-3

A few new radicals and related constructions

Introduction 3.1

The Lower radical and the upper radical constructions of Kurosh have paved the way for introducing new radicals. Leavitt introduce the idea of join and meet of two or more radicals. Majumdar generalized the construction of β from the class of nilpotent rings. He discovered that certain properties on a class C of rings ensure that the lower radical L_c can be described in terms of C exactly in the manner in which β is described in terms of the class nilpotent rings. This led him to the concept of the join radical independently. Using these constructions , a few new radicals have been obtained and studied in this chapter .Radical pairs of Sulinsky and Divinsky [9] ,Majumdar and Paul's [17] work and our own study of a number of new radicals E+J,EJ,$L_{C_1}+L_{C_2}, L_{C_1}L_{C_2}, U_{C_1}+U_{C_2}, U_{C_1}U_{C_2}$, etc. both particular and general have been described.

Definition 3.1.2

Let R_1 and R_2 be two radicals. Then the intersection $R_1 \cap R_2$ is a radical . We call $R_1 \cap R_2$ the meet radical or the product radical and denote it by $R_1 \wedge R_2$ or by R_1R_2 . It is the largest radical containing in both R_1 and R_2. In general, the union $R_1 \cup R_2$ of two radicals is not a radical. As the smallest radical contained R_1 and R_2, construct $L_{R1 \cup R2}$ the lower radical determined by $R_1 \cup R_2$. We call $L_{R1 \cup R2}$ the join radical or the sum radical of R_1 and R_2 and denote it by $R_1 \vee R_2$ or by $R_1 + R_2$. These were first defined by W.G. Leavitt.[29]. Majumdar [24] also defined them independently.

We now describe the $R_1 \vee R_2$ radical of a ring A.

1. The $(R_1 \vee R_2)$-radical of A

Let A_1 be the sum of all ideals of A which are either R_1-ideals or R_2 -ideals or both. Let α be an ordinal $\alpha \geq 2$. If α is not a limit ordinal, define A_α as the ideal of A such that $\frac{A_\alpha}{A_{\alpha-1}}$ is the sum of all ideals of $\frac{A_\alpha}{A_{\alpha-1}}$ for which R_i $(\frac{A}{A_{\alpha-1}}) \neq 0$ at least one i=1 or 2 which are either R_1 -ideals or R_2 -ideals or both.

If α is a limit ordinal τ, define $A_\alpha = \sum_{\beta<\alpha} A_\beta$. Then $A_1 \subset A_2 \subset A_3 \ldots\ldots$, all ordinals. $A_\iota = A_{\iota+1}$ for some ordinal. Then the $R_1 \vee R_2$ radical of A is A_τ . In general, let $\{R_j\}$, $j \in J$ be a set of radical class .Let $A_1=0$ and for a given ordinal α assume that ideals A_α have been defined for all ordinals .Then define $A_\alpha = \sum_{\beta<\alpha} A_\beta$ when β is a limit ordinal. Otherwise by $\frac{A_\alpha}{A_{\alpha-1}} = R_i\,(\frac{A_\alpha}{A_{\alpha-1}})$ for some $i \in J$ (if such exists) for which $R_i(\frac{A_\alpha}{A_{\alpha-1}}) \neq 0$ .Therefore there must exists some ordinal τ such that $R_j(A/A_\tau) = 0$ for all $j \in J$.

We have the following theorem

Theorem (Leavitt [29]) 3.2.2

For a ring A, $A_\tau \subseteq R(A)$, the sum of all ideal of A which belongs to $\bigcup_{j\in J} R_j$ and if A is associative then $A_\tau = R(A)$.

This construction due to Leavitt reminds us of the β-radical of A. We verify that the required radical has indeed such a structure. We proceed after Majumdar as in [24], [25]. We define a general result and from there deduce our required structure.

We begin with a few definitions:

Definition 3.2.3

Let C be a nonempty class of rings .We say that C satisfies

(i) P_1 : If C is homomorphically closed ,

(ii) P_2 : If, for every ring A and for every ideal I of A, A has no nonzero C-ideals implies that I has no nonzero C-ideal.

Let C be a nonempty class of rings, and let A be any ring. Let A_1 be the sum of all C-ideals of A. For an ordinal $\alpha \geq 2$, not a limit ordinal, define A_α to be the ideal of A such that $\frac{A_\alpha}{A_{\alpha-1}}$ is the sum of all C-ideals of $\frac{A}{A_{\alpha-1}}$. If $\alpha \geq 2$ is a limit ordinal, define $A_\alpha = \sum_{\beta<\alpha} A_\beta$. Then, A_γ has been defined for each ordinal γ. The ascending chain $A_1 \subseteq A_2 \subseteq A_3 .... \subseteq A_\alpha \subseteq A_{\alpha+1} \subseteq ..$ must terminate . Let γ be the smallest ordinal such that $A_\gamma = A_{\gamma+1}$. We denote A_γ by $\bar{A}_c$.

We then have

Theorem 3.2.4 (Majumdar)

Let C be a non empty class of rings satisfying the condition P_1, and let A be a ring. Then $\bar{A}_c$ is equal to the intersection Q all ideals Q_i of A such that $\frac{A}{Q_i}$ has no nonzero C-ideals.

Proof:

Since $\frac{A}{Ac}$ has no nonzero C-ideals $Q \subseteq A_C$.

Now let Q_i be an ideal of A such that $\frac{A}{Q_i}$ has no nonzero C-ideal of A. Then $Q_i \supseteq A_1$, otherwise $\frac{A}{Q_i}$ would contain a nonzero ideal $\frac{I}{Q_i}$, where I is a nonzero C-ideal of A, By P_1, $\frac{I}{Q_i}$ is a nonzero C-ideal of $\frac{A}{Q_i}$ which is a contradiction .Let β be any ordinal ≥ 2. Assume that $A_\alpha \subseteq Q_i$, for each $\alpha < \beta$. If β is limit ordinal, then $A_\beta = \sum_{\alpha<\beta} A_\alpha \subseteq Q_i$ .If β is not a limit ordinal then $A_{\beta-1} \subseteq Q_I$ .If

$A_{\beta-1} \neq Q_i$, there exists a C-ideal of $\frac{I}{A_{\beta-1}}$ of $\frac{A}{A_{\beta-1}}$ such that $I \not\subset Q_i$. Then

$0 \neq \frac{I+Q_i}{Q_i} \cong \frac{I}{I \cap Q_i}$, a homomorphic image of $\frac{I}{A_{\beta-1}}$, since $A_{\beta-1} \subseteq I \cap Q_I$. By (P_1)

$\frac{I}{I \cap Q_i}$ is a C-ring . Thus $\frac{I+Q_i}{Q_{ii}}$ is a nonzero C-ideal of $\frac{A}{Q_i}$ which is also

contradiction. Hence $A_\beta \subseteq Q_i$ Thus $\overline{A_C} \subseteq Q_i$ .This completes the proof .

We now arrive at generalization of Baer construction of the B-radical of a ring.

Theorem 3.2.5 (Majumdar)

Let C be a nonempty class of rings satisfying conditions P_1 and P_2 . Then the class R_C of all rings A such $A = \overline{A_C}'$ for some ring A′, is a radical.

Proof :

Let $A \in R_C$ and let J be an ideal of A. We shall show that $\frac{A}{J} \in R_C$. We may assume. $J \neq A$.

Suppose, if possible, that $0 \neq \frac{A}{J} \notin R_c$. Since $\frac{A}{J} \neq \overline{(\frac{A}{J})c}$, then exists an ideal M of A, $M \neq A$, such that $\frac{M}{J} = \overline{(\frac{A}{J})c}$. By the isomorphism $\frac{A}{M} \cong \frac{A/J}{M/J} \cdot \frac{A}{M}$ does not contain any nonzero C-ideal. Hence $M \supseteq \overline{A_c}$, through $A = \overline{A'c}$, for some other ring A′. Thus, $A = \Sigma A'_\alpha$. Clearly there exists an ordinal $\beta \geq 2$ which is not a limit ordinal and is such that $Á_\beta \subseteq A_C$, but $Á_\beta \not\subset \overline{A'c}$. Then $\frac{A_C}{A'_{\beta-1}}$ does not contain some C-ideal of $\frac{A'}{A'_{\beta-1}}$,say $\frac{I}{A'_{\beta-1}}$. Hence $I \subseteq A'_\beta \subseteq A$, and $I \not\subset \overline{A'c}$. By the construction of A'_C, I be an ideal of A′, since $A \subseteq Á$, I be an ideal of A, Hence the isomorphism $\frac{A_C + I}{A_C} \cong \frac{I}{A_C \cap I}$ shows that $\frac{A_C + I}{A_C}$ is a non zero C-ideal of

$\frac{A}{A_C}, \frac{I}{A_C \cap I}$ being a homomorphic image of $\frac{I'}{A'_{\beta-1}}$, is a C-ring. This being absurd $\frac{A}{J} \subseteq R_C$. Thus R_c is homomorphically closed.

It is clear from the proof in the last paragraph that A is an R_c-ring if and only if A = $\overline{A'_C}$. Hence a ring A has no nonzero R_c-ideals if and only if A has no nonzero C-ideals.

Hence for any ring A, the R_c-ideal $\overline{A'c}$ contains all R_c-ideals of A with I $\not\subset \overline{A'c}$, then $0 \neq \frac{A_C + I}{A_C} \cong \frac{I}{A_C \cap I}$ is a nonzero R_c-ideal of $\frac{A}{Ac}$ which is a contradiction. For the same reason, $\frac{A}{Ac}$ has no nonzero R_c-ideals.

As consequence we get the following result.

Theorem 3.2.6 (Majumdar)

If C is a nonempty class of rings satisfying conditions P_1 and P_2 and if L_c is the lower radical of Kurosh determined by C, then $L_c = R_C$.

Proof:

It is clear from the constructions of $\overline{A_C}$ and L_C that $\overline{A_C} \subseteq L_c(A)$, R_C being a radical containing C, $L_C = R_C$ and so, $L_C(A)$ Hence $L_C = R_C$.

If R_1 and R_2 are two radicals $R_1 \cup R_2$ satisfies condition P_1 and P_2. Here $R_{R_1 \cup R_2}$ = The class of all rings A such that A = A′ (see).

Therefore we immediately have from Theorem 3.2.5 & Theorem 3.2.6

Corollary 3.2.7

$L_{R_1 \cup R_2} = R_{R_1 \cup R_2}$

Corollary 3.2.8

A_ι is the $(R_1 \vee R_2)$ - radical of A.

The $(R_1 \wedge R_2)$-radical of A

We shall now describe the $(R_1 \wedge R_2)$- radical of a ring A.

Let A be a ring. Define A_1=A. Let $\alpha \geq 2$ be an ordinal . Define $A_\alpha = \bigcap_{\beta<\alpha} A_\beta$ if α is a limit ordinal and $A_\alpha = R_i(A_{\alpha-1})$ for some i (if such exists) for which $R_i(A_{\alpha-1}) \neq A_{\alpha-1}$.

As particular cases of C_1 we may have C_1=

(i) R_1, a radical class,(ii)N, the class of all nilpotent rings(ii)I, the class of all rings every ideal of which is idempotent,(iv), the class of rings A for which every ideal of every ideal of A is an ideal of A .

Also as particular cases of C_2, We may have C_2 = (i) R_2, a hereditary radical, (ii) N, (iii) I, or (iv) H.

We verify that I and H are hereditary.

Let A$\in$I. Let I be an ideal of A and J an ideal of I. Let J be the ideal of A generated by J. Then, $J^3 \subseteq$ J. Since $J^2 = J^1 = J^3 \subseteq J$, i.e., J^3 = J. Hence J^2 =J. Thus I$\in$J.

Next let A $\in$ H and let I be an ideal of A. Let J be an ideal of I and K and ideal of J, Since A$\in$H J is an ideal of K and so K is an ideal of A, and hence, in ideal of I.I, L radical rings

(α) If A is a nonzero C_1-radical ring. Then by lemma 3 of Sulinsky ,Anderson & Divinsky[9] either (i) A has a nonzero accessible C_1-subring or (ii) there exists a nonzero accessible subring of A which can not be mapped homomorphically onto a nonzero C_1 -ring.

It is clear that L -rings. and hence, in particular, C_1-rings and C_2-rings are L_{C1}-rings.

We now consider the following particular situations. The result have better and still better forms as we pass to special and more special situations.

(1) Let R_1 be a radical and R_2 a hereditary radical. We consider $R_1 \cap R_2$-radical rings

A is $R_1 \vee R_2$-radical ring either (i) A has a nonzero accessible R_1-subring or (ii) there exists a nonzero accessible subring of A which can not be mapped onto a nonzero R_2- ring.

(2) Let R be a hereditary radical

<u>$R \vee U_R$-radical rings</u>

A is an (<u>$R \vee U_R$</u>)-radical ring either (i) has a non-zero accessible R-subring or (ii) there exists an accessible subring of A which can not be mapped homomorphically onto a nonzero R-rings.

The (<u>$R \vee U_R$</u>)-radical rings

(1) If a ring A is R-radical then (i) A has a nonzero accessible C_1-subring and (ii) A can not be mapped homomorphically onto a nonzero C_2-ring.

Since if A is (<u>$R \vee U_R$</u>)-radical, then A is both L_{C1} radical and R-radical.

<u>(2) ($R \vee U_R$)-radical rings.</u>

A ring A is (<u>$R \vee U_R$</u>)-radical if and only if (I) A is an R_1-ring and (ii) A can not be mapped homomorphically onto a non-zero R-ring.

We now consider the more special situations where $C_1=C_2=R=$ a hereditary radical.

(1) (<u>$R \vee U_R$</u>)-radical rings

If a ring A is (<u>$R \vee U_R$</u>)-radical, then either (i) A has a nonzero a accessible R-subring or (ii) there exists a non-zero assemble subring of A which can not be mapped onto a non-zero R-ring .

3. Let C be a nonempty homomorphically closed class satisfying condition (E) of Divinsky [11].

We shall study $L_C \vee U_C$ and $L_C \wedge U_C$.

The radical rings and the semisimple rings for these radicals are described below.

The radical rings

Theorem 3.3.1

The radical rings for $L_C \wedge U_C$ are precisely those rings A for which

(i) $\frac{A}{I} \notin C$ for each $I \lhd A$, $I \neq A$,

(ii) A has a nonzero accessible subring in C

The proof follows from the definition of U_C and theorem 2.1.

The radical rings for $L_C \vee U_C$ are precisely those rings A such that A has a nonzero accessible subring B in $L_C \cup U_C$. Thus, either B can not be mapped homomorphically onto a nonzero ring in C, or B has a non-zero accessible subring in C.

The proof follows from the definitions of U_C, $\vee$ and Theorem 2.2, since $L_C \cup U_C$ is homomorphically closed.

(V) The semisimple rings

The semisimple rings for $L_C \wedge U_C$ are precisely those rings A which are either L_C-semisimple or U_C-semisimple i.e., those rings A for which either

(i) every non-zero ideal of A can be homomorphically onto a non-zero ring in C

(ii) A has no non-zero accessible subring in C.

The semisimple rings for $L_C \wedge U_C$ are precisely those rings A which are both L_C-semisimple and U_C-semisimple i.e., which are such that

(i) every non-zero ideal of A can be mapped homomorphically onto a non-zero ring in C and

(ii) A has no non-zero accessible rings in C .

The proofs follow from the definitions of the terms and theorem 3.1.of [19].

Partition of simple rings

Each radical partitions simple rings into two disjoint classes: the upper class consisting of all simple semisimple rings and the lower class consisting all simple radical rings.

Upper class

For $L_C \wedge U_C$ the upper class consists of all simple rings A which are either in C or not in C, i.e., the upper class consists of all simple rings. Since a simple ring has no nonzero ideals except itself and no non-zero homomorphic image except rings isomorphic to itself , the statements follows.

The lower class is thus empty in this case.

Let α and β be two radicals. Consider the following class of rings:-

I. $(\alpha: \beta)$ = The class of all rings R such that for each ideal A of R, $\alpha(R/A) \geq \beta(R/A)$.

II. $(\alpha;\beta)$ = The class of all rings R such that for each ideal A of R and for each ideal B of A, $\alpha(A/B)\geq\beta(A/B)$.

III. $(\alpha||\beta)$ = The class of all rings R such that for each ideal A of R $\alpha(R|A) = \beta(R|A)$.

IV. $(\alpha \oplus \beta)$ = The class of all rings R such that for each ideal A of R, and for each ideal B of A, $\alpha(A/B)=\beta(A/B)$.

Snider (1972) has studied $(\alpha:\beta)$. He has give some sufficient condition of which $(\alpha:\beta)$ is a radical and has provided with a different characterization of $(\alpha:\beta)$ in this situation. Divinsky and Sulinski [9] have made a deeper study of $(\alpha:\beta)$ and have given a set of sufficient condition for which $(\alpha:\beta)$ is hereditary and a number of characterizations of $(\alpha:\beta)$ under a different set of conditions.

With the same motivation at that of Snider, Divinsky and Sulinskiy [9] viz search for now radical, we study the other classes i.e., the class II, III and IV. In this paper we have obtained conditions for which these classes are (i) radicals and (ii) hereditary class, We have also established the analogues of the characterization-theorems of Divinsky and Sulinsky [9] for these classes.

If α is the Baer lower radical, then $(\alpha:\beta) = (\alpha;\beta) = (\alpha||\beta) = (\alpha\oplus\beta)$ represents the class of Jacobson rings or the class of Brown-McCoy rings according as β is the Jacobson ring and Brown-McCoy rings are interesting classes and have been studied by Kaplansky (1970) and in Processi [4], Watters [7],[8].s.

We shall use the terminology of (Divinsky and Sulinsky 1977) and (Snider 1972).

It is easily seen that the classes I-IV are all homomorphically closed. If 1 represents the radical for which all rings are radical and 0 the radical for which the only radical ring is 0, then, evidently

(i) $\beta \leq \alpha \Rightarrow (\alpha:\beta) = (\alpha;\beta) = 1$,

(ii) $(\alpha: 0) = (\alpha; 0) = 1$,

(iii) $\alpha = \beta \Rightarrow (\alpha|| \beta) = (\alpha\oplus\beta) = 1$,

(iv) $(\alpha||\beta) \leq (\alpha: \beta)$, $(\alpha ; \beta) \leq (\alpha : \beta)$, $(\alpha\oplus\beta) \leq (\alpha || \beta)$ and $(\alpha\oplus\beta) \leq (\alpha;\beta)$.

The examples 1,2,3 given below show that the classes I-IV are all distinct.

Example

Let α be the nil radical. Then $Z \notin (\alpha|| 0)$. Since $(\alpha : 0) = 1$, this implies that $(\alpha || 0) \neq (\alpha: 0)$. Similarly, $(\alpha || 0) \neq (\alpha; 0)$.

Example

Let α be the torsion radical, i.e., $\alpha(R)$ = the additive torsion subgroup of R. Let R be a two-dimensional algebra over the field of rational numbers Q with basis I and a and. let the multiplication be defined by the table.

	1	a
1	1	a
a	a	0

Then R is an associative ring and Ra is the only proper ideal of R. Let $I = Za$, then I is an ideal of Ra with $I \neq Ra$. Then, $\alpha(Ra/I) = Ra/I$, and so, R is not contained in $(0;\alpha)$. But, since $\alpha(R) = 0$ and $\alpha(R/Ra) = 0$, R is contained in $(0 : \alpha)$. Hence $(0 : \alpha) \neq (0 ; \alpha)$.

The same example clearly shows that R is contained in $(\alpha||0)$ but not contained in $(\alpha \oplus 0)$. Therefore, $(\alpha||0) \neq (\alpha \oplus 0)$

Example

We now consider Sasiada's example of a simple non-trivial Jacobson radical ring. Let x and y be two non-commutative S be the set of all elements of A with Zero constant term. Then S is the Jacobson radical of A. It can be shown that x is not contained in the ideal generated by $x+yx^2y$. By Zorn's lemma, there is an ideal A of A which is maximal with respect to inclusion of $x+ yx^2y$ and exclusion of x. Then S/M is subdirectly irreducible since $x+M$ belongs to every non-zero ideal S/M . Let J be the heart of S/M .Then J is a simple non-trivial Jacobson radical ring.

Now, let β = Baer lower radical and α = Jacobson radical. Then, $\beta(J) \leq \alpha(J)$. But $\beta(J) \neq \alpha(J)$.Hence $J \in (\alpha;\beta)$.But $J \notin (\alpha \oplus \beta)$. Thus $(\alpha \oplus \beta) \neq (\alpha;\beta)$. Also $J \not\subset (\alpha \,||\, \beta)$, and so, $(\alpha \,||\, \beta) \neq (\alpha;\beta)$.

We note that if α is hereditary, then $\alpha \leq (\alpha;\beta)$. The following example shows that if α is not hereditary than α may not be contained in $(\alpha;\beta)$.

Example

Let R be an algebra over the field $F=Z_2$ with a basis {1, a, b} where the multiplication is defined by the table

	1	a	b
1	1	a	b
a	a	a	b
b	b	b	a

Then R is an associative ring and $a = F\ (a + b)$ is an ideal of R.

Let α be the radical for which a ring S is radical if and only if $S = S^2$. Then α is not hereditary, since Z∈α, but 2Z ∉ α. Let β be the Baer lower radical. Then, α (A) = A, so that R is not in (α;β). However, R is in α.

The radical α of this example is identical with the with the radical y for which a ring S is radical if and only if S cannot be mapped homomorphically onto a non-zero nilpotent ring (Majumdar 1977). For, if S is an α.-ring then every non-zero homomorphic image is again an α.-ring and so cannot be nilpotent, and if S is not an ∝-ring, then S/S^2 is a non-zero nilpotent homomorphic image of S.

The radical y has been used by Anderson, Divinsky, and Sulinsky [9] and the radical α (in case of na-rings) by Armerdariz and Leavitt (1967) for construction of counter examples.

1. We now give certain conditions on α and β for which some of these classes *I-IV* coincide with one another.

Theorem .3.3.1

If α is supernilpotent and β hereditary, then (α : β) = (α ; β).

Proof.

It is sufficient to prove that (α: β) ≤ (α; β). Let R be in (α ; β). Let A be an ideal of R, B an ideal of A, B* the ideal of R generated by B. Then, $B^{*3} \subseteq B$, by lemma 61 of Divinsky [11]due to Andrunakievic (Andruakievic 1958). Thus B*/B is nilpotent. hence B*/B is in α. Let β(A/B)=I/B. Then B*/B

Similar arguments establish

(A) If α and β are supernilpotent then $(\alpha\beta) = (\alpha \oplus \beta)$. Also, we have evidently.

(B) If $\alpha \leq \beta$, then $(\alpha \quad \beta) = (\alpha{:}\ \beta)$ and $(\alpha \oplus \beta \quad) = (\alpha;\ \beta)$.

2. Divinsky and Sulinski [9] have shown that although $(0 : \alpha) = (0 \oplus \alpha) = (\alpha \oplus 0)$ is closed under extensions it need not in general be a radical. We similarly prove.

Theorem 3.3.2

$(0\ ;\ \alpha) = (0 \oplus \alpha) = (\alpha \oplus 0)$ *is closed under extensions.*

Proof.

Let R be a ring and I an ideal of R such that both I and R/I are in (0; α). Let A be an ideal of R and B an ideal of A. However (0 ; α) = (0 : α) = (α ⊕ 0) is not in general a radical as is shown by the following example.

Example

Let α be the radical for which a ring R is radical if and only if $R2 = R$, and let R be the algebra over the field of real numbers, with a basis $\{x_\alpha : 0 \ <1\}$ and multiplication defined by

$x_\alpha x_\beta = x_{\alpha+\beta}$; if $\alpha+\beta<1$,

$= 0$, otherwise.

Then R is an associative ring $< x_\alpha>$, the ideal generated by x_α, is a nilpotent ideal of R and R is the union of these ideals. Each $<x_\alpha>$ is in $(0\ ;\ \alpha)$, but R is not in $(0\ ;\ \alpha)$ since R is in α. Hence $(0\ ;\ \alpha) = (0 \oplus \alpha) = (\alpha \oplus 0)$ is not a radical.

The following theorem gives a set of sufficient conditions for which the classes II-IV are radicals.

Theorem 3.3.3

.If α and β are hereditary then $(\alpha;\beta)$, $(\alpha \parallel \beta)$ and $(\alpha \oplus \beta)$ are radicals.

Suppose R is a radical and A an ideal of a ring R. The *radical closure of* A, written $\sqrt[\alpha]{A}$ is defined as the ideal of A given by $\sqrt[\alpha]{A}/A$

An ideal A of a ring R is called an **ideal** if $\alpha(R/A) = 0$.

A radical is said to be *superior* if any α-semisimple ring can be expressed as a subdirect sum of subdirectly irreducible α-semisimple rings.

A class M of rings is called *regular* if every non-zero ideal of a ring in M can be mapped homomorphically onto a non-zero ring in M.

Lemma . (Divinsky and Suliniski [9]) **3.3.4**

If α is superior and β is hereditary, then the class M of all α-semisimple subdirectly irreducible rings with β-radical hearts is regular.

Lemma 3.3.5

If α is superior, β hereditary, A an ideal of a ring R and B an ideal of A, then if * B can be represented as an intersection of α-ideals I_k of A such that A/I_k is subdirectly irreducible withβ-semisimple hearts, then R is in $(\alpha;\beta)$).

Theorem 3.3.6

H is a radical.

Theorem 3.3.7

Let M be the class of subdirectly irreducible α-semisimple rings with β-radical hearts and M the class of subdirectly irreducible β-semisimple rings with radical hearts. Then,

(i) if α is superior and β hereditary, then $(\alpha;\beta) = HU_M$,U_M being the upper radical determined by M;

(ii) if α and β are both superior and hereditary, then $(\alpha\ \beta) = U_M \cap U_{M'}$ and $(\alpha\ \beta) = HU_M \cap HU_{M'}$ $\oplus$

Theorem 3.3.8

If α and β are hereditary, then for any ring R,

Proof

We shall only prove that $(\alpha;\beta)(R) \cap \beta(R) \subseteq \alpha(R)$ .The other proofs are similar.

Let I be an ideal of R which is contained in $(\alpha;\beta)(R) \cap \beta(R)$. Then, I is an ideal of $(\alpha;\beta)(R)$ which is an $(\alpha;\beta)$-ring. Hence $\alpha(R) \geq \beta(R) = I \cap \beta(R) = I$. Thus, $I = \alpha(R) = I \cap \alpha(R) \subseteq \alpha(R)$.

Divinsky and Sulinisky [9] have introduced the concept of a mutagenic radical – a radical which is very far form being hereditary – to decide when $(0;\alpha)=(0| \alpha)$ is a radical. Analogously, we define a radical α to be strongly mutagenic if there exists a ring R such that (i) R is the union of an ascending chain of ideals I_k:

$$0 \subseteq I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots\ldots\ldots \subseteq R = \bigcup_k I_k$$

where each I_k is in $(0;\alpha)=(0\,;\alpha)$ and $\alpha(R) \neq 0$.

It is clear that every strongly mutagenic radical is mutagenic.

Let α be the radical of example 5 and R be the ring of the same example with the only difference that the basis of the algebra now is the set of all rationals r with $0 < r < 1$. The basis being countable, we may write it as a sequence $\{r_1, r_2, r_3, \ldots\}$. Let I_k denote the ideal of R generated by $\{ r_1, r_2, r_3, \ldots \}$. Then, $0 \subseteq I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots\ldots\ldots \subseteq R = \bigcup_k I_k$ is an ascending chain of ideals of R such that each I_k is in $(0;\alpha)$. Also we have $\alpha(R)=R$. Hence α is strongly mutagenic.

Theorem 3.3.9

$(0;\alpha)$ is a radical if and only if $(0 \oplus \alpha)$ is not strongly mutagenic.

Proof

Let $(0;\alpha)$ be a radical. If possible, Let α be strongly mutagenic. Then there exists a ring satisfying (i) and (ii). Then (i) implies that R is in $(0;\alpha)$ but (ii) contradicts this. Hence α is not strongly mutagenic.

Conversely, let α be strongly mutagenic. Then there exists a ring R satisfying (i) and (ii) Then every non-zero homomorphic image of R has a non-zero ideal in (0 ; α). For, if $\frac{R}{A} \neq 0$, then for some n, $0 \neq \frac{I_n + A}{A} \cong \frac{I_n}{I_n \cap A}$, and the latter is in (0 ;α) . However R is not in (0 ; α). Hence (0;α) is not a radical, by theorem 1 of (Divinsky [11]).

This theorem also shows that the radical α of example 5 of [18] is strongly mutagenic.

Theorem 3.3.10

If α is a weakly supernilpotent radical, then α is strongly mutagenic if and only if there exists a non-zero α-ring R such that

$$0 \subseteq J_1 \subseteq J_2 \subseteq J_3 \subseteq \dots\dots\dots \subseteq R = \bigcup_k J_k$$

where each J_k is an ideal of R and is in (0 ; α) .

Proof

The sufficiency of the condition is obvious from the definition of a strongly mutagenic radical. We only prove the necessity.

Let α be strongly mutagenic. Then there exists a ring R with an ascending chain of ideals.

$$0 \subseteq I_1 \subseteq I_2 \subseteq I_3 \subseteq \dots\dots\dots \subseteq R = \bigcup_k I_k$$

where I_k is in (0 ; α) and $\alpha(R) \neq 0$. Now, $\alpha(R) = R \cap \alpha(R) = \bigcup_k I_k \cap \alpha(R) = \bigcup_k (I_k \cap \alpha(R))$ .Write $J_k = I_k \cap \alpha(R)$ .By theorem 5, (0;α) is hereditary, and so each J_k is in (0;α), Then, $\alpha(R) = \bigcup_k J_k$, with each J_k in (0;α). Since $\alpha(R)$ is a non-zero α-ring, the proof is complete.

Corollary 3.3.11

If α is a weakly supernilpotent radical and if there does not exist an α-ring which is not the union of an ascending chain of (0 ; α) ideals then (0 ; α) is a hereditary radical.

Chapter - 4

Semi-simple Classes

Introduction 4.1

In this chapter we study semisimple classes of rings . Different interesting properties of semisimple classes and a number of their characterizations due to different authors have been given. Semisimple classes corresponding to specific interesting classes of radicals have been described and structures of semisimple rings for certain important classes of radicals have been determined . Conditions for which a semisimple class is minimal have been described .

Properties of Semisimple Class 4.2.1

We recall that a non-empty class **C** of rings is called a **semisimple class** if C is the class of all semisimple rings with respect to some radical R. We denote C by S(R).

Every semisimple class is hereditary. To show this we proceed as follows.

Lemma (Divinsky[11]) 4.2.2

Let A be an ideal of a ring R and let B be an ideal of A .Let B* be the ideal of R generated by B. Then $B^{*3} \subseteq B$.

Proof:

$$B^{*3} \subseteq AB^{*}A = A\,(B+RB+BR+RBR)A \subseteq ABA \subseteq B.$$

Theorem (Divinisky [11]) 4.2.3

If R is any radical, then for any ring A and any ideal I of A , R(I) is an ideal of A .

Proof :

If R(I) is not an ideal of A, then there exists an element x of A such that either x.R(I) or R(I).x is not contained in R(I). Assume first that $xR(I) \not\subset R(I)$. Then xR(I)+R(I) properly contains R(I). It is contained in I for I is an ideal of A and $R(I) \subseteq I$. Furthermore xR(I)+R(I) is an ideal of I , because R(I) is an ideal of I and $IxR(I) = Ix\, R(I) \subseteq I.R(I) \subseteq R(I)$.

Since I/R(I) is R-semisimple, the ideal [xR(I) + R(I)]/R(I) cannot be an R-ring. But this ideal is a homomorphic image of the R-ring R(I) and is therefore an R-ring. This contradiction proves that $xR(I) \subseteq R(I)$ for every x in A. Similarly $R(I).x \subseteq R(I)$ and R(I) is an ideal of A.

To set up the homomorphism, let y be any element in R(I) and define $\theta(y) = xy + R(I)$. Thus, θ is a mapping from R(I) to [xR(I)+R(I)]/ R(I). Clearly θ is an onto mapping, and θ preserves addition. To see that $\theta(y_1y_2) = \theta(y_1).\theta(y_2)$. We shall show that both of these are the zero coset. First, $\theta(y_1y_2) = xy_1\, y_2 + R(I)$. Since y_1 is in $R(I) \subseteq I$, $xy_1 \in I$ and thus $xy_1y_2 \in I$. $R(I) \subseteq R(I)$. Thus $\theta(y_1y_2) = 0 + R(I)$.

On the other hand,

$\theta(y_1).\theta(y_2) = [xy_1+R(I)]\,[xy_2+R(I)] = xy_1xy_2+R(I)$. However, xy_1x is in I and xy_1xy_2 is in $I.R(I) \subseteq R(I)$. Thus, $\theta(y_1).\theta(y_2) = 0+R(I) = \theta(y_1y_2)$. This proves that θ is a homomorphism, that [xR(I)+R(I)]/R(I) is an *R*-radical ring, and that $xR(I) \subseteq R(I)$.Thus the theorem is proved.

Corollary 4.2.4

An ideal of an R-semisimple ring is R-semisimple. Thus, ***a semisimple class is hereditary.***

Our next result is

Theorem (Wiegandt [13]) **4.2.5**

Every semisimple class S in closed under subdirect sum i.e. If $A_\alpha \in S$. $\alpha \in \Lambda$. then subdirect $\sum A_\alpha \in S$).

Proof :

If $0 \neq B \triangleleft \sum(A_{\alpha}: A_{\alpha} \in S) \xrightarrow{\pi_\alpha} A_{\alpha}$, then there exists an index α such that $\pi_{\alpha}(B) \neq 0$ and $\pi_{\alpha}(B) \triangleleft A_{\alpha} \in S$ .Since, if $A \in S$, then for every $0 \neq B$ A there exists a $B \rightarrow C$ such that $0 \neq C \in S$: so $\pi_{\alpha}(B)$ has a nonzero homomorphic image of B. Thus, if for a given A and every $0 \neq B \triangleleft A$ there exists a $B \rightarrow C$ such that $0 \neq C \in S$, then $A \in S$.

Theorem (Wiegandt [13]) **4.2.6**

Every semisimple class S has the coinductive property: If $I_1 \supseteq I_2 \supseteq \ldots\ldots$ is a descending chain of ideals of a ring A such that each $A/I_{\alpha} \in S$, then also $A/\cap I_{\alpha} \in S$.

Proof :

Since $A/\cap I_{\alpha} \cong \sum_{\text{subdirect}} A/I_{\alpha}$.

Lemma (Wiegandt [13]) **4.2.7**

Every semisimple class S is closed under extensions $B \in S, A/B \in S$ and $B \triangleleft A$ imply $A \in S$.

Proof :

Suppose that B, $A/B \in S$. We have $(B+US(A))/B \cong US(A)/(B \cap US(A)) \in US$ and $(B+US(A))/B \triangleleft A/B \in S$.

If $(B+US(A))/B \neq 0$, then by the condition (S_1) if $A \in S$. then for every $0 \neq B$ A there exists a $B \rightarrow C$ such that $0 \neq C \in S$. it has a nonzero homomorphic image in S which is impossible as $(B+US(A))/B \in US$. Hence $B+US(A) \subseteq B$, that $US(A) \subseteq B \in S$. A similar arguments shows that $US(A) = 0$, hence $A \in S$.

Lemma (Wiegandt [13]) **4.2.8**

If S is a semisimple class then $US(A)=(A)S$ where $(A)S=\cap(I_{\alpha} \triangleleft A; A/I_{\alpha} \in S)$.

Proof:

Since $R(A/R(A))=0$ for every ring A where R is a radical, $US(A/US(A))=0$, hence it follows $A/S(A) \in SUS = S$. Thus $(A)S \subseteq US(A)$. Since S is closed under subdirect sums we have $A/(A)S \in S$. Since $US(A)/(A)S \in US$ and $US(A)/(A)S \triangleleft A/(A)S \in S$. Hence $S(A)/(A)=0$, i.e., $S(A)=(A)S$.

For a special class M every S_M –semisimple ring is a subdirect sum of rings from M. To see this we first prove

Theorem 4.2.9

The special radical S_M of any ring K is equal to the intersection of all ideals T_α of K such that K/T_α is a ring in the special class M.

Proof:

If T_α is an ideal of K such that K/T_α is in M, then K/T_α is an S_M-semisimple ring, for all ring is M are S_M-semisimple. Therefore $S_M \subseteq \cap T_\alpha$.

On the other hand, let T be defined and $\cap T_\alpha$. If T is S_M-radical, then $T \subseteq S_M$ and $S_M = T$. However T is not S_M radical, then T can be mapped homomorphically onto a ring of M. Let I be an ideal of T such that T/I is in M. Then I is an ideal of K for $(IK+I)/I.T/I$ is the zero ideal in T/I. However T/I being in M is a prime ring and thus $IK \subseteq I$. Similarly $KI \subseteq I$ and I is an ideal of K. Thus K/I has an ideal of K/I which is in M. Thus $(K/I)/T(I)^*$ is itself in M where $T(I)^*$ is the anhilator of T(I) .

Let $Q = \{x \in K, xT \subseteq I \text{ and } Tx \subseteq I\}$. Then $K/Q \cong (K/I)\ (T/I)$. Clearly $I \in Q$. Furthermore (T/I) is the set of elements of K/I which multiply T/I, on either side into I/I. Thus $(T/I)^* \cong V/I$ where $V=\{x \in K: xT \in I \text{ and } Tx \subseteq I\}$. Thus $V=Q$ and $(T/I) \cong Q/I$. Then $(K/I)\ (T/I) \cong (K/I)\ (Q/I) \cong K/Q$

Since K/Q is in M, Q must be equal to one of T_α's. Thus $T = \cap T_\alpha \subseteq Q$ .However if $T \subseteq Q$, Then $TT \subseteq T$ and this means that T/I is nilpotent ring. Since

T/I is in M, it is a prime ring and therefore this is impossible .Thus T must be an S_M-radical ring. $T \subseteq S_M$.

Then immediately we have

Corollary 4.2.10

Every S_M-semisimple ring is a subdirect sum of rings from M.

Characterizations of Semisimple Classes

We now give a few characterizations of semisimple classes .We start with the following definitions .Let A be a ring .Let $I_1(A)$ denote the class of all ideals of A. For each $k \geq 1$, define $I_{k+1}(A) = \cup I_1(B)$. Define $I(A) = \bigcup_k I_k(A)$.

Let C be a nonempty class of rings .We consider the following conditions on C :

(α) Every nonzero ideal of a ring in C can be mapped homomorphically onto a nonzero ring in C.

(α') If $A \in C$, every nonzero ring in I(A) can be mapped homomorphically onto a non-zero ring in C .

(β) If every non zero ideal of a ring A can be mapped homomorphically onto a non-zero ring in C, then $A \in C$.

(β') If A is a ring such that every nonzero ring in I(A) can be mapped homomorphically onto a non-zero ring in C, then $A \in C$.

(β'') If every non zero subring of a ring A can be mapped homomorphically onto a non-zero ring in C, then $A \in C$.

Theorem (Majumdar[23]) **4.3.1**

A nonempty class C of rings is a semisimple class if and only if C satisfies . Since (α) or (α') together with (β)or (β') .

Proof :

Since $(\alpha) \Rightarrow (\alpha')$ and $(\beta) \Rightarrow (\beta')$ it will be sufficient to prove (i) if C is a semisimple class, then C satisfies (α') and if C satisfies $(\alpha\ )$ and (β'), then C satisfies (β).

First , let C be a semi-simple class, and let $A \in C$. Using Corollary 2 of Theorem 1 [ ] and induction, we see that $I(A) \subseteq C$. Hence C satisfies (α').

Next let C satisfy (α) and (β'). Let A be a ring such that every non-zero ideal of A can be mapped homomorphically onto a non-zero ring in C. Then, clearly, A is U_C-semi-simple, where U_C is the upper radical class determined by C. Using Corollary 2 of Theorem I [ ] and induction, we see that of $I \in I(A)$, then I is U_C-semi-simple. Therefore every non-zero ring in I (A) can be mapped homomorphically onto a nonzero ring in C. By (β'), $A \in C$. Thus C satisfies (β).

Theorem (Majumdar[23]) **4.3.2**

A nonempty strongly hereditary class C of rings is a semi-simple class if and only if C satisfies (β'').

Proof :

It will be sufficient to prove that in the situation (β'') implies (β). So, suppose C satisfies (β'') . Let A be a ring such that every non-zero ideal can be mapped homomorphically onto a non-zero ring in C. Let S be a subring of A and $\overline{S}$ be the ideal of A generated by S. Then there exists an ideal I of $\overline{S}$ such that $0 \neq S/I \in C$. since C is strongly hereditary, $\frac{S+1}{i} \in C$. Since $\overline{S}$+I is generated by $\frac{S+I}{I}, \frac{S+I}{I} \neq 0$. Thus every non-zero subring of A can be mapped homomorphically onto a non-zero ring in C. Hence (β''), $A \in C$. Therefore C satisfies (β).

Some conditions related to semisimple classes of rings are described below

(S1) if $A \in S$, then for every $O \neq B \triangleleft A$ there exists a $B \to C$ such that $0 \neq C \in S$.

(S2) If for a given A and every $O \neq B \triangleleft A$ there exists a $B \to C$ such that $0 \neq C \in S$, then $A \in S$.

(a) R is homomorphically closed

(a*) S is hereditary i.e. $I \triangleleft A \in S$ implies $I \in S$.

(b) R has the inductive property, if $I_1, \subseteq I_2$, C.. is ascending chair of ideals of A such that $I\alpha$ is in r, then also $UI\alpha \in R$.

(b*) $R(A) = \Sigma (I\alpha \Delta A: I\alpha \in R) \in R$

(b*) R has the coinductive property:

It $I_1 \geq I_2 \geq I\alpha \geq$ is a descending chain of ideals of a ring A such that each $A/I\alpha \in S$ then also. $A/\cap I\alpha \in S$.

(b*) S is closed under subdirect sums: If $A\alpha \in S$, $\alpha \in \Lambda$, then $\Sigma A\alpha \in S$.

(c) R is closed under extensions, $B \in R$, $A/B \in R$ and $A \in A$ imply $A \in R$.

(d) $R(A/R(A)) = 0$ for every ring A

Theorem (Gardner and Wiegandt [3]) **4.3.3**

The following conditions are equivalent for a class S

(I) S is the semisimple class of a supernilpotent radical which has the intersection property relative to $C \cap S$.

(II) S is an essentially closed and subdirectly closed class of rings satisfying (s) It $A \in S$, then every nonzero ideal of A has a nonzero C-ideal in S and the upper radical of S has the intersection property of $C \cap S$.

(III) S is essentially closed, subdirectly closed, regular class of rings satisfying the condition (t) every S-ring is a subdirect product of $C\cap S$ rings.

Proof :

First suppose (i) holds. Since S is the semiprime class of a hereditary radical, S is essentially closed as well as subdirectly closed. If I be a nonzero ideal of A in S, then $I\in S$, so by the intersection property I is a subdirect product of $C\cap S$-rings, in particular, I has nonzero $C\cap S$-factor. Thus the condition (s) satisfied thus (i)$\Rightarrow$(ii).

Suppose (ii) holds. It follows readily from the condition (s) that S is regular. Let A be an S-ring. Let $T=C\cap S$. Then $U_{T(A)}$ has no nonzero T-factors so (S) implies that $U_{T(A)}=0$. Thus A is a subdirect products of T-rings and the condition (t) has been established thus (ii)$\Rightarrow$(iii).

Finally let (III) holds. By Corollary 2 ([ ]) an essentially closed, S of rings is the semisimple class of the hereditary radical class U.S Since C consists of semiprime rings by condition (t) Us is supernilpotent. Moreover (i) also says that Us has the intersection property relative to $C\cap S$.

Theorem (Weigandt[13]) **4.3.4**

The following conditions are equivalent

(i) S is a semisimple class

(ii) S satisfies the condition (a) it $A \in S$, then for every $0 \neq B \triangleleft A$ there exists a $B\to I$ such that $0 \neq I \in S$.

(b) S is closed under subdirect sums if $A_\alpha \in S$, $\alpha \in \Lambda$, then $\sum A_\alpha \in S$.

(c) S is closed under extensions: $B\in S$, $A/B \in S$ and $A\in A$ implies $A\in S$

(d) $((A)S)S \triangleleft A$.

(iii) S satisfies conditions (S1) (b*), (C) and (e) if $K \triangleleft I \triangleleft A$, and I and K are minimal with respect of $A/I \in S$ and $I/K \in S$, respectively, then $K \triangleleft A$.

Proof :

From propositions 8,10,11 and 12 of Weigandt [13] (i) $\Rightarrow$(ii).

Let (ii) holds. Since (b*) implies conductive property trivially. By (b*), the ideal I=(A)S is the unique one which is minimal with respect to A/I $\in$S, and the corresponding assertion applies to K=(i)S $\triangleleft$ I. Hence using coinductive (d), the get K=(I)S = ((A)S) S $\triangleleft$ A.

Thus also condition (e) holds.

Finally let (III) holds. We have to show the show the validity of (S2). Let A be a ring such that every nonzero ideal of A has a nonzero homomorphic image in s. By condition (b*), Zorn's lemma is applicable and so there exists an ideal I of A which is minimal relative to A/I$\in$S. If I$\neq$0 then by assumption there exist an ideal K of I such that 0$\neq$I/K $\in$S.

Moreover, by (b*), K can be chosen such that K is minimal with respect to I/K $\in$S. Then by condition (c), we have K$\triangleleft$ A, and so

$\frac{A/K}{I/K} \cong$ A/I $\in$S holds .

Using condition (c), me get A/K $\in$S, and therefore by the chosen of I, it following I$\subseteq$K, contradicting I/K$\neq$0 . Thus necessarily I=0 and A$\in$C hold proving the validity of condition (S2).

Theorem (Weigandt [13]) **4.3.5**

R and S are corresponding radical and semisimple classes if and only if

(i) R$\cap$S =0

(ii) A$\in$R and A$\rightarrow$B$\neq$0 imply B$\notin$S .

(iii) A$\notin$S and 0 $\neq$B$\triangleleft$ A imply B$\notin$R .

(iv) each ring A has an ideal B such that B$\in$R and A/B$\in$S.

Proof :

The necessity is obvious (in (iv) take B=R(A)). For the sufficiency, (ii) $\Rightarrow$ R $\subseteq$ US and (III) implies S $\subseteq$ S_R. Let A$\in$ S_R. By (iv) There is a B$\triangleleft$A such that $B \in R$ and $A/B \in S$. Since $A \in S_R$ if follows $B=0$, hence $A \in S$. Thus $S_R \subseteq S$ holds implies $S=S_R$. Similarly one can conclude $R=US$ and by (iii) $R=US$ is a radical class and hence S is its semisimple class.

Lemma (Weigandt [13]) **4.3.6**

Let $K \triangleleft I \triangleleft A$ and $A \in A$. Then

(i) $aK+K \triangleleft I$

(ii) The mapping ϕ: $K \rightarrow (aK+K)/K$ defined by $\Phi(x) = ax + K$,$\forall x \in K$ is onto homomorphism.

(iii) $(aK+K)^2 \subseteq K$

(iv) $\text{Ker}\, \Phi \triangleleft I$

Theorem (Weigandt [13]) **4.3.7**

A class S of rings is a semisimple class if and only if S satisfies conditions (S1). (b*) and (c).

Proof :

In view of Theorem [30] it suffices to show that condition (c) follows from (S1), (B*) and (c). By (S1) and (b*) there exists ideals I of A and K of I such that they are minimal relative to $A/I \in S$ and $I/K \in S$ respectively. We have to prove that $K \triangleleft A$. Assume that this is not the case and that $aK \not\subset K$ for an element $a \in A$. By Lemma [31] we have

Φ: $K \rightarrow (aK+K)/K \triangleleft I/K \in S$ and $(aK+K)^2 \subseteq K$.

Hence Lemma [31] yields $0 \neq K/\text{Ker}\Phi \cong (aK+K)/K \in S$. By Lemma [31] (iv) $\ker\Phi \triangleleft I$, and

so $\dfrac{I/Ker\Phi}{K/Ker\Phi} \cong I/K \in S$ holds. Applying condition (c) we get $I/\text{Ker}\Phi \in S$ and so by the minimality of K it follows $\text{Ker}\Phi=K$ contradicting $K/\text{Ker}\Phi \neq 0$. Thus $K \triangleleft A$.

In view of Theorem [30] we have also proved that condition (d) is a consequence of (S1), (b*) and (c).

Corollary (Weigandt [13]) **4.3.8**

A class S is semisimple class if and only it S satisfies (a*), (b*) and (c).

Corollary (Weigandt [13]) **4.3.9**

R and S are corresponding radical and semisimple class if and only if

(i) $R \cap S = 0$

(ii) R is homomorphically closed

(iii) S is hereditary

(iv) each ring A has an ideal B such that $B \in R$ and $A/B \in S$

Lemma (Wiegandt[13]) 4.3.10

If R is a hereditary radical, then for any $I \triangleleft A$, $R(I)=R(A) \cap I$.

Theorem (Wiegandt[13]) 4.3.11

S is the semisimple class of a hereditary radical if and only if S satisfies conditions (S1), (b*) and (λ) S is closed under essential extensions: I $\Delta \in$ and I $\in S$ imply $A \in S$.

Corollary (Weigandt [13]) 4.3.12

S is the semisimple class of a hereditary radical if and only of S satisfies condition (a*), (b*) and (λ).

Theorem (Wiegandt[13]) 4.3.13

A proper subclass S of associative ring is the semisimple class of a supernilpotent radical if and only if S satisfies condition (S1), (b), (λ) and (α) S is weakly homomorphically closed: $I \triangleleft A \in S$ and $I^2=0$ imply $A/I \in S$.

Chapter - 5

Semisimple Radical Classes

Introduction 5.1

This chapter deals with semisimple radical classes. Characterizations of such classes by Stewart and Majumdar have been described. Stewart has obtained a complete description of semisimple radical classes .Majumdar too has obtained a family of such classes .We have given a detailed description of their works.

Definition 5.2.1

A non-empty class C of rings is called a semi-simple radical class if C is both a semi-simple class and a radical. Thus C is a semi-simple radical class if and only if $C = R_1$ and $C = S(R_2)$, for some radicals R_1 and R_2. Obviously, if $R_1 = R_2$, then C contains only the zero rings.

In the discussion of semisimple radical classes a certain class of rings, called B_1-rings by Stewart [12], plays a very significant role. We give below a detailed description and characterization of B_1-rings.

Definition 5.2.2

Let R be a ring and $x \in R$. Let $[x]$ = the subring of R generated by x. A ring R is called a B_1-ring if for all $x \in R$ and $[x] = [x]^2$.

Lemma 5.2.3

The class of B_1-rings denoted by B_1 is a radical class.

Proof:

Let R be a ring and $x \in R$. Clearly $[x] = [x]^2$ if and only if $x \in [x]^2$, i.e., if and only if there are integers $a_1, a_2 \dots a_k$ such that $x = \Sigma a_i x^i$ Using this, it is clear that every homomorphic images of B_1-rings are B_1-rings and it A/B and B are B_1-rings then A is a B_1-rings. Thus by theorem 1.3.3 is a radical.

Lemma 5.2.4

A nonzero B_1-ring without proper divisor of zero is a field of prime characteristic which is algebraic over its prime subfield.

Proof :

Let R be a nonzero B_1-ring without proper divisors of zero. If x is a nonzero element of R there are integers a_1, ... a_k such that $x=\sum_{i=2}^{k} a_i x^i$ hence $ex = \sum_{i=2}^{k} a_i x^i$ is an identity for [x]. Since x is not a zero divisor ex is an identity for R. If $w \in R$, $w \neq 0$ $ew \in [w] = [w]^2$, so $ew \in [w]$. $w \subseteq Rw$ $R=Rw$. Since R is nonzero, R is a division ring.

Let e be the identify of R. Then $[2e] = [2e]^2 = 4e$, so Ne =0 for some positive integer N. Consequently the characteristic of R is a prime and since $e=ew \in [w]$ for all nonzero $w \in R$, R is algebraic over its prime subfield. Therefore by Theorem 2 of Jacobson [10] R is a field.

Theorem (Strewart [12]) **5.2.5**

A ring R is a B_1-ring, if and only if every finitely generated subring of R is isomorphic to a finite direct sum of finite fields.

Proof:

Let $R \in B_1$ and R' be a finitely generated subring of R. Then $R' \in B_1$ and hence is commutative, so by the Hilbert Basis Theorem R' has maximum condition on ideals. If $P' \neq R'$ and P' is a prime ideal of R' then P' is a maximal ideal of R' since by Lemma 3.2 of [12] R'/P' is a field. Since R' is finitely generated commutative and has an identify for each generator g of R'. R' has an identify. Then by Theorem 2 [32] R' has minimum condition on ideals. But then R' is a commutative Wedderburn ring. so R' is isomorphic to a finite direct sum of fields each of which must be finite since they are finitely generated algebraic and of prime characteristic.

The converse is obvious. In fact, if $x \in R$ and R is isomorphic to a finite direct sum of finite fields then there is an integer n (x) ≥ 2 such that $x^{n(x)} = x$..

Corollary (Strewart[12]) **5.2.6**

R is a B_1-ring if and only if for each $x \in R$ there exists an integer $n(x) \geq 2$ such that $x^{n(x)} = x$.

Definition 5.2.7

A class of rings C is said to be **strongly hereditary** if S is a subring of $R \in C$ then $S \in C$.

Proposition 5.2.8

If F is a strongly hereditary finite set of finite fields then a ring R is isomorphic to a subdirect sum of fields in F if and only if every finitely generated subring of R is isomorphic to a finite direct sum of fields in F.

Proof:

Since F is a finite set of finite fields there exists an integer $N \geq 2$ such that $x^N = x$ for all $x \in F \in F$.

Let R have ideals $I_\alpha : \alpha \in \Lambda$ such that $R/I_\alpha \cong F_\alpha \in F$ and $\cap \{ I_\alpha : \alpha \in \Lambda \} = (0)$. Let R' be a finitely generated subring of R. Then $R \in B_1$ since $x^N = x$ for all $x \in R \supseteq R'$, so by lemma [12] $R' \cong A_1 \oplus \ldots \oplus A_k$ and A_i are finite fields. Choose $a_i \in R'$ such that $[a_i] \cong A_i$. Then $a_i \neq 0$ so $a_i \in I\, \beta_i$ for $\beta_i \in A_i$ but $I\beta_1 \cap [a_i] \triangleleft [a_i]$ so $I/\beta_i \cap [a_i] = (0)$. Therefore, $A_i \cong [a_i] \cong [a_i] + I\beta_i / I\beta_i$ is isomorphic to a subring $F\beta_i$. Since F is strongly hereditary R' is isomorphic to a finite direct sum of fields in F.

Conversely if every finitely generated subring of R is isomorphic to a finite direct sum of fields in F, R must be a B_1-ring since again $x^N = x$ for all

$x \in R$. Thus by lemma 3.3 of [12] there are ideals I_α: $\alpha \in \Lambda$ of R such that $\cap\{I_\alpha:\alpha \in \Lambda\} = (0)$ and R/I_α is a field of prime characteristic moreover, R/I_α must be a finite field since $x^N - x = 0 \in I_\alpha$ for all $x \in R$. Therefore for each $\alpha \in \Lambda$, there exists $x_\alpha \in R$ such that $[x_\alpha] + I_\alpha + I_\alpha = R/I_\alpha$. but then R/I_α is a homomorphic image of $[x_\alpha]$ so R/I_α is isomorphic to a field in F.

Lemma (Strewart[12]) **5.2.9**

If C is a class of rings such that subdirect sums of ring in C are in C and C is closed under homomorphic image then C is strongly hereditary.

Proof

Let $R \in C$ and S be a subring of R. Set $R_i = R$ for all $i \in Z^+$=the set of positive integers. Now the direct sum $\Sigma\{R_i \; i \in Z^+\}$ is an ideal of the direct product $\Pi\{R_i : i \in Z^+\}$. If $S \in S$ let $S(i) = S$ for all $i \in Z^+$. Then $S \rightarrow \lhd(s) = \{S:S \in S\}$ is an embedding of S into π $\{Ri : i \in Z^+\}$. $\lhd(s) + \Sigma\{Ri : i \in Z^+\}$ is a subdirect sum of copies of R and hence in C so

$$S \cong \lhd(s) \cong \lhd'(s)$$

Thus C is strongly hereditary we now state and prove the central result of Stewart.

Theorem : [ ]

If C is a semisimple radical class and $C \not\subset B_1$ then C consists of all rings.

Proof:

Let C be a semisimple radical class. If $C \not\subset B_1$, then there is a $R \in C$ and $x \in R$ such that $[x] \neq [x]^2$. In Kurosh [33] shows that for any semisimple class F, subdirect sums of rings in F are in F. Thus $[x] \in C$ and since $[x]^2 \lhd [x]$, $[x]/[x]^2 \in C$. Now $[x] / [x]^2$ is a zero ring on a cyclic group and since C satisfies (S_2 property), C^∞ = the zero ring on the infinite cyclic group is in C. This implies that C contains all nilpotent rings. Since C is a semisimple class ([ ] [], []) C is hereditary.

The following theorem gives a few characterizations, due to Stewart, of semisimple radical classes other than the class of all rings.

Theorem (Strewart[12]) **5.2.11**

If C is not the class of all rings then the following are equivalent (1) C is a semisimple radical class (2) There is a strongly hereditary finite set C (F) of finite fields such that $R \in C$ if and only if every finitely generated subrings of R is isomorphic to a finite direct sum of fields in C(F).

Proof :

Assume that B satisfies condition (3). Clearly C satisfies (A) and (E).

If $B \triangleleft A$ and both A/B and B are in C and A' is a finitely generated subring of A then $A'+B/B \cong A/A \cap B$ is isomorphic to a finite direct sum of fields in C (F). A slight modification of the proof given for Proposition 1 on page 241 of Jacobson [10] shows that $A \cap B$ is finitely generated as a ring. Thus $A \cap B$ is also isomorphic to a finite direct sum of fields in C)F) and so $A \cong A/A \cap B \oplus A \cap B$. Therefore, $A \in C$. From this it is easy to show that if C(R) = the sum of all ideals of R which are in C then C(R) $\in C$ and C(R) / C(R) = (0). Thus C satisfies (B) and (C).

If every nonzero ideal of a ring R can be homomorphically mapped onto a nonzero ring in C then by 3.7, every nonzero ideal of R can be homomorphically mapped onto a ring in C(F). Sulinsky [9] (see also [6], Theorem 46) shows that this implies that R is isomorphic to a subdirect sum of rings in C(F) and hence by 3.7 again, $R \in C$. So C satisfies (F) and hence C is a semisimple radical class.

Conversely, suppose C satisfies condition (1). Let C(F) = the class of all fields which are in C and define A=II $\{R \in C\ (F)\}$. Since C is a semi-simple class subdirect sums of rings in C are in C; thus $A \in C$. By hypothesis, $C \subseteq \beta_1$ so by 3.4 all elements of A must be torsion. From this it follows that there is a finite number of primes $p_1, \ldots, p_N$ such that every field in C (F) is of characteristic p_i for some $1 \leq i \leq N$. For each finite field $R \in C$ (F) choose a(R) such that [a(R)] = R and for each infinite field $R \in C$ (F) set a (R) = 0. Then $a = \{a(R)\}_{R \in C(R)}$ is in A and by 3.5 $a^k = a$ for some integer $K \geq 2$. thus, for all finite fields R in C(F), the dimension of R over its prime subfield is $\leq k-1$. Hence there is only a finite number of finite fields in C(F). Suppose there is an infinite field $R \in C(F)$. By 3.2 r is of prime characteristic and is algebraic over its prime subfield so R has an infinite number of non-isomorphic finite subfields. All these subfields are in C(F) since C is strongly hereditary by 4.1. This is impossible since there is only a finite number of finite fields in C(F).. Therefore, C(F) is a strongly hereditary finite set of finite fields. If $R \in C$ then $R \in B_1$ so by 3.3 R is isomorphic to a subdirect sum of fields all of which are in C(F) since C satisfies (A). Conversely, any ring isomorphic to a subdirect sum of rings in C(F) is in C since C is semisimple class. Thus C satisfies (2).

3. We shall conclude the chapter with Majumdar's characterization of semisimple radical classes and his example of a semisimple radical class. these are given below :

For a ring A, define I(A) to be class of all intersections of non-empty collections of rings in I(A).

Lemma 5.3.1

Let C be a non-trivial semi-simple radical class, and let $A \in C$. Then, $\bar{I}(A) = I(A) = I_1(A)$.

Proof

Let $B \in I_k(A)$, $k > 1$, Then there is a chain $A = A_0 \supset A_1 \supseteq A_2 \supseteq \ldots \supseteq A_k = B$, where each A_{i+1} is an ideal of A_2. $x \in B$ and $a \in A$. Since A belongs to the non-trivial semi-simple radical class C it follows from corollary and Theorem 4.2 of (8) that $[x] = [x]^2 = [x]^k$, where [x] is the subring of A generated by x. Thus, $x = P_1(x)\, P_2(x) .. P_k(x)$, where each $P_i(x) \in [x]$. Since each $P_i(x)$ belongs to each

A_j, it follows that $aP_1(x) \in A_1$, $aP_1(x)P_2(x) \in A_2$.........$ax=aP_1(x)\,P_2(x)$....... $P_k(x)$, $\in A_k=B$. Similarly $xa \in B$ .Hence $I(A)=I_1(A)$ .Since the intersection of every nonempty collection of ideals of A is an ideal of A, $\bar{I}(A)=I(A)=I_1(A)$.

Theorem 5.3.2

A non-empty class C of rings is a semi-simple radical class if and only if C satisfies the following conditions:

(1) C is homomorphically closed.

(2) C is closed under extensions.

(3) C is strongly hereditary.

(4) If $A_1 \subseteq A_2 \subseteq A_3 \subseteq$....................

is an ascending chain of rings in I(A) for a ring A such that each $A_\beta \in$**C**, then $\bigcup_\beta A_\beta \in C$.

(5) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq$..... is a descending sequence of subrings of a ring A such that (i) for each i, A_{i+1} is an ideal of A_i and $\frac{A_i}{A_i + I} \in C$ and (ii) $\bigcap_i A_i \in C$, for some i.

Proof

Suppose C is semi-simple radical class. Without loss of generality, we may assume that C is a non-trivial class. By Theorem 3, C satisfies (1) and (2) since every semisimple class is closed under subdirect sums (7), Lemma 4.1 of (8) shows that C satisfies (3), Suppose $A_1 \subseteq A_2 \subseteq A_3 \subseteq$.....is an ascending chain of rings in $\bar{I}(A)$, for a ring A, such that each $A_\beta \in C$. Let $B = \bigcup_\beta A_\beta$, and let C(B) denote the C-radical of B. By lemma 1,A_β is an ideal of A, and so, $A_\beta \subseteq C(B)$, for each A_β. Hence $B=C(B)$, i.e., $B \in C$. Thus C satisfies (4). Now suppose $A_1 \supseteq A_2 \supseteq A_3 \supseteq$..... is descending sequence of subrings of a ring A such that (i) for each i, A_{i+1} is an ideal of A_1 and $\frac{A_i}{A_i + I} \in$ C, and (ii) $\bigcap_i A_i \in$ C. Let R denote the upper radical class determined by C. Then, for each i , $R(A_i) \subseteq A_{i+1}$, since $\frac{A_i}{A_i + I} \in C$. Now $R(A_i)$ being ideal of A_{i+1}, $\frac{R(A_i)}{R(A_{i+1})}$ is R-semisimple, by Corollary 2 to Theorem 1 of [3]. This implies $R(A_i) = R(A_{i+1})$. Thus $R(A_i) = \bigcap_i R_i$, for each i. Now $\bigcap_j R(A_j)$ being a subring of $\bigcap_i A_i$, the former belongs to C, by (ii) and (3). Thus for each i. $R(A_i) \in$ C, so that $R(A_i) = 0$, i, e, $A_i \in C$, Hence C satisfied (5).

Conversely, suppose C satisfied the conditions 1-5. Theorem 3 shows that C is a radical class. Let A be a ring such that every non-zero subring of A can be mapped homomorphically onto a non-zero ring in C. By Theorem 2 it will be sufficient to show that $A \in C$.

Define I_1=A: Let β be any ordinal ≥ 2, and assume that I_α has been defined for each ordinal $\alpha < \beta$. If β is not a limit ordinal, either $I_{\beta-1}=0$, or there exists an ideal I of $I_{\beta-1}$ such that $0 \neq \frac{I_{\beta-1}}{I} \in C$. If $I_{\beta-1}=0$, define $I_\beta=0$ and if $I_{\beta-1} \neq 0$ define I_β=I. If β is a limit ordinal, define $\bigcap_{\alpha<\beta} I_\beta = I_\alpha$ Then I_β has been defined for every ordinal. It follows from the construction that $I_\iota \in C$, for some ordinal ι .

Define $J_1 = I_\iota$. Let γ be any ordinal ≥ 2. Assume that J_α has been defined to be some $I_\alpha \in C$, for each ordinal $\alpha < \gamma$ .First, let γ be not a limit ordinal. Let $J_{\gamma-1} = I_\alpha$. If $\alpha=1$, define $J_\gamma = I_\alpha$. If α is not a limit ordinal and $\alpha \neq 1$, then define $J_\gamma = I_{\alpha-1}$. By (2), $I_{\alpha-1} \in C$. If $\alpha=\omega$, I_α is the intersection of the descending sequence of subrings $I_1 \supseteq I_2 \supseteq I_3 \supseteq$..... of A, and hence by (5) there exists $\alpha < \omega$ such that $I_\alpha \in C$. Define $J_\alpha = I_\alpha$. If α is a limit ordinal $> \omega$, then I_α can be regarded as the intersection of all I_α's, where α is less than α but greater than every limit ordinal less than α. such I_α' s form a descending sequence of subrings of A, and so, by (5), there exists an ordinal α as described above such that $I_\alpha \in C$. Define $J_\alpha = I_\alpha$. Next, let γ be a limit ordinal. Define $J_\gamma = \bigcup_{\alpha<\gamma} J_\alpha$. It is easy to see by transfinite induction that each I_β and hence each J_α, belongs to $\bar{I}(A)$. So, by (4). $J_\gamma \in C$. Also $J_\gamma = I_\alpha$ where $\alpha = \min \{\alpha,: J_\alpha = I_\alpha, \alpha<\gamma\}$ .Thus, for every ordinal γ ,J_γ has been defined as some $I_\alpha \in C$. It follows from the construction that $J_\gamma = I_1 = A$, for some ordinal γ. Hence $A \in C$. The proof is complete.

We conclude by observing the following improvement .

A radical class R is a semi-simple class if and only if R is closed under subdirect sums.

Since every semi-simple class is hereditary this is an immediate consequence of Lemma 4.1 of [8] and Theorem 4.5 of [5].

Theorem 5.4.1

Let p be a prime greater than 2, and let C_p be the class of all rings A such that, for each $x \in A$, $x^p = x$ and $px = 0$. Then, C_p is a semisimple radical class.

For proof we need the following lemmas

Lemma 5.4.2

Let a and b be two elements of a ring such that ab=ba. Then, for each n ≥ 2,

$$(a-b)^n = (-1)^n.\, b^n + (-1)^{n-1},\, {}^nC_{n-1}.\, ab^{\,n-2}\,(b-a)$$

$$+ (-1)^{\,n-2},\, ({}^nC_{n-2} - {}^nC_{n-1})\, a^2 b^{n-3}\,(b-a)$$

$$+ (-1)^{\,n-3}\, ({}^nC_{n-3} - {}^nC_{n-2} + {}^nC_{n-1}),\, a^3 b^{n-4}\,(b-a)$$

$$+$$

$$+ ({}^nC_1 - {}^nC_2 + {}^nC_3 \,\ldots\ldots + (-1)^{\,n-1},\, {}^nC_{n-1}),\, a^{n-1}\,(b-a)$$

$\pm a^n$, ... (A)

the positive or the negative sign occurring according as n is odd or even.

Proof

An obvious rearrangement of the terms in the expansion of $(a-b)^n$ yields (A), since ${}^nC_1 - {}^nC_2 + {}^nC_3$ ...... $+ (-1)^{n-1}$, ${}^nC_{n-1} = 0$ or 2. according as n is odd or even, as is easily seen by considering the expansion of the left-hand side of the identity $(1-1)^n = 0$.

We now prove

Lemma 5.4.3

C_p is closed under extensions.

Proof

Let A be a ring and I an ideal of A such that I and $\frac{A}{I}$ are in C_p. Let $x \in A$. Then, $x - x^p \in I$, and so,

$x\text{-}x^p = (x\text{-}x^p)^p = x^p - x^{p^2}$, by Lemma 1,

$= x^{p^2} - x^{p^3}$, by Lemma 1,

$= x^{p^{p-1}} - x^{p^p}$, similarly

Hence, $0 = p\,(x\text{-}x^p) = (x\text{-}x^p) + (x^p\text{-}x^{p^2}) + \ldots\ldots + (x^{p^{p-1}} - x^{p^p})$

$= x - x^{p^p}$,

i.e., $x^{p^p} = x$ .... (1)

Since $\frac{A}{I}$ is in C_p, $px \in I$,

and so, $p^2x = p(px) = 0$. Thus, $(px)^{p^p} = p^2x.x.\ (px)^{p^p-2} = 0$.

Hence by (1), $px = 0$ .... (2)

By virtue of (1) and (2), [x], the subring of A generated by x, is commutative, finite and nil semisimple, and the number of elements in [x] divides $p(p^p-1)$. Also, [x] is the direct sum of a finite number of finite fields, say, $F_1, F_2, \ldots\ldots F_r$. By (2), the number of element in F1 is pni, for some positive integer n_i, i=1,2, ..., r., Hence $p^{n_1+n_2+\ldots\ldots+n_r}$ divides of (p^p-1). Thus, $n_1 + n_2 + \ldots + n_r = 1$, i.e., r=1 and $n_1=1$. Therefore, [x] is a field of p elements, and so, $x^p=x$.

Lemma 5.4.4

C_p is closed subdirect sums.

Proof

Let $\{A_\alpha\}$ be a non-empty set of rings A_α in C_α, and let A be the subdirect sum of $(A_\alpha\}$. Let $A' = \prod_\alpha A_\alpha$, the direct product of $\{A_\alpha\}$. Clearly, $x^p = x$ and $px = 0$, for each $x \in A'$. A being a subring of A', it follows that A is in C_p.

Proof of the Theorem

It is clear that (i) if a ring A is in C_p so are every subring of A and every homomorphic image of A and (ii) if every term in an ascending chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots\ldots$ of ideals of a ring A is in C_p, so is their union $\bigcup_\alpha I_\alpha$. This completes the proof.

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