2013

Asymptotic Method for Time Dependent Nonlinear Differential Systems with Slowly Varying Coefficients

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Asymptotic Method for Time Dependent Nonlinear Differential Systems with Slowly Varying Coefficients

A dissertation Thesis Submitted to the Department of Mathematics, University of Rajshahi, Rajshahi-6205, Bangladesh, for the Degree of Doctor of Philosophy in Mathematics

By

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July, 2013

STATEMENT OF ORIGINALITY

I declare that the contents in my Ph. D Thesis entitled "Asymptotic Method for Time Dependent Nonlinear Differential Systems with Slowly Varying Coefficients" is original and accurate to the best of my knowledge. I also declare that the materials contained in my research work have not been previously published or written any person for degree or diploma.

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ACKONOWLEDGEMENT

I would like to convey an immense grateful to a number of individuals for their kind advices, directions, suggestions and co-operation, which have enabled me to complete my thesis successfully.

I am bound to express my deepest respect and thanks to my respected supervisor professor Dr. M. Zulfikar Ali, Department of Mathematics, University of Rajshahi, Bangladesh for his day-to-day monitoring, supervision and maximum assistance for the completion of the present study successfully. I also thankful to Dr. Pinaki Day, Assistant professor, Department of Mathematics,...................., for providing me some important source of information and materials about my study.

I wish to thank the Chairman, Department of Mathematics, University of Rajshahi, for providing facilities during the course of thesis work.

I am thankful to Dr. M Ali Akbar, Associate professor, Department of Applied Mathematics, University of Rajshahi, Bangladesh, for providing me the source of information and materials about my study. I also wish to express my sincere admiration, appreciation and gratitude to all the teacher of the Department of Mathematics, University of Rajshahi, Bangladesh, for providing me invaluable suggestions during the period of my research work.

A special thank will go to the shoulders of my colleagues and friends at Pabna university of Science and Technology, Pabna, for friendly support to overcome the hurdles during the Ph.D program.

Finally, I am ever grateful to my parents, beloved daughter and wife for their moral and constant blessing, good advice and encouragement.

(Harun-Or-Roshid)

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June, 2013

CERTIFICATE

Certified that the Ph.D. thesis entitled “Asymptotic Method for Time Dependent Nonlinear Differential Systems with Slowly Varying Coefficients” submitted by **Harun-Or-Roshid** in fulfillment of the requirement for the degree of Ph.D. in Mathematics, University of Rajshahi, Rajshahi-6205, Bangladesh has been completed under our supervision. We believe that the research work is an original one and it has not been submitted elsewhere for any degree.

We wish him a bright future and every success in life.

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Abstract

Almost all perturbation methods are developed to find periodic solutions of nonlinear system where transients are not considered. First Krylov and Bogoliubov introduced a perturbation method which is well known as "asymptotic averaging method" to discuss the transients in the second order autonomous systems with small nonlinearities. Later, this method has been amplified and justified by Bogoliubov and Mitropolskii. Mitropolskii has extended the method for slowly varying coefficients to determine the steady state periodic motions and transient process. In this dissertation, we have modified and extended the KBM method to investigate some fifth order and second order nonlinear systems in both cases with constant and slowly varying coefficients.

At first, a fifth order damped nonlinear autonomous differential system is considered and a perturbation solution is developed. Then a procedure is developed for the same system with damped taking three of eigenvalues are real. After then we considered fifth order systems for over damped with small nonlinearity to obtain the transient response. We also developed a formula for fifth order critically damped nonlinear systems to control micro vibration, in micro and nano-technological industries that bring the system to equilibrium as quickly as possible without oscillating. After then we presented an analytical technique based on the extended Krylov-Bogoliubov-Mitropolskii method (by Popov) to determine approximate solutions of nonlinear differential systems whose coefficients change slowly and periodically with time. Furthermore, a non-autonomous case also investigated in which an external force acts in this system. At last, Krylov-Bogoliubov-Mitropolskii (KBM) method has been extended to certain damped-oscillatory nonlinear systems with varying coefficients. The implementations of the methods are illustrated by several examples.

Contents

Introduction

Nonlinear physical science and nonlinear mathematics focuses on recent advances of fundamental theories and principles, analytical and symbolic approaches, as well as computational techniques with engineering applications. Almost all nonlinear phenomena can be modeled by dynamical system. Dynamical system although is very complicated to study but very interesting both mathematically and physically. It is complicated because of nonlinearity of the equation but physically interesting because we can easily visualize how changes occur due to the variation of the parameters. In nonlinear dynamical system, we mainly deal with nonlinear differential equations of the governed system. In this system, it is very difficult to get an exact solution other than some special cases. In general, approximate solutions are obvious to accept but attempt should be made to get more accurate solutions. Most of the nonlinear oscillating systems show complex behavior such as strong attractor's chaos and bifurcations. Recently, people have been trying to solve various types of dynamical system by different methods and having to find the graphs for different parametric values in the equation, through which they can comment on the nature of the system.

To find analytical approximate solutions to the nonlinear oscillator equations many methods are established by many researchers. The common methods for finding analytical approximate solutions to the nonlinear oscillator equations are the perturbation methods. The Lindstedt method [66], Poincare method [88], WKB method [18, 60, 126], Multi-time-scale method [47, 74], the Krylov-Bogoliubov-Mitropolskii method [22, 63] etc. are some well known perturbation methods to obtain analytical approximate solutions of non-linear systems. Among the above methods KBM method is the particularly convenient and is the widely used technique to obtain analytical approximate solution of non-linear systems. Krylov and Bogoliubov [63] originally developed a perturbation method for obtaining periodic solutions that was amplified and justified by Bogoliubov and Mitropolskii [22] and later Popov [91] and Mendelson [69] extended the method for damped nonlinear oscillations. Murty [74] has presented a unified KBM method for solving second order nonlinear systems in the cases un-damped, under-damped and over-damped system with constant coefficients. Sattar [99] studied a third order over damped nonlinear system and Bojadziev [24] studied the damped oscillations modeled by a three dimensional nonlinear system. Shamsul and Sattar [103] have presented a method for critically damped nonlinear systems and Islam and Akbar [55] obtained a new solution of more critically damped third order nonlinear systems. Shamsul and Sattar [109] presented a unified KBM method for solving third order nonlinear systems. Akbar et al. [4] has presented a method for solving the fourth order over damped nonlinear systems which is easier, simple and less laborious than Murty et al. [75]. Later, Akbar et al. [6] pull out the method presented in [4] for the damped oscillatory systems. Akbar [5] investigated the solutions of fourth order more critically damped differential systems. Rahman et al. [93] obtained fourth order nonlinear oscillatory systems when two of the eigenvalues are real and negative and the other two are complex numbers. Akbar and Siddique [9] presented a method to obtain solutions of fifth order weakly nonlinear oscillatory systems.

The purpose of this dissertation is to introduce some new extended KBM method in nonlinear physics to explore different nonlinear dynamical systems in both case of oscillatory and non-oscillatory differential systems with constant and slowly varying coefficients. Some of their equivalent formulations along with various new characterizations and results concerning the existing ones are presented here. The implementation of the presented methods is illustrated by its applications via cubic nonlinear Duffing type oscillator. Figures are provided to compare validation and usefulness of the solutions obtained by the presented method for different initial conditions with the corresponding numerical solutions obtained by the fourth order Runge-Kutta method.

We aim to develop formulae based on KBM method of nonlinear dynamical systems with constant and slowly varying coefficients for both oscillators with and without damp. We also tried to extend some formulae on non-oscillatory systems in both over damp and critical damp cases. In our work, the materials have been divided into seven chapters, a brief scenario of which we present as follows.

In the first chapter, we incorporates Gauge functions, Order symbols , Expansions of functions and their convergence conditions, some of the basic definitions, and some existing perturbation techniques for analytical approximate solutions to nonlinear dynamical systems whose results are available in given references.

Our work starts from second chapter. In this chapter, we tried to find an analytical periodic solutions by constructing a formula for a fifth order Duffing type oscillatory systems in the presence of damping effects with small non-linearity, using a perturbation method. The implementation of the work is illustrated by giving a suitable example. We have also provided some figures to test the correctness of our results in contrast with corresponding numerical results and have computed the Pearson correlation between the results.

In third chapter, we add a formula to explain a different type of natural damp oscillatory system. This investigation of hereditary, productive and projective formula is good extension to non linear physics. Here we solved an example to show that the proposed method is quite efficient. Accuracy of this formula is examined providing some figures of our results in contrast with corresponding numerical results and strongly correlated is corroborated by computing the Pearson correlation between the results.

A procedure is presented in the fourth chapter to solve a non oscillatory nonlinear differential systems which is over damped. We employed the proposed method to an example. Some figures are provided here and also correlation between our results and numerical results are calculated in this chapter.

In fifth chapter, an analytical approximate procedure is investigated for obtaining the transient response of a system in the case of pair wise equal eigenvalues. We solved an example to show the implementations of our method. Three sets of results (depicted by figures) are given here between perturbation and corresponding numerical results to show reliability and advantages of the proposed technique.

In the previous chapter, we established some procedure with classical KBM method but from this chapter we start with the differential equations with slowly varying coefficients. We have presented an analytical technique based on the extended Krylov-Bogoliubov-Mitropolskii method (by Popov) [91] to determine approximate solutions of nonlinear differential systems whose coefficients change slowly and periodically with time. Furthermore, a non-autonomous case also explored in which an external force acts in this systems.

In seventh chapter, we have given a procedure which is simpler than other classical KBM method to solve nonlinear damp oscillatory differential systems with slowly varying coefficients. To show the reliability and advantages of the proposed technique, we have applied it to an example. We also delivered three figures of our results comparing with corresponding numerical results to test exactitude of our solutions.

Chapter-One

Mathematical Preliminaries

1.1 Introduction:

Perturbation theory comprises mathematical methods that are used to find an approximate solution to a problem which cannot be solved exactly, by starting from the exact solution of a related problem. Perturbation theory is applicable if the problem at hand can be formulated by adding a small term to the mathematical description of the exactly solvable problem.

Perturbation theory leads to an expression for the desired solution in terms of a formal power series (i. e. asymptotic series) in some small parameter known as a perturbation series that quantifies the deviation from the exactly solvable problem. The leading term in this power series is the solution of the exactly solvable problem, while further terms describe the deviation in the solution, due to the deviation from the initial problem. Formally, we have for the approximation to the full solution x, a series in the small parameter (here called ε), like the following:

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$$

In this example, x_0 would be the known solution to the exactly solvable initial problem and x_1, $x_2, \cdots$ represent the higher-order terms which may be found iteratively by some systematic procedure. For small ε these higher-order terms in the series become successively smaller. An approximate "perturbation solution" is obtained by truncating the series, usually by keeping only the first two terms, the initial solution and the "first-order" perturbation correction:

$$x = x_0 + \varepsilon x_1$$

Perturbation is important tool for describing nonlinear systems, as it turns out to be very difficult to find exact solutions to the Duffing equation, Schrödinger equation for Hamiltonians of even moderate complexity etc. The Hamiltonians to which we know exact solutions, such as the hydrogen atom, the quantum harmonic oscillator and the particle in a box and different oscillatory systems, are too idealized to adequately describe most systems. Using perturbation theory, we can use the known solutions of these simple Hamiltonians to generate solutions for a range of more complicated systems.

In this chapter, we have describe asymptotic expansions of a function and when the expansion is uniformly convergent. Finally, we tried to explain some well-known exist perturbation techniques such as Van Der Pol's [122] technique, Krylov-Bogoliubov [63] method and Krylov-Bogoliubov-Mitropolski (KBM) [22, 63] method.

1.2 Gauge Functions and Order Symbols:

Let $f(\varepsilon)$ be a function of the real parameter ε . If the limit of $f(\varepsilon)$ exists as $\varepsilon \to 0$, then there are three possibilities $f(\varepsilon) \to 0$, $f(\varepsilon) \to A$, $f(\varepsilon) \to \infty$ with $0 < |A| < \infty$. In the first and second cases we may express the rates at which $f(\varepsilon) \to 0$ and $f(\varepsilon) \to A$ by comparing $f(\varepsilon)$ with known functions called Gauge Functions. The simplest and most useful gauge functions are members of the set $[\varepsilon^n]$, where n is an integer. Other gauge functions often used are $\sin\varepsilon$, $\log\varepsilon$ etc. The behavior of a function $f(\varepsilon)$ as $\varepsilon \to \infty$, may be compared with a gauge function $g(\varepsilon)$ by employing the Landu symbols: O and o.

The symbol O:

The symbol O (big 'O') is defined as follows: Let $f(\varepsilon)$ be a function of the parameter ε and let $g(\varepsilon)$ be a gauge function. Let there exists a positive number A independent of ε and $\varepsilon_0 > 0$, such that $|f(\varepsilon)| \le A|g(\varepsilon)|$ for all $|\varepsilon| \le \varepsilon_0$ (1.1) then $f(\varepsilon) = O[g(\varepsilon)]$ as $\varepsilon \to 0$ (1.2)

The condition given in (1.2) may be replaced by

$$\lim_{\varepsilon \to 0} \left| \frac{f(\varepsilon)}{g(\varepsilon)} \right| < \infty \tag{1.3}$$

Let $f(x,\varepsilon)$ be a function of the variable x as well as the parameter ε, and let $g(x,\varepsilon)$ be a gauge function. We write $f(x,\varepsilon) = O[g(x,\varepsilon)]$ as $\varepsilon \to 0$ (1.4)

if there exists a positive number A independent of ε and $\varepsilon_0 > 0$ such that

$$|f(x,\varepsilon)| \le A|g(x,\varepsilon)| \text{ for all } |\varepsilon| \le \varepsilon_0 \tag{1.5}$$

If A and ε_0 are independent of x, the relationship is said to hold uniformly.

The symbol o:

The symbol o (small 'o') is defined as follows: Let $f(\varepsilon)$ be a function of the parameter ε and let $g(\varepsilon)$ be a gauge function. Let there exists an positive $\varepsilon_0 > 0$ and let for every positive number δ independent of ε, the following condition hold

$$|f(\varepsilon)| \le \delta|g(\varepsilon)| \text{ for all } |\varepsilon| \le \varepsilon_0 \tag{1.6}$$

then $f(\varepsilon) = o[g(\varepsilon)]$ as $\varepsilon \to 0$ (1.7)

The condition given in (1.7) may be replaced by

$$\lim_{\varepsilon \to 0} \left| \frac{f(\varepsilon)}{g(\varepsilon)} \right| = 0 \tag{1.8}$$

Let $f(x,\varepsilon)$ be a function of the variable x as well as the parameter ε, and let $g(x,\varepsilon)$ be a gauge function. We write $f(x,\varepsilon) = o[g(x,\varepsilon)]$ as $\varepsilon \to 0$ (1.9)

If for every positive number δ, independent of ε there exists an ε_0 such that

$$|f(x,\varepsilon)| \le \delta|g(x,\varepsilon)| \text{ for all } |\varepsilon| \le \varepsilon_0 \tag{1.10}$$

If δ and ε_0 are independent of x, the equation (1.10) is said to hold uniformly.

1.3. Asymptotic Expansions:

Let $[\delta_n(\varepsilon)]$ be a sequence of functions such that $\delta_n(\varepsilon) = o[\delta_{n-1}(\varepsilon)]$ as $\varepsilon \to 0$ (1.11)

Such a sequence is called an asymptotic sequence.

Consider the series $\sum_{m=0}^{\infty} a_m \delta_m(\varepsilon)$ (1.12)

where a_m are independent of ε, and $[\delta_m(\varepsilon)]$ is an asymptotic sequence. We say that this expansion is an asymptotic expansion and denote it by

$$y \approx \sum_{m=0}^{\infty} a_m \delta_m(\varepsilon) \text{ as } \varepsilon \to 0 \tag{1.13}$$

if and only if $y = \sum_{m=0}^{n-1} a_m \delta_m(\varepsilon) + O[\delta_n(\varepsilon)]$ as $\varepsilon \to 0$ (1.14)

The expansion given by (1.13) may diverge. However, if the series is an asymptotic expansion, then although (1.13) may diverge, for fixed n the first n terms in the expansion can represent y with an error that can be made arbitrarily small by taking ε sufficiently small. Thus the error committed in truncating the series after n terms is numerically less than the first neglected term, namely the $(n+1)$ th term.

Given a function $y(\varepsilon)$, the asymptotic expansion of $y(\varepsilon)$ as $\varepsilon \to 0$, is not unique. In fact, y can be represented by an infinite number of asymptotic expansions because there exists an infinite number of asymptotic sequences that can be used. However, once we choose a particular asymptotic sequence $[\delta_m(\varepsilon)]$, the representation of y in terms of this sequence is unique. Thus, if $y(\varepsilon)$ is an asymptotic expansion, for the given sequence $[\delta_m(\varepsilon)]$, we have

$$y \approx \sum_{n=0}^{\infty} a_n \delta_n(\varepsilon) \text{ as } \varepsilon \to 0$$

where the coefficient a_n are given uniquely by $a_n = \lim_{\varepsilon \to 0} \dfrac{y(\varepsilon) - \sum_{m=0}^{n-1} a_m \delta_m(\varepsilon)}{\delta_n(\varepsilon)}$ (1.15)

1.4. Uniform Expansions:

Let y be a function of the variable x as well as the parameter ε and develop y in an asymptotic expansion in terms of the asymptotic sequence $[\delta_m(\varepsilon)]$, we have

$$y(x,\varepsilon) \approx \sum_{m=0}^{\infty} a_m(x)\delta_m(\varepsilon) \text{ as } \varepsilon \to 0 \tag{1.16}$$

where the coefficients a_m are the function of x only. This expansion is said to be uniformly

valid if $y(x,\varepsilon) = \sum_{m=0}^{n-1} a_m(x)\delta_m(\varepsilon) + R_n(x,\varepsilon)]$ (1.17)

where $R_n(x,\varepsilon) = O[\delta_n(\varepsilon)]$ (1.18)

Uniformly, for all x of interest. If these conditions do not hold, then the expansion is said to be non-uniformly valid. For the expansion to be uniformly valid, the term $a_m(x)\delta_m(\varepsilon)$ must be small compared with the preceding term $a_{m-1}(x)\delta_{m-1}(\varepsilon)$ for each m.

Since $\delta_m(\varepsilon) = o[\delta_{m-1}(\varepsilon)]$ as $\varepsilon \to 0$

We require that $a_m(x)$ be no more singular than $a_{m-1}(x)$, for all values of x of interest , if the expansion is to be uniform. In practical terms, this means that $\frac{a_m(x)}{a_{m-1}(x)}$ is bounded.

Thus each term in the expansion given by (1.17).

where a_m are independent of ε, and $[\delta_m(\varepsilon)]$ is an asymptotic sequence. We say that this expansion is an asymptotic expansion and denote it by (1.16) must be a small correction to the preceding term irrespective of the value of x.

For a real valued function $f(x)$ of real variable x containing a number x_0 in its domain of definition , there is a power series expansion of the form $\sum_{j=0}^{\infty} a_j(x-x_0)^j$ (1.19)

With nonzero radius of convergence which provides a valid representation for f on I, the interval of convergence of the series if $f(x)$ has uniformly bowled derivatives of all orders at each point in I.

Further, the power series is uniformly determined and

$$a_j = \frac{f^{(j)}(x_0)}{j!} \tag{1.20}$$

where $f^{(j)}(x_0)$ denotes the j-th derivative of $f(x)$ evaluated at x_m. In this case the power series expansion (1.19) is called the Taylor series of the function $f(x)$ about the point x_0 and is uniquely determined. Also, if $x_0 < x_1$, the closed interval $[x_0, x_1]$ is in the domain of $f(x)$ and $f^{(n+1)}(x)$ exists for all $x \in [x_0, x_1]$, then the Taylor theorem states that there is an $\tilde{x} \in (x_0, x_1)$ such that

$$f(x_1) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x_1 - x_0)^j + R_n \tag{1.21}$$

where $R_n = \dfrac{f^{(n+1)}(\tilde{x})}{(n+1)!}(x_1 - x_0)^{n+1}$ (1.22)

Let B be the uniform bound for $f^{(n+1)}(x)$ on (x_0, x_1) i.e, $\left|f^{(n+1)}(x)\right| \le B$ for all $x \in (x_0, x_1)$.

Then from (1.22)

$$|R_n| = \left|\frac{f^{(n+1)}(\tilde{x})}{(n+1)!}(x_1 - x_0)^{n+1}\right| \le \frac{1}{(n+1)!}(x_1 - x_0)^{n+1}\left|f^{(n+1)}(\tilde{x})\right| \le \frac{B}{(n+1)!}(x_1 - x_0)^{n+1}$$

and thus the error introduced by using only n terms of the Taylor series for $f(x_1)$ is of the same order of magnitude as the first term in the series which is neglected.

Unfortunately, the theoretical and computational use of a Taylor series representation often posses serious problems. Suppose for example, that we are given the Taylor series for $f(x)$ about x_0 and a point $a > x_0$ in both the domain D of f and the interval of convergence I of the power series.

Consider the computational problem: Using relation (1.21), compute $f(a)$ where n in (1.21) is an integer such that $|R_n| \leq \varepsilon << 1$. The constant $\varepsilon > 0$ gives a bound on the allowable error. At the k-th stage of numerical procedure, we must thus compute a bound for $|R_k|$. If $|R_k| \leq \varepsilon$, inserting $n = k$ in (1.21) and hence $\left| f(a) - \sum_{j=0}^{k} \frac{f^{(j)}(x_0)}{j!}(a - x_0)^j \right| \leq \varepsilon$

(1.23)

On the other hand, if $|R_k| > \varepsilon$, we must compute at least one additional term in the Taylor series. If $a - x_0$ is large, it may be necessary to compute a large number of terms in the Taylor series before satisfying condition (1.23). This may not be practical even with the aid of a modern high speed computer. In such cases, it is natural to seek a different representation for $f(x)$ which makes the computational problem more manageable.

Often, such an alternative representation takes the form of an asymptotic expansion $\sum_{n=0}^{\infty} a_n \delta_n(x)$, where the function $g_n(x)$ are determined by the nature of the computational problem.

1.5. General description of the perturbation method

Perturbation method is a technique in which the solution can be expanded of a power series in a small parameter. This approximation method will be applied to obtain periodic solutions to second-order nonlinear differential equation of the form

$$\ddot{y} + y + \varepsilon F(y, \dot{y}) = 0 \tag{1.24}$$

where over dot represent derivative with respect to t, ε is a small parameter and F is assumed to be analytic nonlinear function of y and $\dot{y}$.

Let us assume that a periodic solution of the equation (1.24) can be written as a power series in terms of the small parameter ε of the form

$$y(t)=y_0(t)+\varepsilon y_1(t)+\varepsilon^2 y_2(t)+\cdots\cdots\cdots+\varepsilon^n y_n(t)+\cdots\cdots \tag{1.25}$$

where the coefficients of the powers of the parameter ε are functions of the independent variable *t*. If ε is sufficiently small, the series in equation (1.25) converges. The functions $y_n(t)$ are found by substituting equation (1.25) into the equation (1.24) and equating the coefficients of like powers of ε. This leads to an infinite set of linear non-homogeneous differential equations that may be solved recursively.

To illustrate this perturbation method, we consider a nonlinear differential equation of the form

$$\ddot{y}+y+\varepsilon y^2=0\,,\ t>0 \tag{1.26}$$

with $0<\varepsilon\le 1$.

Consider initial condition

$$y(0)=A,\quad \dot{y}(0)=0 \tag{1.27}$$

Substituting equation (1.25) into the equation (1.26), we get

$$\begin{aligned}&\left(\ddot{y}_0+\varepsilon\ddot{y}_1+\varepsilon^2\ddot{y}_2+\cdots\cdots\right)+\left(y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots\cdots\cdots\right)\\&\qquad+\varepsilon\left(y_0+\varepsilon y_1+\varepsilon^2 y_2+\cdots\cdots\cdots\right)^2=0\end{aligned} \tag{1.28}$$

Simplifying the equation (1.28), we get

$$\left(\ddot{y}_0+y_0\right)+\varepsilon\left(\ddot{y}_1+y_1+y_0^2\right)+\varepsilon^2\left(\ddot{y}_2+y_2+2y_0y_1\right)+\cdots\cdots=0 \tag{1.29}$$

Since the equation (1.29) is a power series in ε that is identically equal to zero, the coefficients of the various power of ε must be zero. Thus we obtain

$$\begin{aligned}
&\ddot{y}_0 + y_0 = 0 \\
&\ddot{y}_1 + y_1 = -y_0^2 \\
&\ddot{y}_2 + y_2 = -2y_0 y_1 \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\ddot{y}_n + y_n = F(y_0, y_1, \cdots\cdots\cdots, y_{n-1}) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{1.30}$$

where F_n is a polynomial in $y_0, y_1, \cdots\cdots\cdots, y_{n-1}$.

Under substitution the initial conditions (1.27) of the equation (1.25) translate into the following initial conditions on $y_n(t)$:

$$y_0(0) = A,\ y_i(0) = 0 \ \text{ for } i \geq 1,\ \dot{y}_k(0) = 0 \ \text{ for } k \geq 0 \tag{1.31}$$

The genearal solution of the first equation of (1.27) is

$$y_0(t) = c_1 \cos t + c_2 \sin t \tag{1.32}$$

where c_1 and c_2 are arbitrary constants.

Differentiating (1.32) with respect to t , we get

$$\dot{y}_0(t) = -c_1 \sin t + c_2 \cos t \tag{1.33}$$

Using (1.31) in (1.32) and (1.33), we get

$$c_1 = A \text{ and } c_2 = 0.$$

Thus the solution for the first equations of (1.27) becomes

$$y_0(t) = A \cos t \tag{1.34}$$

The equation for y_1 is

$$\ddot{y}_1(t) + y_1 = -y_0^2 = -(A\cos t)^2 = -\frac{A^2}{2} - \frac{A^2}{2}\cos 2t \tag{1.35}$$

This is the linear second-order non-homogeneous differential equation with constant coefficients. So the complementary function of (1.35) is

$y_1^c(t) = c_1' \cos t + c_2' \sin t$, where c_1' and c_2' are arbitrary constants

and the particular solution of this equation is

$$y_1^p(t) = -\frac{A^2}{2} + \frac{A^2}{6}\cos 2t$$

Therefore the complete solution of (1.35) is

$$y_1(t) = c_1' \cos t + c_2' \sin t - \frac{A^2}{2} + \frac{A^2}{6}\cos 2t \tag{1.36}$$

Differentiating (1.36) with respect to t , we get

$$y_1(t) = -c_1' \sin t + c_2' \cos t - \frac{A^2}{3}\sin 2t \tag{1.37}$$

The initial condition $y_1(0) = 0$ implies that $c_1' = A^2/3$ and $\dot{y}_1(0) = 0$ implies that $c_2' = 0$.

Therefore the equation (1.36) becomes

$$y_1(t) = \frac{A^2}{3}\cos t - \frac{A^2}{2} + \frac{A^2}{6}\cos 2t \tag{1.38}$$

Thus to order ε, the solution of the equation (1.26) is

$$y(t) = A\cos t + \varepsilon \frac{A^2}{6}(\cos 2t + 2\cos t - 3) \tag{1.39}$$

1.6 . Secular Terms

The calculation of section 1.5 has shown that we cannot try a solution of the form in equation (1.25) to obtain a periodic solution of the equation (1.24) if we remain only a finite number of terms. This is because the resulting approximation may be aperiodic. This lack of periodically comes about because even if y is a periodic function of t, the retention of only a finite number of terms in equation (1.25) may give a functions that is not periodic. Such a situation occurs for the expansion of the periodic function $\sin(1+\varepsilon)t$. That is

$$\sin(1+\varepsilon)t = \sin t + \varepsilon t \cos t - \frac{\varepsilon^2 t^2}{2}\sin t + \cdots\cdots\cdots \tag{1.40}$$

From the right hand side of equation (1.40), we have seen that the retentions of a finite numbers of terms gives rise to be a function that not only a periodic but also unbounded as $t \to \infty$.

Terms like $t^n \cos t$ or $t^n \sin t$ are called secular terms. Secular terms arise because the series solution given by equation (1.25) is non-uniformly valid. It is clear that the existence of such expressions, which become unbounded as $t \to \infty$, destroys the periodicity of the expression, equation (1.25), when we keep only a finite number of its terms. In applications, calculations or time considerations usually force us to consider only a small numbers of terms. Therefore, to obtain a uniformly valid solution, an approximation is needed that will eliminates the secular terms.

1.7. Poincaré–Lindstedt method

In perturbation theory, the Poincaré Lindstedt [66, 88] method or Lindstedt Poincaré method is a technique for uniformly approximating periodic solutions to ordinary differential equations, when regular perturbation approaches fail. The method removes secular terms when terms growing without bound and arising in the straightforward application of perturbation theory to weakly nonlinear problems with finite oscillatory solutions. The method is named after Henri Poincaré, and Anders Lindstedt. This approximation method will be applied for obtaining uniformly valid solutions of the nonlinear differential equation

$$\frac{d^2 y}{dt^2} + y + \varepsilon F\left(y, \frac{dy}{dt}\right) = 0 \qquad (1.41)$$

where ε is a small parameter and F is assumed to be analytic nonlinear function of y and dy/dt.

The essence of the method is to introduce a transformation of the independent variable. This transformation will allow us to avoid the occurrence of the secular terms in the perturbation series solution of the equation (1.41).

The fundamental idea comes from the astronomer Lindstedt and is based on the observation that one of the effects of the nonlinear term in equation (1.41) is to change the frequency of the system from the nonlinear value $\omega_0 = 1$ to $\omega(\varepsilon)$. To account for this change in frequency, a new variable $\theta = \omega\, t$ is introduced and both y and ω are expanded in power series of ε of the form

$$\begin{aligned} y(\theta,\varepsilon) &= y_0(\theta) + \varepsilon y_1(\theta) + \varepsilon^2 y_2(\theta) + \cdots\cdots\cdots + \varepsilon^n y_n(\theta) + \cdots\cdots \\ \omega(\varepsilon) &= 1 + \varepsilon\omega_1 + \cdots\cdots\cdots\cdots\cdots\cdots + \varepsilon^n \omega_n + \cdots\cdots \end{aligned} \qquad (1.42)$$

where ω_i are unknown constants.

If we substitute (1.42) into (1.41) and equate the coefficient of the various powers of ε equal to zero, then we obtain the equations for y_n:

$$\begin{aligned} &\ddot{y}_0 + y_0 = 0 \\ &\ddot{y}_1 + y_1 = -2\omega_1 \ddot{y}_0 - F(y_0, \dot{y}_0) \\ &\ddot{y}_2 + y_2 = -2\omega_1 \ddot{y}_1 - \left(\omega_1^2 + 2\omega_2\right)\ddot{y}_0 - F_y(y_0, \dot{y}_0) y_1 + F_{\dot{y}}(y_0, \dot{y}_0)(\omega_1 \dot{y}_0 + \dot{y}_1) \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &\ddot{y}_n + y_n = G(y_0, y_1, \cdots\cdots\cdots, y_{n-1}; \dot{y}_0, \dot{y}_1, \cdots\cdots\cdots, \dot{y}_{n-1}) \end{aligned} \qquad (1.43)$$

where

$$\dot{y} = \frac{dy}{d\theta},\ \ddot{y} = \frac{d^2 y}{d\theta^2}$$

and

$$F_y(y_0, \dot{y}_0) = \frac{\partial F(y_0, \dot{y}_0)}{\partial y}, F_{\dot{y}}(y_0, \dot{y}_0) = \frac{\partial F(y_0, \dot{y}_0)}{\partial \dot{y}}.$$

If $F_n(y, dy/dt)$ is a polynomial function of y and dy/dt, then G_n is also a polynomial function of its arguments.

Now the periodically condition in the new variable can be written as

$$y(\theta) = y(\theta + 2\pi).$$

The corresponding condition for $y_n(\theta)$ is

$$y_n(\theta) = y_n(\theta + 2\pi)$$

If the equation (1.42) is to be a periodic solution of (1.41), then the right hand side of the equations in (1.43) must contain no multiple of either $\sin\theta$ or $\cos\theta$; otherwise secular terms would arise. Thus to be able to choose any given $y_n(\theta)$ periodic involves satisfying conditions and consequently at each step of the procedure two free parameters are needed. It is easy to seen that in equation for $y_n(\theta)$ one of the constant is ω_n. The only other place from where a second constant can come is from the initial conditions on y_{n-1}. This means that the initial conditions take of the following form as

$$\begin{aligned} &y(0) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots\cdots\cdots\cdots \\ &\frac{dy(0)}{d\theta} = 0 \end{aligned} \qquad (1.44)$$

where A_n's are unknown constants. The periodicity requirement turns second equation in (1.43) into pair of equations linear in ω_1 and nonlinear in A_0. Likewise, it is easily to seen that for $n \geq 1$, the periodicity condition on $y_n(\theta)$ became a pair of linear equations for ω_n and A_{n-1}. This, at a given step in our calculations, we may determine simultaneously ω_n, A_{n-1} and $y_n(\theta)$. In this way we can determine a series solution of the form given in (1.42) for in generally infinite many solutions of (1.41).

To illustrate this method, consider the undamped, unforced nonlinear Duffing equation be

$$\frac{d^2 y}{dt^2} + y + \varepsilon y^3 = 0 \tag{1.45}$$

with initial conditions $y(0) = 1$ and $dy(0)/dt = 0$.

If we change to the new independent variable $\theta = \omega\, t$ and expand in power of ε, we obtain the following inhomogeneous linear differential equations to solve:

$$\ddot{y}_0 + y_0 = 0 \qquad , y_0(0) = 1, \dot{y}_0(0) = 0 \tag{1.46}$$

$$\ddot{y}_1 + y_1 = -2\omega_1 \ddot{y}_0 - y_0^3 \qquad , y_1(0) = \dot{y}_1(0) = 0 \tag{1.47}$$

$$\ddot{y}_2 + y_2 = -2\omega_1 \ddot{y}_1 - \left(\omega_1^2 + 2\omega_2\right)\ddot{y}_0 - 3y_0^2 y_1, y_2(0) = \dot{y}_2(0) = 0 \tag{1.48}$$

where we used $F = y^3$, $\dot{y} = \dfrac{dy}{d\theta}$ and $\ddot{y} = \dfrac{d^2 y}{d\theta^2}$. This can be used directly of the equations in (1.43) to obtain the proceeding relation.

Equation (1.46) can be solved easily, giving $y_0(\theta) = \cos\theta$. If we substitute $y_0(\theta) = \cos\theta$ into the equation (1.47), the resulting equation becomes

$$\ddot{y}_1 + y_1 = \left(2\omega_1 - \frac{3}{4}\right)\cos\theta - \frac{1}{4}\cos 3\theta \tag{1.49}$$

The secular term may be eliminated if the coefficient of $\cos\theta$ is zero. This implies that $\omega_1 = \frac{3}{8}$. Equation (1.49) can be solved subject to the initial conditions $y_1(0) = \dot{y}_1(0) = 0$.

The solution is thus $y_1(\theta) = \frac{1}{32}(-\cos\theta + \cos 3\theta)$.

Again if we substitute $y_0(\theta)$, $y_1(\theta)$ and $\omega_1 = \frac{3}{8}$ into the equation (1.49), the resulting equation becomes

$$\ddot{y}_2 + y_2 = \left(\frac{21}{128} + 2\omega_2\right)\cos\theta + \frac{3}{16}\cos 3\theta - \frac{3}{128}\cos 5\theta \tag{1.50}$$

No secular term requires $\omega_2 = -\frac{21}{256}$.

The solution for $y_2(\theta)$ subject to the initial conditions $y_2(0) = \dot{y}_2(0) = 0$ is

$$y_2(\theta) = \frac{1}{1024}(23\cos\theta - 24\cos 3\theta + \cos 5\theta).$$

Thus to the third approximation, the solution of the equation (1.41) is

$$\begin{aligned} y(\theta,\in) = \cos\theta + \frac{\varepsilon}{32}(-\cos\theta + \cos 3\theta) + \frac{\varepsilon^2}{1024} \times \\ (23\cos\theta - 24\cos 3\theta + \cos 5\theta) + O(\varepsilon^3) \end{aligned} \tag{1.51}$$

where $\theta = \omega\, t$ and $\omega(\varepsilon) = 1 + \frac{\varepsilon}{8} - \frac{21\varepsilon^2}{256} + O(\varepsilon^3)$.

1.8 Van Der Pol's Technique

Van der Pol [122] devised a technique to investigate the periodic solutions of the equation

$$\ddot{x} + \omega_0^{\,2} x = \varepsilon(1 - x^2)\dot{x} + \varepsilon k\lambda \cos\lambda t \tag{1.52}$$

where the over-dots denote differentiation with respect to t. In eq.(1.52) ε is assumed to be small, and λ (the frequency of the excitation) is assumed to differ from ω_0 (the natural frequency) by a small quantity which is of the order of ε. Under these assumptions the solution of the eq.(1.52) is assumed to have the form

$$x(t) = a_1(t)\cos\lambda t + a_2(t)\sin\lambda t \tag{1.53}$$

where $a_1(t)$ and $a_2(t)$ are assumed to be slowly varying functions of time; that is,

$\dot{a}_i = O(\varepsilon)$ and $\ddot{a}_i = O(\varepsilon^2)$.

Differentiating (1.53) two times and substituting these into eq.(1.52), neglecting terms of order higher than ε, keeping in mind that $\dot{a}_i = O(\varepsilon)$ while $\ddot{a}_i = O(\varepsilon^2)$ and equating the coefficients of $\cos\lambda t$ and $\sin\lambda t$ on both sides, we obtain

$$2\dot{a}_1 + \frac{\lambda^2 - \omega_0^{\ 2}}{\lambda} a_2 - \varepsilon a_1(1-\rho) = 0 \tag{1.54}$$

$$2d_2 - \frac{\lambda^2 - \omega_0^{\ 2}}{\lambda} a_1 - \varepsilon a_2(1-\rho) = \varepsilon k \tag{1.55}$$

where $\rho = \frac{a^2}{4} = \frac{a_1^{\ 2} + a_2^{\ 2}}{4}$ (1.56)

To analyze the periodic solutions of (1.52), we note that they correspond to the stationary solutions of the form eq. (1.54) and eq. (1.55); i.e., they correspond to the solutions of

$$2\sigma a_{20} - a_{10}(1-\rho_0) = 0 \tag{1.57}$$

$$-2\sigma a_{10} - a_{20}(1-\rho_0) = k \tag{1.58}$$

where σ is the detuning factor, and it is given by $\sigma = \frac{\lambda - \omega_0}{\varepsilon}$ (1.59)

Terms of $O(\varepsilon^2)$ in (1.54) and (1.55) have been neglected. By adding the squares of (1.57) and (1.58) and using (1.56), we obtain the frequency response equation

$$\rho_0[4\sigma^2 + (1-\rho_0)^2] = \frac{k^2}{4} \tag{1.60}$$

1.9. The Krylov-Bogoliubov Technique

Here we discuss this technique in connection with the general weakly nonlinear second-order equation of the form

$$\ddot{x} + \omega_0^{\,2} x = \varepsilon f(x, \dot{x}) \qquad (1.61)$$

where ε is a sufficiently small parameter so that the nonlinear term $\varepsilon f(x, \dot{x})$ is respectively small.

When $\varepsilon = 0$, the equation reduces to linear, then the solution of (1.61) can be written as

$$x = a\cos(\omega_0 t + \theta) \qquad (1.62)$$

where a and θ are constants. To determine an approximate solution to eq.(1.61) for ε small but different from zero. Krylov and Bogoliubov [63] assumed that the solution is still given by eq. (1.62) but with time varying a and θ, and subject to the condition

$$\dot{x} = -a\omega_0 \sin\phi, \quad \phi = \omega_0 t + \theta \qquad (1.63)$$

If $\varepsilon = 0$ but sufficiently small, one might reasonably assume that the nonlinear equation (1.61) also has a solution of the form Eq. (1.62), provided that a and θ now be regarded as functions of t rather then constants. This is precisely what we shall do in applying the Krylov-Bogoliubov [63] Technique. That is, we assume a solution of Eq. (1.62) of the form

$$x = a(t)\cos(\omega_0 t + \theta(t)) \qquad (1.64)$$

Thus, this technique is similar to Vander Pol's [122] technique which was discussed in the previous section. The only difference is in the first term.

Differentiating eq. (1.62) with respect to t gives $\dot{x} = -a\omega_0 \sin\phi + \dot{a}\cos\phi - a\dot{\theta}\sin\phi$.

Hence $\dot{a}\cos\phi - a\dot{\theta}\sin\phi = 0$ (1.65)

on account of (1.63). Differentiating eq. (1.63) with respect to t, we obtain

$\ddot{x} = -a\omega_0^{\,2}\cos\phi - \omega_0 \dot{a}\sin\phi - a\omega_0\dot{\theta}\cos\phi$. Substituting this expression into (1.61) and using (1.62), we obtain

$$\omega_0 \dot{a}\sin\phi + a\omega_0\dot{\theta}\cos\phi = -\varepsilon f[a\cos\phi, -a\omega_0 \sin\phi] \qquad (1.66)$$

Solving (1.65) and (1.66) for $\dot{a}$ and $\dot{\theta}$ yields

$$\dot{a} = -\frac{\varepsilon}{\omega_0}\sin\phi\, f[a\cos\phi,\ -a\omega_0\sin\phi] \tag{1.67}$$

$$\dot{\theta} = -\frac{\varepsilon}{a\omega_0}\cos\phi\, f[a\cos\phi,\ -a\omega_0\sin\phi] \tag{1.68}$$

Thus the original second-order differential equation (1.61) has been replaced by the two first-order differential equations (1.67) and (1.68) for the amplitude a and the phase θ.

To solve (1.67) and (1.68) , we note that the right-hand sides of these equations are periodic with respect to the variable ϕ, hence $\dot{a} = O(\varepsilon)$ and $\dot{\theta} = O(\varepsilon)$. Thus a and θ are slowly varying functions of time because ε is small; hence they change very little during the time $T = 2\pi/\omega_0$ (the period of the terms on the right-hand sides of these equations). Averaging (1.67) and (1.68) over the interval $[t, t+T]$, during which a and θ can be taken to be constants on the right-hand side of these equations, we obtain

$$\dot{a} = -\frac{\varepsilon}{2\omega_0} f_1(a) \tag{1.69}$$

$$\dot{\theta} = -\frac{\varepsilon}{2a\omega_0} g_1(a) \tag{1.70}$$

where $f_1(a) = \frac{2}{T}\int_0^T \sin\phi\, f[a\cos\phi, -a\omega_0\sin\phi]\,dt$

$$= \frac{1}{\pi}\int_0^{2\pi} \sin\phi\, f[a\cos\phi, -a\omega_0\sin\phi]\,d\phi \tag{1.71}$$

$$g_1(a) = \frac{1}{\pi}\int_0^{2\pi} \cos\phi\, f[a\cos\phi, -a\omega_0\sin\phi]\,d\phi \tag{1.72}$$

Note that f_1 and g_1 are simply two coefficients of the Fourier series expansion of f.

1.9.1 Example:

As an example, let us consider Duffing's equation $\ddot{x} + x = -\varepsilon x^3$, $x(0) = a$, $\dot{x}(0) = 0$ in which

$$f(x, \ddot{x}) = -x^3 \tag{1.73}$$

Hence according to the above process, we have

$$f_1(a) = 0, \quad g_1(a) = -\frac{3}{4}a^3 \tag{1.74}$$

Consequently, a = a constant from (1.69), and $\theta = \frac{3}{8}\varepsilon\frac{a^2}{\omega_0}t + \theta_0$ …………..(1.75), from (1.70). Therefore, to first approximation

$$u = a\cos\omega_0[1 + \frac{3}{8}\varepsilon\frac{a^2}{\omega_0{}^2}]t + O(\varepsilon) \tag{1.76}$$

1.10. The Krylov-Bogoliubov-Mitropolski Technique

In the course of refinement of the first approximation for $\ddot{x} + \omega_0{}^2 x = \varepsilon f(x, \dot{x})$, (1.77)

Krylov and Bogoliubov [63] developed a technique for determining the solution to any approximation. This technique has been amplified and justified by Bogoliubov and Mitropolski [22] and extended to non-stationary vibrations by Mitropolski [73].They assumed an asymptotic expansion of the form

$$x = a\cos\psi + \sum_{n=1}^{N}\varepsilon^n x_n(a, \psi) + O(\varepsilon^{N+1}) \tag{1.78}$$

where each x_n is a periodic function of ψ with a period 2π, and a and ψ are assumed to vary with time according to

$$\dot{a} = \sum_{n=1}^{N}\varepsilon^n A_n(a) + O(\varepsilon^{N+1}) \tag{1.79}$$

and $\dot{\psi} = \omega_0 + \sum_{n=1}^{N}\varepsilon^n \psi_n(a) + O(\varepsilon^{N+1})$ (1.80)

where the function x_n, A_n and ψ_n are chosen such that (1.78) through (1.80) satisfy the differential equation (1.63). In order to uniquely determine A_n and ψ_n, we require that no contains $\cos\psi$. The derivatives are transformed according to

$$\frac{d}{dt} = \frac{da}{dt}\frac{\partial}{\partial a} + \frac{d\psi}{dt}\frac{\partial}{\partial\psi}$$

$$\frac{d^2}{dt^2} = \left(\frac{da}{dt}\right)^2\frac{\partial^2}{\partial a^2} + \frac{d^2a}{dt^2}\frac{\partial}{\partial a} + 2\frac{da}{dt}\frac{d\psi}{dt}\frac{\partial^2}{\partial a\partial\psi} + \left(\frac{d\psi}{dt}\right)^2\frac{\partial^2}{\partial\psi^2} + \frac{d^2\psi}{dt^2}\frac{\partial}{\partial\psi}$$

$$\frac{d^2a}{dt^2} = \frac{d}{dt}\left(\frac{da}{dt}\right) = \frac{da}{dt}\frac{d}{da}\left(\frac{da}{dt}\right) = \frac{da}{dt}\sum_{n=1}^{N}\varepsilon^n \frac{dA_n}{da} = \varepsilon^2 A_1 \frac{dA_1}{da} + O(\varepsilon^3)$$

$$\frac{d^2\psi}{dt^2} = \frac{d}{dt}\left(\frac{d\psi}{dt}\right) = \frac{da}{dt}\frac{d}{da}\left(\frac{d\psi}{dt}\right) = \frac{da}{dt}\sum_{n=1}^{N}\varepsilon^n \frac{d\psi_n}{da} = \varepsilon^2 A_1 \frac{d\psi_1}{da} + O(\varepsilon^3)$$

For convenience the calculation is omitted.

1.11 Conclusion

We have discussed some method. Among these the KBM is the most accurate and suitable for approximate solutions of nonlinear problems. Thus we would like to use this method for my dissertation.

Chapter-Two

Perturbation Solutions for Fifth Order Nonlinear System with Damping Effects

2.1 Introduction

The world around us is inherently nonlinear and nonlinear differential equations are widely used as models to describe the complex physical phenomena. The approximate solutions of nonlinear differential equations play a vital role in nonlinear science and engineering. Nonlinear Physical Science focuses on recent advances of fundamental theories and principles, analytical and symbolic approaches, as well as computational techniques with engineering applications. Topics of interest in nonlinear physical Science include but are not limited to new findings and discoveries in nonlinear physics and mathematics, nonlinearity, complexity and mathematical structures in nonlinear physics, nonlinear phenomena and observations in nature and engineering, lie group analysis, stability, bifurcation, chaos and fractals in physical science and engineering, nonlinear chemical and biological physics.

We investigate the master nonlinear fifth order partial differential equation that governs the evolution of shear-free spherically symmetric charged fluids. By making dimensionless the fifth order partial differential equations can be converted to fifth order ordinary differential equations. Some of the converted equations reduce in the forms of nonlinear differential equation with damping effects. The common methods for finding analytical approximate solutions to these nonlinear oscillator equations are the perturbation methods. The Krylov-Bogoliubov-Mitropolskii [22, 63] etc. are well known perturbation methods to obtain analytical approximate solutions of non-linear systems and is the widely used technique to obtain analytical approximate solution of non-linear systems with damping effects. Krylov and Bogoliubov (KB) [63] originally developed a perturbation method for obtaining periodic solutions was amplified and justified by Bogoliubov and Mitropolskii [22] and the KB method has been extended by Kruskal [61]. Later Popov [91] and Mendelson [69] extended the method for damped nonlinear oscillations. Volosov [123, 124], Zebreiko [130] also obtained higher order approximations. Most probably, Osiniskii [79] first extended the KBM method to a third order nonlinear differential equation. Making use of KBM method, Bojadziev [25] has invastigated nonlinear damped oscillatory systems with small time lag. Bojadziev [30] has also studied the damped forced nonlinear vibrations with small time delay. Bojadziev [31] applied the Krylov- Bogoliubov-Mitropolskii method to models of Population dynamics. Bojadziev and Chan [32] has found asymptotic solutions of differential equations with delay in population dynamics. Bojadziev [33] presented a damped oscillating processes in Biological and Biochemical systems. Shamsul and Sattar [109] presented a unified KBM method for solving third order nonlinear systems. Later, Akbar et al. [6] extended the method presented in [4] for the damped oscillatory systems. Akbar and Siddique [9] presented a method to obtain solutions of fifth order weakly nonlinear oscillatory systems.

In this chapter, we employ the perturbation method to obtain analytical approximate solutions. Even, many engineering problems and physical phenomena arise in the nature of fifth degrees of freedom are oscillatory and their governing equations are fifth order nonlinear differential systems with damping effects. For this reason, we have extended the KBM method, an approximate technique to obtain the analytical solutions of fifth order nonlinear oscillatory systems with damping effects. Figures are provided to compare the solutions obtained by the presented method with the corresponding numerical solutions obtained by the fourth order Runge- Kutta method.

2.2. The Method

Since by making dimensionless the fifth order partial differential equations can be converted to fifth order ordinary differential equations or in the couple system of pendulum lead to higher degrees of freedom, let us consider a governing equation of a fifth order nonlinear damped oscillatory system,

$$\frac{d^5 x}{dt^5} + \sum_{i=1}^{4} c_i \frac{d^i x}{dt^i} + c_5 x = -\varepsilon f(x,t) \tag{2.1}$$

where ε is a small parameter, $f(x,t)$ is the given nonlinear function, $c_i\,; i = 1,2,..,5$ are the characteristic parameters of the system defined by $c_1 = \sum_{i=1}^{5} \lambda_i$, $c_2 = \sum_{\substack{i,j=1 \\ i \neq j}}^{5} \lambda_i \lambda_j$, $c_3 = \sum_{\substack{i,j,k=1 \\ i \neq j \neq k}}^{5} \lambda_i \lambda_j \lambda_k$, $c_4 = \sum_{\substack{i,j,k,l=1 \\ i \neq j \neq k \neq l}}^{5} \lambda_i \lambda_j \lambda_k \lambda_l$ and $c_5 = \prod_{i=1}^{5} \lambda_i$ where $-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4, -\lambda_5$ are the eigenvalues of the unperturbed equation of (2.1).

When $\varepsilon = 0$ i.e, the equation (2.1) lead to linear or unperturbed equation and then the solution is

$$x(t,0) = \sum_{j=1}^{5} a_{j,0} e^{-\lambda_j t}, \tag{2.2}$$

where $a_{j,0}$, $j = 1,2....,5$ are arbitrary constants.

When $\varepsilon \neq 0$, the powerful perturbation approximant solutions will be investigated in which amplitude and phase are not arbitrary but time varying functions. First, we will discuss the construction of approximants for functions and polynomials. Next, we will explore the implementation of approximants with initial value problems. Polynomials are frequently used to approximate power series. However, polynomials tend to exhibit oscillations that may produce an approximation error bounds and this makes the singularities. To overcome these difficulties, the Taylor series is best manipulated by our certain approximants for approximations. We seek a solution in accordance with Shamsul [108] or Murty and Deekshatulu [76] or the KBM [22, 63] method, of the form

$$x(t,\varepsilon)=\sum_{j=1}^{5}a_j(t)\,e^{-\lambda_j t}+\varepsilon\,u_1(a_1,a_2,\cdots,a_5,t)+\cdots \tag{2.3}$$

where each $a_j;\, j=1,2,\cdots,5$, satisfies the equations

$$\frac{da_j(t)}{dt}=\varepsilon A_j(a_1,a_2,\cdots,a_5,t)+\cdots \tag{2.4}$$

Confining our concentration to the first some terms $1,2,\cdots,m$ in the series expansions of equations (2.3) and (2.4), we calculate the functions u_1 and $A_j;\, j=1,2,\cdots,5$ such that $a_j;\, j=1,2,\cdots,5$, appearing in eq. (2.3) and eq. (2.4), satisfy the differential equation (2.1) with an accuracy of ε^{m+1}. Though the solution can be obtained up to the accuracy of any order of approximation, but to avoid the rapidly-growing algebraic complexity for the derivation, the solution, in general, confining to first order [74]. In order to determine these unknown functions, it is assumed that the function u_1 exclude fundamental terms which are included in the series expansion (2.3) at order ε^0.

Differentiating $x(t,\varepsilon)$ five times with respect to t and substituting $x(t,\varepsilon)$ and their derivatives in the eq. (2.1), using the relations in eq. (2.4) and equating the coefficients of ε, we obtain

$$\prod_{j=1}^{5}\left(\frac{d}{dt}+\lambda_j\right)u_1+\sum_{j=1}^{5}e^{-\lambda_j t}\left(\prod_{k=1,\,j\neq k}^{5}\left(\frac{d}{dt}-\lambda_j+\lambda_k\right)\right)A_j=-f^{(0)}(a_1,a_2,\cdots,a_5,t) \tag{2.5}$$

where $f^{(0)}=f(x_0)$ and $x_0=\sum_{j=1}^{5}a_j(t)\,e^{-\lambda_j t}$

The function $f^{(0)}$ can be expanded in a Taylor series (see Murty and Deekshatulu [76] for details) as:

$$f^{(0)}=\sum_{m_1=-\infty\cdots m_5=-\infty}^{\infty.....\infty}F_{m_1,\cdots m_5}\sum_{i=1}^{5}a_i^{m_i}e^{-(m_1\lambda_1+m_2\lambda_2+\cdots+m_5\lambda_5)t}$$

To obtain the solution of eq.(2.1), it has been proposed that u_1 exclude the fundamental terms. To do this, we have to separated the eq.(2.5) into six equations for unknown functions u_1 and $A_j; j = 1,2,.....,5$ (see [108] for details).

Substituting the functional values and equating the coefficients of $e^{-\lambda_j t}; j = 1,2,...,5$, we obtain

$$e^{-\lambda_1 t}\sum_{i=2}^{5}(\frac{d}{dt} - \lambda_1 + \lambda_i)A_1 = -\sum_{\substack{m_1=-\infty,\cdots,m_5=-\infty \\ m_3=m_4,\, m_1=m_2+1}}^{\infty.....\infty} F_{m_1,\cdots m_5}\sum_{i=1}^{5} a_i^{m_i} e^{-(m_1\lambda_1 + m_2\lambda_2 + \cdots + m_5\lambda_5)t} \tag{2.6}$$

$$e^{-\lambda_2 t}\sum_{i=1, i\neq 2}^{5}(\frac{d}{dt} - \lambda_2 + \lambda_i)A_2 = -\sum_{\substack{m_1=-\infty,\cdots,m_5=-\infty \\ m_3=m_4,\, m_1=m_2-1}}^{\infty.....\infty} F_{m_1,\cdots m_5}\sum_{i=1}^{5} a_i^{m_i} e^{-(m_1\lambda_1 + m_2\lambda_2 + \cdots + m_5\lambda_5)t} \tag{2.7}$$

$$e^{-\lambda_3 t}\sum_{i=1, i\neq 3}^{5}(\frac{d}{dt} - \lambda_3 + \lambda_i)A_3 = -\sum_{\substack{m_1=-\infty,\cdots,m_5=-\infty \\ m_1=m_2,\, m_3=m_4+1}}^{\infty.....\infty} F_{m_1,\cdots m_5}\sum_{i=1}^{5} a_i^{m_i} e^{-(m_1\lambda_1 + m_2\lambda_2 + \cdots + m_5\lambda_5)t} \tag{2.8}$$

$$e^{-\lambda_4 t}\sum_{i=1, i\neq 4}^{5}(\frac{d}{dt} - \lambda_4 + \lambda_i)A_4 = -\sum_{\substack{m_1=-\infty, m_5=-\infty \\ {}_1=m_2,\, m_3=m_4-1}}^{\infty.....\infty} F_{m_1,\cdots m_5}\sum_{i=1}^{5} a_i^{m_i} e^{-(m_1\lambda_1 + m_2\lambda_2 + ... + m_5\lambda_5)t} \tag{2.9}$$

$$e^{-\lambda_5 t}\sum_{i=1}^{4}(\frac{d}{dt} - \lambda_5 + \lambda_i)A_5 = -\sum_{\substack{m_1=-\infty\cdots m_5=-\infty \\ m_1=m_2,\, m_3=m_4}}^{\infty.....\infty} F_{m_1,\cdots m_5}\sum_{i=1}^{5} a_i^{m_i} e^{-(m_1\lambda_1 + m_2\lambda_2 + ... + m_5\lambda_5)t} \tag{2.10}$$

and

$$\sum_{i=1}^{5}(\frac{d}{dt} + \lambda_i)u_1 = -\sum_{m_1=-\infty\cdots m_5=-\infty.}^{\infty.....\infty} F_{m_1,\cdots m_5}\sum_{i=1}^{5} a_i^{m_i} e^{-(m_1\lambda_1 + m_2\lambda_2 + ... + m_5\lambda_5)t} \tag{2.11}$$

where u_1 avoid the terms for $m_1 = m_2 \pm 1,\ m_3 = m_4 \pm 1,\ m_1 = m_2,\ m_3 = m_4$.

Solving Eq. (2.6) to Eq. (2.11), we obtain $A_1, A_2,..., A_5$ and u_1.

For the suitable form of the result we can transform equation (2.3) to the exact formal KBM [4, 6, 9, 108] solution by inserting $a_1 = \frac{a}{2}e^{i\varphi_1}$, $a_2 = \frac{a}{2}e^{-i\varphi_1}$, $a_3 = \frac{b}{2}e^{i\varphi_2}$ and $a_4 = \frac{b}{2}e^{-i\varphi_2}$. Herein a, b are amplitudes and φ_1, φ_2 are phase variables which are time dependent i. e, slowly varying function of time.

2.3. Example

As an example of the above procedure, we are going to consider the Duffing type equation of fifth order

$$\frac{d^5x}{dt^5}+\sum_{i=1}^{4}c_i\frac{d^ix}{dt^i}+c_5x=-\varepsilon x^3 \tag{2.12}$$

here $f(x,t)=x^3$.

We have $f^{(0)}=(\sum_{i=1}^{5}a_ie^{-\lambda_it})^3$

or

$$\begin{aligned}f^{(0)}&=a_1^3e^{-3\lambda_1t}+3a_1^2a_2e^{-(2\lambda_1+\lambda_2)t}+3a_1a_2^2e^{-(\lambda_1+2\lambda_2)t}+a_2^3e^{-3\lambda_2t}\\&+3a_1^2a_3e^{-(2\lambda_1+\lambda_3)t}+3a_1^2a_4e^{-(2\lambda_1+\lambda_4)t}+3a_1^2a_5e^{-(2\lambda_1+\lambda_5)t}\\&+6a_1a_2a_3e^{-(\lambda_1+\lambda_2+\lambda_3)t}+6a_1a_2a_4e^{-(\lambda_1+\lambda_2+\lambda_4)t}+6a_1a_2a_5e^{-(\lambda_1+\lambda_2+\lambda_5)t}\\&+3a_2^2a_3e^{-(2\lambda_2+\lambda_3)t}+3a_2^2a_4e^{-(2\lambda_2+\lambda_4)t}+3a_2^2a_5e^{-(2\lambda_2+\lambda_5)t}\\&+3a_1a_3^2e^{-(\lambda_1+2\lambda_3)t}+3a_1a_4^2e^{-(\lambda_1+2\lambda_4)t}+3a_1a_5^2e^{-(\lambda_1+2\lambda_5)t}\\&+6a_1a_3a_4e^{-(\lambda_1+\lambda_3+\lambda_4)t}+6a_1a_3a_5e^{-(\lambda_1+\lambda_3+\lambda_5)t}+6a_1a_4a_5e^{-(\lambda_1+\lambda_4+\lambda_5)t}\\&+3a_2a_3^2e^{-(\lambda_2+2\lambda_3)t}+3a_2a_4^2e^{-(\lambda_2+2\lambda_4)t}+3a_2a_5^2e^{-(\lambda_2+2\lambda_5)t}\\&+6a_2a_3a_4e^{-(\lambda_2+\lambda_3+\lambda_4)t}+6a_2a_4a_5e^{-(\lambda_2+\lambda_4+\lambda_5)t}+6a_2a_3a_5e^{-(\lambda_2+\lambda_3+\lambda_5)t}\\&+a_3^3e^{-3\lambda_3t}+3a_3^2a_4e^{-(2\lambda_3+\lambda_4)t}+3a_3a_4^2e^{-(\lambda_3+2\lambda_4)t}+a_4^3e^{-3\lambda_4t}\\&+3a_3^2a_5e^{-(2\lambda_3+\lambda_5)t}+6a_3a_4a_5e^{-(\lambda_3+\lambda_4+\lambda_5)t}+3a_4^2a_5e^{-(2\lambda_4+\lambda_5)t}\\&+3a_3a_5^2e^{-(\lambda_3+2\lambda_5)t}+3a_4a_5^2e^{-(\lambda_4+2\lambda_5)t}+3a_5^3e^{-3\lambda_5t}\end{aligned} \tag{2.13}$$

Thus the equations (2.6) to (2.11) takes the form

$$e^{-\lambda_1t}\sum_{i=2}^{5}(\frac{d}{dt}-\lambda_1+\lambda_i)A_1=-3a_1^2a_2e^{-(2\lambda_1+\lambda_2)t}-6a_1a_3a_4e^{-(\lambda_1+\lambda_3+\lambda_4)t} \tag{2.14}$$

$$e^{-\lambda_2t}\sum_{i=1,i\neq2}^{5}(\frac{d}{dt}-\lambda_2+\lambda_i)A_2=-3a_1a_2^2e^{-(\lambda_1+2\lambda_2)t}-6a_2a_3a_4e^{-(\lambda_2+\lambda_3+\lambda_4)t} \tag{2.15}$$

$$e^{-\lambda_3t}\sum_{i=1,i\neq3}^{5}(\frac{d}{dt}-\lambda_3+\lambda_i)A_3=-3a_3^2a_4e^{-(2\lambda_3+\lambda_4)t}-6a_1a_2a_3e^{-(\lambda_1+\lambda_2+\lambda_3)t} \tag{2.16}$$

$$e^{-\lambda_4t}\sum_{i=1,i\neq4}^{5}(\frac{d}{dt}-\lambda_4+\lambda_i)A_4=-3a_3a_4^2e^{-(\lambda_3+2\lambda_4)t}-6a_1a_2a_4e^{-(\lambda_1+\lambda_2+\lambda_4)t} \tag{2.17}$$

$$e^{-\lambda_5t}\sum_{i=1}^{4}(\frac{d}{dt}-\lambda_5+\lambda_i)A_5=-6a_1a_2a_5e^{-(\lambda_1+\lambda_2+\lambda_5)t}-6a_3a_4a_5e^{-(\lambda_3+\lambda_4+\lambda_5)t} \tag{2.18}$$

and

$$\sum_{i=1}^{5}(\frac{d}{dt}+\lambda_i)u_1=-(a_1^3e^{-3\lambda_1 t}+a_2^3e^{-3\lambda_2 t}+3a_1^2a_3e^{-(2\lambda_1+\lambda_3)t}+3a_1^2a_4e^{-(2\lambda_1+\lambda_4)t}+3a_1^2a_5e^{-(2\lambda_1+\lambda_5)t}$$
$$+3a_2^2a_3e^{-(2\lambda_2+\lambda_3)t}+3a_2^2a_4e^{-(2\lambda_2+\lambda_4)t}+3a_2^2a_5e^{-(2\lambda_2+\lambda_5)t}$$
$$+3a_1a_3^2e^{-(\lambda_1+2\lambda_3)t}+3a_1a_4^2e^{-(\lambda_1+2\lambda_4)t}+3a_1a_5^2e^{-(\lambda_1+2\lambda_5)t}$$
$$+6a_1a_3a_5e^{-(\lambda_1+\lambda_3+\lambda_5)t}+6a_1a_4a_5e^{-(\lambda_1+\lambda_4+\lambda_5)t}$$
$$+3a_2a_3^2e^{-(\lambda_2+2\lambda_3)t}+3a_2a_4^2e^{-(\lambda_2+2\lambda_4)t}+3a_2a_5^2e^{-(\lambda_2+2\lambda_5)t}$$
$$+6a_2a_4a_5e^{-(\lambda_2+\lambda_4+\lambda_5)t}+6a_2a_3a_5e^{-(\lambda_2+\lambda_3+\lambda_5)t}$$
$$+a_3^3e^{-3\lambda_3 t}+a_4^3e^{-3\lambda_4 t}+3a_3^2a_5e^{-(2\lambda_3+\lambda_5)t}+3a_4^2a_5e^{-(2\lambda_4+\lambda_5)t}$$
$$+3a_3a_5^2e^{-(\lambda_3+2\lambda_5)t}+3a_4a_5^2e^{-(\lambda_4+2\lambda_5)t}+3a_5^3e^{-3\lambda_5 t}) \tag{2.19}$$

Again solving the equations (2.14) to (2.18) and inserting $\lambda_1=k_1-i\omega_1$, $\lambda_2=k_1+i\omega_1$, $\lambda_3=k_2-i\omega_2$, $\lambda_4=k_2+i\omega_2$ and $\lambda_5=\xi$, we obtain

$$A_1=-\frac{3a_1^2a_2e^{-2k_1t}}{2(k_1-i\omega_1)\{(3k_1-k_2)-i(\omega_1-\omega_2)\}\{(3k_1-k_2)-i(\omega_1+\omega_2)\}\{(3k_1-\xi)-i\omega_1\}}$$
$$-\frac{6a_1a_3a_4e^{-2k_2t}}{2(k_2-i\omega_1)\{(k_1+k_2)-i(\omega_1-\omega_2)\}\{(k_1+k_2)-i(\omega_1+\omega_2)\}\{(k_1+2k_2-\xi)-i\omega_1\}}$$

$$A_2=-\frac{3a_1a_2^2e^{-2k_1t}}{2(k_1+i\omega_1)\{(3k_1-k_2)+i(\omega_1+\omega_2)\}\{(3k_1-k_2)+i(\omega_1-\omega_2)\}\{(3k_1-\xi)+i\omega_1\}}$$
$$-\frac{6a_2a_3a_4e^{-2k_2t}}{2(k_2+i\omega_1)\{(k_1+k_2)+i(\omega_1+\omega_2)\}\{(k_1+k_2)+i(\omega_1-\omega_2)\}\{(k_1+2k_2-\xi)+i\omega_1\}}$$

$$A_3=-\frac{3a_3^2a_4e^{-2k_2t}}{2(k_2-i\omega_2)\{(3k_2-k_1)+i(\omega_1-\omega_2)\}\{(3k_2-k_1)-i(\omega_1+\omega_2)\}\{(3k_2-\xi)-i\omega_2\}}$$
$$-\frac{6a_1a_3a_4e^{-2k_1t}}{2(k_1-i\omega_2)\{(k_1+k_2)+i(\omega_1-\omega_2)\}\{(k_1+k_2)-i(\omega_1+\omega_2)\}\{(2k_1+k_2-\xi)-i\omega_2\}}$$

$$A_4=-\frac{3a_3a_4^2e^{-2k_2t}}{2(k_2+i\omega_2)\{(3k_2-k_1)+i(\omega_1+\omega_2)\}\{(3k_2-k_1)-i(\omega_1-\omega_2)\}\{(3k_2-\xi)+i\omega_2\}}$$
$$-\frac{6a_1a_2a_4e^{-2k_1t}}{2(k_1+i\omega_2)\{(k_1+k_2)+i(\omega_1+\omega_2)\}\{(k_1+k_2)-i(\omega_1-\omega_2)\}\{(2k_1+k_2-\xi)+i\omega_2\}}$$

$$A_5=-\frac{6a_1a_2a_5e^{-2k_1t}}{\{(k_1+\xi)^2+\omega_1^2\}\{(2k_1-k_2+\xi)^2+\omega_2^2\}}-\frac{6a_3a_4a_5e^{-2k_2t}}{\{(k_2+\xi)^2+\omega_2^2\}\{(2k_2-k_1+\xi)^2+\omega_2^2\}}$$

Now, inserting $A_j; j=1,2,..,5$ in the equations (2.4) and using

$a_1 = \frac{1}{2} a e^{i\varphi_1}, a_2 = \frac{1}{2} a e^{-i\varphi_1}$, $a_3 = \frac{1}{2} b e^{i\varphi_2}$, $a_4 = \frac{1}{2} b e^{-i\varphi_2}$ *and* $a_5 = c$, to convert actual form of KBM solution, we obtain the differential equations for amplitudes and phases are

$$\frac{da}{dt} = \varepsilon(l_1 a^3 e^{-2k_1 t} + l_2 a b^2 e^{-2k_2 t}) \qquad \frac{db}{dt} = \varepsilon(n_1 b^3 e^{-2k_2 t} + n_2 a^2 b e^{-2k_1 t})$$

$$\frac{d\varphi_1}{dt} = \varepsilon(m_1 a^2 e^{-2k_1 t} + m_2 b^2 e^{-2k_2 t}) \qquad \frac{d\varphi_2}{dt} = \varepsilon(p_1 b^2 e^{-2k_2 t} + p_2 a^2 e^{-2k_1 t})$$

and

$$\frac{dc}{dt} = \varepsilon(q_1 a^2 c e^{-2k_1 t} + q_2 b^2 c e^{-2k_2 t}) \tag{2.20}$$

where

$$l_1 = -\frac{3}{8}\left\{ \frac{\begin{array}{c}(3k_1^2 - k_1 k_2 - \omega_1^2 - \omega_1\omega_2)(9k_1^2 - 3k_1k_2 - 3k_1\xi + k_2\xi - \omega_1^2 + \omega_1\omega_2) - \\ (4k_1\omega_1 + k_1\omega_2 - k_2\omega_1)(6k_1\omega_1 - k_2\omega_1 - 3k_1\omega_2 - \omega_1\xi + \omega_2\xi)\end{array}}{(k_1^2 + \omega_1^2)\{(3k_1 - k_2)^2 + (\omega_1 + \omega_2)^2\}\{(3k_1 - k_2)^2 + (\omega_1 - \omega_2)^2\}\{(3k_1 - \xi)^2 + \omega_1^2\}} \right\}$$

$$l_2 = -\frac{3}{4}\left\{ \frac{\begin{array}{c}(k_1k_2 + k_2^2 - \omega_1^2 + \omega_1\omega_2)\{(k_1 + k_2)(k_1 + 2k_2 - \xi) - \omega_1^2 + \omega_1\omega_2\} - \\ (2k_2\omega_1 - k_2\omega_2 + k_1\omega_1)(2k_1\omega_1 + 3k_2\omega_1 + k_1\omega_2 + 2k_2\omega_2 - \omega_1\xi - \omega_2\xi)\end{array}}{(k_2^2 + \omega_1^2)\{(k_1 + k_2)^2 + (\omega_1 - \omega_2)^2\}\{(k_1 + k_2)^2 + (\omega_1 + \omega_2)^2\}\{(k_1 + 2k_2 - \xi)^2 + \omega_1^2\}} \right\}$$

$$m_1 = -\frac{3}{8}\left\{ \frac{\begin{array}{c}(3k_2^2 - k_1k_2 - \omega_2^2 - \omega_1\omega_2)(9k_2^2 - 3k_1k_2 - 3k_2\xi + k_1\xi - \omega_2^2 + \omega_1\omega_2) - \\ (4k_2\omega_2 - k_1\omega_2 + k_2\omega_1)(6k_2\omega_2 - k_1\omega_2 - 3k_2\omega_1 + \omega_1\xi - \omega_2\xi)\end{array}}{(k_2^2 + \omega_2^2)\{(3k_2 - k_1)^2 + (\omega_1 - \omega_2)^2\}\{(3k_2 - k_1)^2 + (\omega_1 + \omega_2)^2\}\{(3k_2 - \xi)^2 + \omega_2^2\}} \right\}$$

$$m_2 = -\frac{3}{4}\left\{ \frac{\begin{array}{c}(k_1^2 + k_1k_2 - \omega_2^2 - \omega_1\omega_2)\{(k_1 + k_2)(2k_1 + k_2 - \xi) - \omega_2^2 + \omega_1\omega_2\} - \\ (k_1\omega_1 + 2k_1\omega_2 + k_2\omega_2)(2k_2\omega_2 + 3k_1\omega_2 - 2k_1\omega_1 - k_2\omega_1 + \omega_1\xi - \omega_2\xi)\end{array}}{(k_1^2 + \omega_2^2)\{(k_2 + k_1)^2 + (\omega_1 - \omega_2)^2\}\{(k_2 + k_1)^2 + (\omega_1 + \omega_2)^2\}\{(2k_1 + k_2 - \xi)^2 + \omega_2^2\}} \right\}$$

$$n_1 = -\frac{3}{8}\left\{ \frac{\begin{array}{c}(3k_1^2 - k_1k_2 - \omega_1^2 - \omega_1\omega_2)(6k_1\omega_1 - k_2\omega_1 - 3k_1\omega_2 - \omega_1\xi + \omega_2\xi) + \\ (4k_1\omega_1 + k_1\omega_2 - k_2\omega_1)(9k_1^2 - 3k_1k_2 - 3k_1\xi + k_2\xi - \omega_1^2 + \omega_1\omega_2)\end{array}}{(k_1^2 + \omega_1^2)\{(3k_1 - k_2)^2 + (\omega_1 + \omega_2)^2\}\{(3k_1 - k_2)^2 + (\omega_1 - \omega_2)^2\}\{(3k_1 - \xi)^2 + \omega_1^2\}} \right\}$$

$$n_2 = -\frac{3}{4}\left\{ \frac{\begin{array}{c}(k_1k_2 + k_2^2 - \omega_1^2 + \omega_1\omega_2)(2k_1\omega_1 + 3k_2\omega_1 + k_1\omega_2 + 2k_2\omega_2 - \omega_1\xi - \omega_2\xi) - \\ (2k_2\omega_1 - k_2\omega_2 + k_1\omega_1)\{(k_1 + k_2)(k_1 + 2k_2 - \xi) - \omega_1^2 - \omega_1\omega_2\}\end{array}}{(k_2^2 + \omega_1^2)\{(k_1 + k_2)^2 + (\omega_1 - \omega_2)^2\}\{(k_1 + k_2)^2 + (\omega_1 + \omega_2)^2\}\{(k_1 + 2k_2 - \xi)^2 + \omega_1^2\}} \right\}$$

$$p_1 = -\frac{3}{8}\left\{ \frac{\begin{array}{c}(3k_2^2 - k_1k_2 - \omega_2^2 - \omega_1\omega_2)(6k_2\omega_2 - k_1\omega_2 - 3k_2\omega_1 + \omega_1\xi - \omega_2\xi) - \\ (4k_2\omega_2 - k_1\omega_2 + k_2\omega_1)(9k_2^2 - 3k_1k_2 - 3k_2\xi + k_1\xi - \omega_2^2 + \omega_1\omega_2)\end{array}}{(k_2^2 + \omega_2^2)\{(3k_2 - k_1)^2 + (\omega_1 - \omega_2)^2\}\{(3k_2 - k_1)^2 + (\omega_1 + \omega_2)^2\}\{(3k_2 - \xi)^2 + \omega_2^2\}} \right\}$$

$$p_2 = -\frac{3}{4}\left\{\frac{\begin{array}{c}(k_1^2 + k_1 k_2 - \omega_2^2 - \omega_1\omega_2)(2k_2\omega_2 + 3k_1\omega_2 - 2k_1\omega_1 - k_2\omega_1 + \omega_1\xi - \omega_2\xi) \\ -(k_1\omega_1 + 2k_1\omega_2 + k_2\omega_2)\{(k_1 + k_2)(2k_1 + k_2 - \xi) - \omega_2^2 + \omega_1\omega_2\}\end{array}}{(k_1^2 + \omega_2^2)\{(k_2 + k_1)^2 + (\omega_1 - \omega_2)^2\}\{(k_2 + k_1)^2 + (\omega_1 + \omega_2)^2\}\{(2k_1 + k_2 - \xi)^2 + \omega_2^2\}}\right\}$$

$$q_1 = -\frac{3}{2}\frac{1}{\{(k_1 + \xi)^2 + \omega_1^2)\}\{(2k_1 - k_2 + \xi)^2 + \omega_2^2\}}$$

and

$$q_2 = -\frac{3}{2}\frac{1}{\{(k_2 + \xi)^2 + \omega_2^2)\}\{(2k_2 - k_1 + \xi)^2 + \omega_2^2\}}$$

Equations (2.20) are nonlinear and have no exact solutions. We can solve (2.20) considering a, b, c, φ_1 and φ_2 are constants in the right-hand sides of (2.20) (as ε is small) $\frac{da}{dt}, \frac{db}{dt}, \frac{dc}{dt}, \frac{d\varphi_1}{dt}$ and $\frac{d\varphi_2}{dt}$ are slowly varying function of time. This assumption was used by Murty et al. [75, 76] to solve the similar nonlinear equations. The solution is thus

$$a(t) = a_0 + \varepsilon(l_1 a_0{}^3 \frac{(1 - e^{-2k_1 t})}{2k_1} + l_2 a_0 b_0{}^2 \frac{(1 - e^{-2k_2 t})}{2k_2})$$
$$b(t) = b_0 + \varepsilon(n_1 b_0{}^3 \frac{(1 - e^{-2k_2 t})}{2k_2} + n_2 a_0{}^2 b_0 \frac{(1 - e^{-2k_1 t})}{2k_1})$$
$$\varphi_1(t) = \varphi_{1,0} + \varepsilon(m_1 a_0{}^2 \frac{(1 - e^{-2k_1 t})}{2k_1} + m_2 b_0{}^2 \frac{(1 - e^{-2k_2 t})}{2k_2})$$
$$\varphi_2(t) = \varphi_{2,0} + \varepsilon(p_1 b_0{}^2 \frac{(1 - e^{-2k_2 t})}{2k_2} + p_2 a_0{}^2 \frac{(1 - e^{-2k_1 t})}{2k_1})$$

and

$$c(t) = c_0 + \varepsilon(q_1 a_0{}^2 c_0 \frac{(1 - e^{-2k_1 t})}{2k_1} + q_2 b_0{}^2 c_0 \frac{(1 - e^{-2k_2 t})}{2k_2}) \tag{2.21}$$

Here, we neglect the calculation of u_1 for small contribution in our solutions.

Finally, we obtain the solution in the form

$$x(t) = a\cos(\omega_1 t + \varphi_1) + b\cos(\omega_2 t + \varphi_2) + ce^{-\xi t}\,. \tag{2.22}$$

Here eq. (2.22) is the first order approximate solution of eq. (2.12), where a, b, c, φ_1 and φ_2 are given by the eq. (2.21).

2.4 . Results and Discussions

On the basis of KBM method, an approximate solution of fifth order time dependent damped nonlinear system with constant has been found. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. Here we obtained the approximate solution to the first order of accuracy. In contrast with the numerical solution, one can easily verify accuracy of approximate solution obtained by a certain perturbation method. We have compared our obtained results (by perturbation) to those obtained by the fourth order Runge-Kutta method for different sets of initial conditions as well as different sets of eigenvalues in this chapter. We have also computed the Pearson correlation between the perturbation results and the corresponding numerical results. From provided the figures, we observed that our perturbation solution agree with numerical results nicely for different initial conditions.

At first, for $k_1 = 0.15,\ k_2 = 0.25,\ \omega_1 = 2.0,\ \omega_2 = 1.35,\ \xi = 0.005$ and $\varepsilon = 0.1$, $x(t,\varepsilon)$ has been computed (2.22), in which a, b, c, φ_1 and φ_2 by the equation (2.21) with initial conditions

$$a_0 = 0.05, b_0 = 0.08, c_0 = 0.0075, \varphi_{1,0} = \frac{2\pi}{3} \ and \ \varphi_{2,0} = \frac{\pi}{2}$$

i.e., $[\,x(0) = -0.0175,\ \frac{dx(0)}{dt} = \text{-}0.190893,\ \frac{d^2x(0)}{dt^2} = 0.1794134,$

$\frac{d^3x(0)}{dt^3} = 0.472252,\ \frac{d^4x(0)}{dt^4} = \text{-}0.783315\,]$

Then the perturbation results obtained by the solution (2.22) and the corresponding numerical results obtained by a fourth order Runge-Kutta method with a small time increment .05, are plotted (Fig. 2.1). The correlation between the results is 0.999999995.

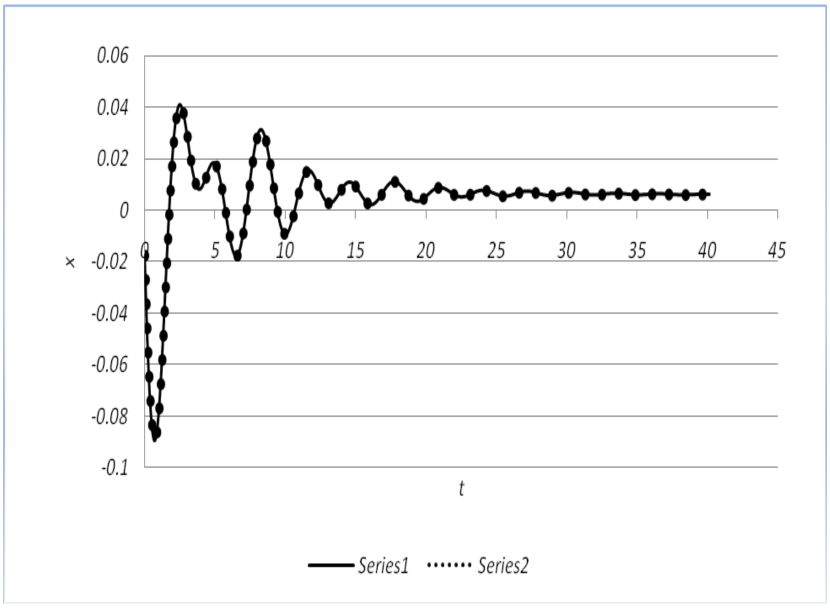

Fig.2.1. Perturbation solution plotted by solid line and numerical solution plotted dotted line.

Secondly, for $k_1 = 0.25$, $k_2 = 0.35, \omega_1 = 1.89, \omega_2 = 2.0, \xi = 0.25$ and $\varepsilon = 0.1$, $x(t,\varepsilon)$ has been computed (2.22), in which a, b, c, φ_1 *and* φ_2 by the equation (2.21)with initial conditions

$$a_0 = 0.25, b_0 = 0.35, c_0 = 0.4, \varphi_{1,0} = \frac{\pi}{2} \; and \; \varphi_{2,0} = \frac{\pi}{3}$$

[i.e., $x(0) = 0.575$, $\frac{dx(0)}{dt} = \text{-}1.246161$, $\frac{d^2x(0)}{dt^2} = 0.046513$,

$$\frac{d^3x(0)}{dt^3} = 4.549864, \frac{d^4x(0)}{dt^4} = \text{-}3.011512\,]$$

Then the perturbation results obtained by the solution (2.22) and the corresponding numerical results obtained by a fourth order Runge-Kutta method with a small time increment .05, are plotted (Fig. 2.2). The correlation between the results is 0.9999975.

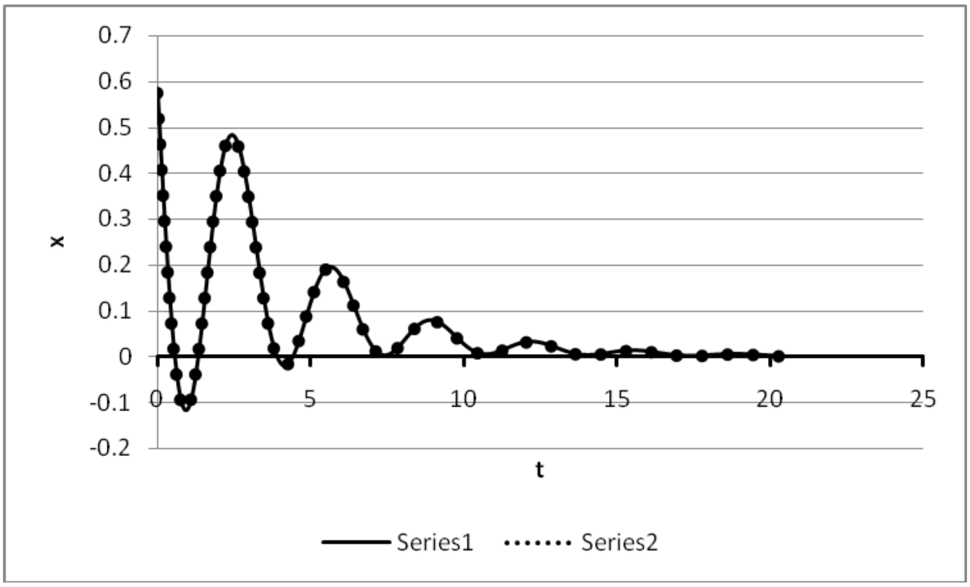

Fig. 2.2. Perturbation solution plotted by solid line and numerical solution plotted dotted line.

Finally, for $k_1 = 0.5$, $k_2 = 0.2$, $\omega_1 = \sqrt{2}$, $\omega_2 = \sqrt{3}$, $\xi = 0.4$ and $\varepsilon = 0.1$, $x(t,\varepsilon)$ has been computed (2.22), in which a, b, c, φ_1 *and* φ_2 by the equation (2.21)with initial conditions

$$a_0 = 0.44, b_0 = 0.25, c_0 = 0.36, \varphi_{1,0} = 0.0 \textit{ and } \varphi_{2,0} = 3.12$$

[i.e., $x(0) = 0.550058$, $\dfrac{dx(0)}{dt} = -.323975$, $\dfrac{d^2x(0)}{dt^2} = 0.033432$,

$\dfrac{d^3x(0)}{dt^3} = 0.81419$, $\dfrac{d^4x(0)}{dt^4} = -1.60889$]

Then the perturbation results obtained by the solution (2.22) and the corresponding numerical results obtained by a fourth order Runge-Kutta method with a small time increment .05, are plotted (Fig. 2.3). The correlation between the results is 0.999999673.

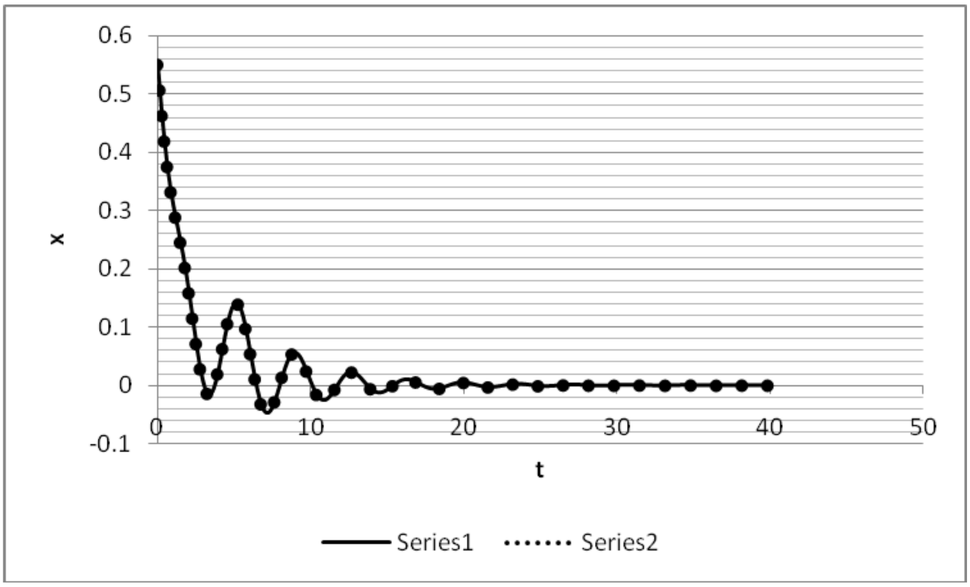

Fig. 2.3. Perturbation solution plotted by solid line and numerical solution plotted dotted line.

2.5. Conclusion

In this chapter, a procedure is founded to obtain the analytical approximate solution of fifth order nonlinear differential systems based on the KBM [22, 63] method. The correlation has been calculated between the results obtained by the perturbation solution and the fourth order Runge-Kutta method of the same problem. The results obtained for different initial conditions, show a good coincidence with corresponding numerical results and they are strongly correlated.

Chapter-Three

Perturbation Solutions for Fifth Order Damped-oscillatory Nonlinear Systems with Only One Pair of Eigen Values are Complex

3.1. Introduction

Oscillation is the rhythmic variation, usually in time, of some measure about a central value (often a point of equilibrium) or between two or more different states. Common examples include a swinging pendulum and AC power. The term vibration is sometimes used more narrowly to mean a mechanical oscillation but sometimes is used to be synonymous with "oscillation" which arise not only in physical systems but also in biological systems and in human society. The harmonic oscillator and the systems it models have a single degree of freedom but more complicated systems have more degrees of freedom, for example two masses and three springs (each mass being attached to fixed points and to each other). In such cases, the behavior of each variable influences that of the others and leads to a coupling of the oscillations of the individual degrees of freedom. This phenomenon was first observed by Christiaan Huygens in 1665[123]. The apparent motions of the compound oscillations normally come out very complicated but a more economic, computationally simpler and conceptually deeper description is given by resolving the motion into normal modes. More special cases are the coupled oscillators where the energy alternates between two forms of oscillation.

As the number of degrees of freedom becomes arbitrarily large, a system approaches continuity; examples include a string or the surface of a body of water. Such systems have (in the classical limit) an infinite number of normal modes and their oscillations occur in the form of waves that can characteristically propagate. Most of their dynamical equations are nonlinear with higher degrees of freedom.

In the last few decades, many efficient and powerful methods have been developed by a diverse group of researchers to construct the analytical approximate solutions of physically important non-linear equations arise in many phenomena in physics and engineering branches. Among the methods used to study nonlinear systems with a small nonlinearity, the Krylov-Bogoliubov-Mitropolskii (KBM) (Bogoliubov and Mitropolskii [22], Krylov and Bogoliubov, [63] method is a vastly used technique to investigate an analytical approximate solutions. However, the process was devised for obtaining the periodic solutions of second order nonlinear differential systems with small nonlinearities, Popov [91] extended the method to explore the solutions of damped oscillatory nonlinear systems. Owing to physical significance, Mendelson [69] rediscovered Popov's results. Murty [74] offered a unified KBM method for obtaining approximate solutions of second order nonlinear systems, which covers the un-damped, damped and over-damped cases. Bojadziev and Hung [35] employed the KBM method to search approximate solutions of damped oscillations modeled by a 3-dimensional time dependent system. Shamsul [111] proposed a new perturbation technique to find the analytical approximate solution of nonlinear systems with large damping. Shamsul and Sattar [109] presented a unified method for obtaining solution of third order damped oscillatory and over-damped nonlinear systems. Later, Akbar et al. [6] extended the technique for damped oscillatory nonlinear systems in the case when the four eigen-values are complex conjugates. Rahman et al. [93] investigated solution of fourth order nonlinear systems in which two of the eigen-values are real, negative and the rest of the two are complex conjugates. Recently, Akbar and Siddique [9] investigate a technique on the basis of KBM Method to obtain the analytical approximate solutions of fifth-order weakly nonlinear oscillatory systems by extending the KBM method. Siddique and Akbar [115] also investigated an asymptotic solution of fifth-order over-damped symmetrical nonlinear system based on the KBM method and the work of Akbar et al. [6].

In the previous chapter, we consider fifth order damped oscillatory which has two pair of roots are complex and one is real. But the purpose of this chapter is to investigate solutions of fifth order damped oscillatory nonlinear systems [51] when two of the eigen-values are complex conjugates and the other three are real and negative. The presented method is illustrated by its applications via cubic nonlinear Duffing type damped-oscillatory differential system which are used to model different nonlinear phenomena. The results obtained by the presented technique agree with the numerical solutions obtained by means of the fourth order Runge-Kutta method nicely.

3.2. Materials and Methods

Consider a fifth order weakly nonlinear damped-oscillatory ordinary differential system

$$\frac{d^5x}{dt^5}+\sum_{i=1}^{4}k_i\frac{d^ix}{dt^i}+k_5x=-\varepsilon f(x,t) \tag{3.1}$$

where ε is a small parameter, $f(x,t)$ is the nonlinear function, $k_i\,;i=1,2,..,5$ are the characteristic parameters of the system defined by $k_1=\sum_{i=1}^{5}\lambda_i$, $k_2=\sum_{\substack{i,j=1\\ i\neq j}}^{5}\lambda_i\lambda_j$,

$k_3=\sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{5}\lambda_i\lambda_j\lambda_k$, $k_4=\sum_{\substack{i,j,k,l=1\\ i\neq j\neq k\neq l}}^{5}\lambda_i\lambda_j\lambda_k\lambda_l$ and $k_5=\prod_{i=1}^{5}\lambda_i$ where $-\lambda_1,-\lambda_2,-\lambda_3,-\lambda_4,-\lambda_5$ are the eigenvalues of the unperturbed equation of (3.1). We consider three of the eigen-values say $-\lambda_1$, $-\lambda_2$, $-\lambda_5$ are real and negative and the other two say $-\lambda_3$, $-\lambda_4$ are complex conjugates.

The unperturbed solution (when $\varepsilon=0$) of the Eq. (3.1) is:

$$x(t,0)=\sum_{j=1}^{5}a_{j,0}e^{-\lambda_j t} \tag{3.2}$$

where $a_{j,0}$, $j = 1,2....,5$ are constants of integration.

If $\varepsilon \neq 0$, following, we seek a solution in accordance with Shamsul [108] or Murty and Deekshatulu [76] or the KBM [22, 63] method, of the form:

$$x(t,\varepsilon) = \sum_{j=1}^{5} a_j(t)\, e^{-\lambda_j t} + \varepsilon\, u_1(a_1, a_2, \cdots, a_5, t) + \cdots \tag{3.3}$$

where each $a_j; j = 1, 2, \cdots, 5$, satisfies the conditions

$$\frac{d}{dt}(a_j(t)) = \varepsilon A_j(a_1, a_2, \cdots, a_5, t) + \varepsilon^2 \cdots \tag{3.4}$$

The analytical approach is very difficult to determine a higher approximation of

equation (3.3). However, a first approximate solution gives the satisfactory results. Confining our attention to the first few terms $1,2,\cdots,m$ in the series expansions of equations (3.3) and (3.4), we calculate the functions u_1 and $A_j; j = 1,2,\cdots,5$ such that $a_j; j = 1,2,\cdots,5$, appearing in eq. (3.3) and eq. (3.4), satisfy the differential equation (3.1) with an accuracy of ε^{m+1}. Theoretically most of the perturbation methods can be proceeded to any order of approximation. But for rapidly growing algebraic complexity of the derivation of the formulae, the methods usually confined to a low order specially the first. It is noted that the determination of higher approximation is also laborious according to Shamsul's [110] and Sattar's [98] techniques. In order to determine these unknown functions, it is obvious that the function u_1 contain secular type terms te^{-t} which are included in the series expansion (3.3) at order ε^0. However, it is customary in KBM method that u_1 does not contain secular type terms like $t\cos t$, $t\sin t$ as well as te^{-t} etc.

Differentiating $x(t,\varepsilon)$ five times with respect to t and substituting $x(t,\varepsilon)$ and their derivatives in the eq. (3.1), using the relations in eq. (3.4) and equating the coefficients of ε, we obtain

$$\prod_{j=1}^{5}(\frac{d}{dt}+\lambda_j)u_1+\sum_{j=1}^{5}e^{-\lambda_j t}(\prod_{k=1,j\neq k}^{5}(\frac{d}{dt}-\lambda_j+\lambda_k))A_j=-f^{(0)}(a_1,a_2,\cdots,a_5,t) \qquad (3.5)$$

where $f^{(0)}=f(x_0)$ and $x_0=\sum_{j=1}^{5}a_j(t)\,e^{-\lambda_j t}$

The function $f^{(0)}$ can be expanded in a Taylor series (see Murty and Deekshatulu [76] for details) as:

$$f^{(0)}=\sum_{m_1=-\infty\cdots m_5=-\infty}^{\infty.....\infty}F_{m_1,\cdots m_5}\sum_{i=1}^{5}a_i^{m_i}e^{-(m_1\lambda_1+m_2\lambda_2+\cdots+m_5\lambda_5)t}$$

To obtain the solution of equation (3.1), it has been proposed in Ref. [108] that u_1 exclude the fundamental terms. To do this, we have to separated the equation (3.5) into six equations for unknown functions u_1 and $A_j; j=1,2,.....,5$ (see [108] for details).

Substituting the functional values and equating the coefficients of $e^{-\lambda_j t}; j=1,2,...,5$, we obtain

$$e^{-\lambda_1 t}\sum_{i=2}^{5}(\frac{d}{dt}-\lambda_1+\lambda_i)A_1=-\sum_{\substack{m_1=-\infty,\cdots,m_5=-\infty\\ m_3=m_4,\,m_1=m_2+1}}^{\infty.....\infty}F_{m_1,\cdots m_5}\sum_{i=1}^{5}a_i^{m_i}e^{-(m_1\lambda_1+m_2\lambda_2+\cdots+m_5\lambda_5)t} \qquad (3.6)$$

$$e^{-\lambda_2 t}\sum_{i=1,i\neq 2}^{5}(\frac{d}{dt}-\lambda_2+\lambda_i)A_2=-\sum_{\substack{m_1=-\infty,\cdots,m_5=-\infty\\ m_3=m_4,\,m_1=m_2-1}}^{\infty.....\infty}F_{m_1,\cdots m_5}\sum_{i=1}^{5}a_i^{m_i}e^{-(m_1\lambda_1+m_2\lambda_2+\cdots+m_5\lambda_5)t} \qquad (3.7)$$

$$e^{-\lambda_3 t}\sum_{i=1,i\neq 3}^{5}(\frac{d}{dt}-\lambda_3+\lambda_i)A_3=-\sum_{\substack{m_1=-\infty,\cdots,m_5=-\infty\\ m_1=m_2,\,m_3=m_4+1}}^{\infty.....\infty}F_{m_1,\cdots m_5}\sum_{i=1}^{5}a_i^{m_i}e^{-(m_1\lambda_1+m_2\lambda_2+\cdots+m_5\lambda_5)t} \qquad (3.8)$$

$$e^{-\lambda_4 t}\sum_{i=1,i\neq 4}^{5}(\frac{d}{dt}-\lambda_4+\lambda_i)A_4=-\sum_{\substack{m_1=-\infty,m_5=-\infty\\ 1=m_2,\,m_3=m_4-1}}^{\infty.....\infty}F_{m_1,\cdots m_5}\sum_{i=1}^{5}a_i^{m_i}e^{-(m_1\lambda_1+m_2\lambda_2+.........+m_5\lambda_5)t} \qquad (3.9)$$

$$e^{-\lambda_5 t}\sum_{i=1}^{4}(\frac{d}{dt}-\lambda_5+\lambda_i)A_5=-\sum_{\substack{m_1=-\infty\cdots m_5=-\infty\\ m_1=m_2,\,m_3=m_4}}^{\infty.....\infty}F_{m_1,\cdots m_5}\sum_{i=1}^{5}a_i^{m_i}e^{-(m_1\lambda_1+m_2\lambda_2+.........+m_5\lambda_5)t} \qquad (3.10)$$

and

$$\sum_{i=1}^{5}(\frac{d}{dt}+\lambda_i)u_1=-\sum_{m_1=-\infty\cdots m_5=-\infty.}^{\infty.....\infty}F_{m_1,\cdots m_5}\sum_{i=1}^{5}a_i^{m_i}e^{-(m_1\lambda_1+m_2\lambda_2+.........+m_5\lambda_5)t} \qquad (3.11)$$

where u_1 keep away from those terms for $m_1=m_2\pm 1,\ m_3=m_4\pm 1,\ m_1=m_2,\ m_3=m_4$.

Solving Eq. (3.6) to Eq. (3.11), we attain $A_1, A_2, ..., A_5$ and u_1.

For the suitable form of the solution, we shall be able to transform Eq. (3.3) to the exact formal KBM [4, 6, 9, 108] solution by substituting $a_1 = \frac{a}{2}e^{\varphi_1}$, $a_2 = \frac{a}{2}e^{-\varphi_1}$, $a_3 = \frac{b}{2}e^{i\varphi_2}$ and $a_4 = \frac{b}{2}e^{-i\varphi_2}$. Herein a, b are amplitudes and φ_1, φ_2 are phase variables.

3.3. Example

To clarify the fact, we apply the method in Duffing type equations: a important in mathematical physics and related to engineering problem. As an example of the above procedure, we consider the Duffing type equation of fifth order

$$\frac{d^5x}{dt^5} + \sum_{i=1}^{4} k_i \frac{d^i x}{dt^i} + k_5 x = -\varepsilon x^3 \tag{3.12}$$

Here $f(x,t) = x^3$.

And therefore, $f^{(0)} = (\sum_{i=1}^{5} a_i e^{-\lambda_i t})^3$

or

$$\begin{aligned} f^{(0)} &= a_1^3 e^{-3\lambda_1 t} + 3a_1^2 a_2 e^{-(2\lambda_1+\lambda_2)t} + 3a_1 a_2^2 e^{-(\lambda_1+2\lambda_2)t} + a_2^3 e^{-3\lambda_2 t} \\ &+ 3a_1^2 a_3 e^{-(2\lambda_1+\lambda_3)t} + 3a_1^2 a_4 e^{-(2\lambda_1+\lambda_4)t} + 3a_1^2 a_5 e^{-(2\lambda_1+\lambda_5)t} \\ &+ 6a_1 a_2 a_3 e^{-(\lambda_1+\lambda_2+\lambda_3)t} + 6a_1 a_2 a_4 e^{-(\lambda_1+\lambda_2+\lambda_4)t} + 6a_1 a_2 a_5 e^{-(\lambda_1+\lambda_2+\lambda_5)t} \\ &+ 3a_2^2 a_3 e^{-(2\lambda_2+\lambda_3)t} + 3a_2^2 a_4 e^{-(2\lambda_2+\lambda_4)t} + 3a_2^2 a_5 e^{-(2\lambda_2+\lambda_5)t} \\ &+ 3a_1 a_3^2 e^{-(\lambda_1+2\lambda_3)t} + 3a_1 a_4^2 e^{-(\lambda_1+2\lambda_4)t} + 3a_1 a_5^2 e^{-(\lambda_1+2\lambda_5)t} \\ &+ 6a_1 a_3 a_4 e^{-(\lambda_1+\lambda_3+\lambda_4)t} + 6a_1 a_3 a_5 e^{-(\lambda_1+\lambda_3+\lambda_5)t} + 6a_1 a_4 a_5 e^{-(\lambda_1+\lambda_4+\lambda_5)t} \\ &+ 3a_2 a_3^2 e^{-(\lambda_2+2\lambda_3)t} + 3a_2 a_4^2 e^{-(\lambda_2+2\lambda_4)t} + 3a_2 a_5^2 e^{-(\lambda_2+2\lambda_5)t} \\ &+ 6a_2 a_3 a_4 e^{-(\lambda_2+\lambda_3+\lambda_4)t} + 6a_2 a_4 a_5 e^{-(\lambda_2+\lambda_4+\lambda_5)t} + 6a_2 a_3 a_5 e^{-(\lambda_2+\lambda_3+\lambda_5)t} \\ &+ a_3^3 e^{-3\lambda_3 t} + 3a_3^2 a_4 e^{-(2\lambda_3+\lambda_4)t} + 3a_3 a_4^2 e^{-(\lambda_3+2\lambda_4)t} + a_4^3 e^{-3\lambda_4 t} \\ &+ 3a_3^2 a_5 e^{-(2\lambda_3+\lambda_5)t} + 6a_3 a_4 a_5 e^{-(\lambda_3+\lambda_4+\lambda_5)t} + 3a_4^2 a_5 e^{-(2\lambda_4+\lambda_5)t} \\ &+ 3a_3 a_5^2 e^{-(\lambda_3+2\lambda_5)t} + 3a_4 a_5^2 e^{-(\lambda_4+2\lambda_5)t} + 3a_5^3 e^{-3\lambda_5 t} \end{aligned} \tag{3.13}$$

Thus, for Eq. (3.13), the Eqs. (3.6) to (3.11) acquire the form

$$e^{-\lambda_1 t}\sum_{i=2}^{5}(\frac{d}{dt}-\lambda_1+\lambda_i)A_1=-3a_1^2a_2e^{-(2\lambda_1+\lambda_2)t}-6a_1a_3a_4e^{-(\lambda_1+\lambda_3+\lambda_4)t} \quad (3.14)$$

$$e^{-\lambda_2 t}\sum_{i=1,i\neq 2}^{5}(\frac{d}{dt}-\lambda_2+\lambda_i)A_2=-3a_1a_2^2e^{-(\lambda_1+2\lambda_2)t}-6a_2a_3a_4e^{-(\lambda_2+\lambda_3+\lambda_4)t} \quad (3.15)$$

$$e^{-\lambda_3 t}\sum_{i=1,i\neq 3}^{5}(\frac{d}{dt}-\lambda_3+\lambda_i)A_3=-3a_3^2a_4e^{-(2\lambda_3+\lambda_4)t}-6a_1a_2a_3e^{-(\lambda_1+\lambda_2+\lambda_3)t} \quad (3.16)$$

$$e^{-\lambda_4 t}\sum_{i=1,i\neq 4}^{5}(\frac{d}{dt}-\lambda_4+\lambda_i)A_4=-3a_3a_4^2e^{-(\lambda_3+2\lambda_4)t}-6a_1a_2a_4e^{-(\lambda_1+\lambda_2+\lambda_4)t} \quad (3.17)$$

$$e^{-\lambda_5 t}\sum_{i=1}^{4}(\frac{d}{dt}-\lambda_5+\lambda_i)A_5=-6a_1a_2a_5e^{-(\lambda_1+\lambda_2+\lambda_5)t}-6a_3a_4a_5e^{-(\lambda_3+\lambda_4+\lambda_5)t} \quad (3.18)$$

and

$$\begin{aligned}\sum_{i=1}^{5}(\frac{d}{dt}+\lambda_i)u_1 = &-(a_1^3e^{-3\lambda_1 t}+a_2^3e^{-3\lambda_2 t}+3a_1^2a_3e^{-(2\lambda_1+\lambda_3)t}+3a_1^2a_4e^{-(2\lambda_1+\lambda_4)t}+3a_1^2a_5e^{-(2\lambda_1+\lambda_5)t}\\ &+3a_2^2a_3e^{-(2\lambda_2+\lambda_3)t}+3a_2^2a_4e^{-(2\lambda_2+\lambda_4)t}+3a_2^2a_5e^{-(2\lambda_2+\lambda_5)t}\\ &+3a_1a_3^2e^{-(\lambda_1+2\lambda_3)t}+3a_1a_4^2e^{-(\lambda_1+2\lambda_4)t}+3a_1a_5^2e^{-(\lambda_1+2\lambda_5)t}\\ &+6a_1a_3a_5e^{-(\lambda_1+\lambda_3+\lambda_5)t}+6a_1a_4a_5e^{-(\lambda_1+\lambda_4+\lambda_5)t}\\ &+3a_2a_3^2e^{-(\lambda_2+2\lambda_3)t}+3a_2a_4^2e^{-(\lambda_2+2\lambda_4)t}+3a_2a_5^2e^{-(\lambda_2+2\lambda_5)t}\\ &+6a_2a_4a_5e^{-(\lambda_2+\lambda_4+\lambda_5)t}+6a_2a_3a_5e^{-(\lambda_2+\lambda_3+\lambda_5)t}\\ &+a_3^3e^{-3\lambda_3 t}+a_4^3e^{-3\lambda_4 t}+3a_3^2a_5e^{-(2\lambda_3+\lambda_5)t}+3a_4^2a_5e^{-(2\lambda_4+\lambda_5)t}\\ &+3a_3a_5^2e^{-(\lambda_3+2\lambda_5)t}+3a_4a_5^2e^{-(\lambda_4+2\lambda_5)t}+3a_5^3e^{-3\lambda_5 t})\end{aligned} \quad (3.19)$$

Inserting $\lambda_1=k_1-\omega_1$, $\lambda_2=k_1+\omega_1$, $\lambda_3=k_2-i\omega_2$, $\lambda_4=k_2+i\omega_2$ and $\lambda_5=\xi$ and solving the Eqs.(3.14)-(3.18), we obtain

$$\begin{aligned}A_1=&-\frac{3a_1^2a_2e^{-2k_1t}}{2(k_1-\omega_1)\{(3k_1-k_2-\omega_1)^2+\omega_2^2)\}\{(3k_1-\xi)-\omega_1\}}\\ &-\frac{6a_1a_3a_4e^{-2k_2t}}{2(k_2-\omega_1)\{(k_1+k_2-\omega_1)^2+\omega_2^2)\}\{(k_1+2k_2-\xi)-\omega_1\}}\end{aligned}$$

$$\begin{aligned}A_2=&-\frac{3a_1a_2^2e^{-2k_1t}}{2(k_1+\omega_1)\{(3k_1-k_2+\omega_1)^2+\omega_2^2)\}\{(3k_1-\xi)+\omega_1\}}\\ &-\frac{6a_2a_3a_4e^{-2k_2t}}{2(k_2+\omega_1)\{(k_1+k_2+\omega_1)^2+\omega_2^2)\}\{(k_1+2k_2-\xi)+\omega_1\}}\end{aligned}$$

$$A_3 = -\frac{3a_3^2 a_4 e^{-2k_2 t}}{2(k_2 - i\omega_2)\{(3k_2 - k_1) + (\omega_1 - i\omega_2)\}\{(3k_2 - k_1) - (\omega_1 + i\omega_2)\}\{(3k_2 - \xi) - i\omega_2\}}$$
$$-\frac{6a_1 a_2 a_3 e^{-2k_1 t}}{2(k_1 - i\omega_2)\{(k_1 + k_2) + (\omega_1 - i\omega_2)\}\{(k_1 + k_2) - (\omega_1 + i\omega_2)\}\{(2k_1 + k_2 - \xi) - i\omega_2\}}$$

$$A_4 = -\frac{3a_3 a_4^2 e^{-2k_2 t}}{2(k_2 + i\omega_2)\{(3k_2 - k_1) + (\omega_1 + i\omega_2)\}\{(3k_2 - k_1) - (\omega_1 - i\omega_2)\}\{(3k_2 - \xi) + i\omega_2\}}$$
$$-\frac{6a_1 a_2 a_4 e^{-2k_1 t}}{2(k_1 + i\omega_2)\{(k_1 + k_2) + (\omega_1 + i\omega_2)\}\{(k_1 + k_2) - (\omega_1 - i\omega_2)\}\{(2k_1 + k_2 - \xi) + i\omega_2\}}$$

$$A_5 = -\frac{6a_1 a_2 a_5 e^{-2k_1 t}}{\{(k_1 + \xi)^2 - \omega_1^2)\}\{(2k_1 - k_2 + \xi)^2 + \omega_2^2\}} - \frac{6a_3 a_4 a_5 e^{-2k_2 t}}{\{(k_2 + \xi)^2 + \omega_2^2)\}\{(2k_2 - k_1 + \xi)^2 - \omega_2^2\}}$$

Here, contribution of the term u_1 in the solution of the considered problem is very small i. e., small correction term, but it is laborious task to solve (3.19) for u_1. So, we ignore u_1 since it is proportional to small parameter ε. Now inserting $A_j; j = 1,2,..,5$ into the equations (3.4) and substituting $a_1 = \frac{1}{2} a e^{\varphi_1}$, $a_2 = \frac{1}{2} a e^{-\varphi_1}$, $a_3 = \frac{1}{2} b e^{i\varphi_2}$, $a_4 = \frac{1}{2} b e^{-i\varphi_2}$ and $a_5 = c$, we obtain

$$\dot{a} = \varepsilon(l_1 a^3 e^{-2k_1 t} + l_2 a b^2 e^{-2k_2 t}) \qquad \dot{b} = \varepsilon(m_1 b^3 e^{-2k_2 t} + m_2 a^2 b e^{-2k_1 t})$$
$$\dot{\varphi}_1 = \varepsilon(n_1 a^2 e^{-2k_1 t} + n_2 b^2 e^{-2k_2 t}) \qquad \dot{\varphi}_2 = \varepsilon(q_1 b^2 e^{-2k_2 t} + q_2 a^2 e^{-2k_1 t})$$

and

$$\dot{c} = \varepsilon(p_1 a^2 c e^{-2k_1 t} + p_2 b^2 c e^{-2k_2 t}) \qquad (3.20)$$

where

$$l_1 = -\frac{3}{16(k_1 - \omega_1)\{(3k_1 - k_2 - \omega_1)^2 + \omega_2^2)\}\{(3k_1 - \xi) - \omega_1\}}$$
$$-\frac{3}{16(k_1 + \omega_1)\{(3k_1 - k_2 + \omega_1)^2 + \omega_2^2)\}\{(3k_1 - \xi) + \omega_1\}}$$

$$l_2 = -\frac{3}{8(k_2 - \omega_1)\{(k_1 + k_2 - \omega_1)^2 + \omega_2^2)\}\{(k_1 + 2k_2 - \xi) - \omega_1\}}$$
$$-\frac{3}{8(k_2 + \omega_1)\{(k_1 + k_2 + \omega_1)^2 + \omega_2^2)\}\{(k_1 + 2k_2 - \xi) + \omega_1\}}$$

$$n_1 = \frac{3}{16(k_1 + \omega_1)\{(3k_1 - k_2 + \omega_1)^2 + \omega_2^2)\}\{(3k_1 - \xi) + \omega_1\}}$$
$$-\frac{3}{16(k_1 - \omega_1)\{(3k_1 - k_2 - \omega_1)^2 + \omega_2^2)\}\{(3k_1 - \xi) - \omega_1\}}$$

$$n_2 = \frac{3}{8(k_2+\omega_1)\{(k_1+k_2+\omega_1)^2+\omega_2^2)\}\{(k_1+2k_2-\xi)+\omega_1\}} - \frac{3}{8(k_2-\omega_1)\{(k_1+k_2-\omega_1)^2+\omega_2^2)\}\{(k_1+2k_2-\xi)-\omega_1\}}$$

$$m_1 = -\frac{3}{8}\left\{\frac{\{k_2(3k_2-\xi)-\omega_2^2\}\{(3k_2-k_1)^2-\omega_1^2-\omega_2^2\}-2\omega_2^2(3k_2-k_1)(4k_2-\xi)}{(k_2^2+\omega_2^2)\{(3k_2-k_1+\omega_1)^2+\omega_2^2\}\{(3k_2-k_1-\omega_1)^2+\omega_2^2\}\{(3k_2-\xi)^2+\omega_2^2\}}\right\}$$

$$m_2 = -\frac{3}{4}\left\{\frac{\{k_1(2k_1+k_2-\xi)-\omega_2^2\}\{(k_1+k_2)^2-\omega_1^2-\omega_2^2\}-2\omega_2^2(k_1+k_2)(3k_1+k_2-\xi)}{(k_1^2+\omega_2^2)\{(k_1+k_2+\omega_1)^2+\omega_2^2\}\{(k_1+k_2-\omega_1)^2+\omega_2^2\}\{(2k_1+k_2-\xi)^2+\omega_2^2\}}\right\}$$

$$q_1 = -\frac{3}{8}\left\{\frac{\omega_2(4k_2-\xi)\{(3k_2-k_1)^2-\omega_1^2-\omega_2^2\}+2\omega_2(3k_2-k_1)\{k_2(3k_2-\xi)-\omega_2^2\}}{(k_2^2+\omega_2^2)\{(3k_2-k_1+\omega_1)^2+\omega_2^2\}\{(3k_2-k_1-\omega_1)^2+\omega_2^2\}\{(3k_2-\xi)^2+\omega_2^2\}}\right\}$$

$$q_2 = -\frac{3}{4}\left\{\frac{\omega_2(3k_1+k_2-\xi)\{(k_1+k_2)^2-\omega_1^2-\omega_2^2\}+2\omega_2(k_1+k_2)\{k_1(2k_1+k_2-\xi)-\omega_2^2\}}{(k_1^2+\omega_2^2)\{(k_1+k_2+\omega_1)^2+\omega_2^2\}\{(k_1+k_2-\omega_1)^2+\omega_2^2\}\{(2k_1+k_2-\xi)^2+\omega_2^2\}}\right\}$$

$$p_1 = -\frac{3}{2}\frac{1}{\{(k_1+\xi)^2-\omega_1^2)\}\{(2k_1-k_2+\xi)^2+\omega_2^2\}}$$

and

$$p_2 = -\frac{3}{2}\frac{1}{\{(k_2+\xi)^2+\omega_2^2)\}\{(2k_2-k_1+\xi)^2-\omega_1^2\}}$$

Equations in (3.20) are nonlinear and therefore have no exact solutions. Since $\dot{a},\dot{b},\dot{c},\dot{\varphi}_1$ and $\dot{\varphi}_2$ are proportional to the small parameter ε, therefore, they are slowly varying function of time t. Therefore, we may assume that a,b,c,φ_1 and φ_2 are constants in the right-hand sides of (3.20). This assumption was used by Murty et al. [75, 76] to solve the similar nonlinear equations. The solution is thus

$$a(t) = a_0 + \varepsilon(l_1 a_0{}^3 \frac{(1-e^{-2k_1 t})}{2k_1} + l_2 a_0 b_0{}^2 \frac{(1-e^{-2k_2 t})}{2k_2})$$

$$b(t) = b_0 + \varepsilon(m_1 b_0{}^3 \frac{(1-e^{-2k_2 t})}{2k_2} + m_2 a_0{}^2 b_0 \frac{(1-e^{-2k_1 t})}{2k_1})$$

$$\varphi_1(t) = \varphi_{1,0} + \varepsilon(n_1 a_0{}^2 \frac{(1-e^{-2k_1 t})}{2k_1} + n_2 b_0{}^2 \frac{(1-e^{-2k_2 t})}{2k_2})$$

$$\varphi_2(t) = \varphi_{2,0} + \varepsilon(q_1 b_0{}^2 \frac{(1-e^{-2k_2 t})}{2k_2} + q_2 a_0{}^2 \frac{(1-e^{-2k_1 t})}{2k_1})$$

and

$$c(t) = c_0 + \varepsilon(p_1 a_0{}^2 c_0 \frac{(1-e^{-2k_1 t})}{2k_1} + p_2 b_0{}^2 c_0 \frac{(1-e^{-2k_2 t})}{2k_2}) \qquad (3.21)$$

Therefore, the solution of Eq. (3.12) is

$$x(t) = a\cosh(\omega_1 t + \varphi_1) + b\cos(\omega_2 t + \varphi_2) + ce^{-\xi t}. \quad (3.22)$$

Here Eq. (3.22) is the first order approximate solution of Eq. (3.12), where a, b, c, φ_1 and φ_2 are given by the Eq. (3.21).

3.4. Results and Discussions

It is customary to compare the perturbation results obtained by a certain perturbation method to the numerical results (considered being exact) to test the accuracy of the method. Two this end, computed $x(t,\varepsilon)$ by (3.22) in which a, b, c, φ_1 and φ_2 are computed by the equation (3.21) by the fourth order Runge-Kutta method for different sets of initial conditions and plotted these results. Beside this, we have also computed the Pearson correlation between the perturbation results and the corresponding numerical results and shows that they are strongly correlated. From the figures we observed that our perturbation solution agree with numerical results suitably for different initial conditions.

At first, for $k_1 = 1/3$, $k_2 = 0.25$, $\omega_1 = 0.15, \omega_2 = \sqrt{5}$, $\xi = 0.5$ and $\varepsilon = 0.1$, $x(t,\varepsilon)$ has been computed by (3.22) in which a, b, c, φ_1 and φ_2 are computed by the equation (3.21) with initial conditions $a_0 = 0.25$, $b_0 = 0.25$, $c_0 = 0.25$, $\varphi_{1,0} = \frac{\pi}{6}$ and $\varphi_{2,0} = \frac{\pi}{6}$ [or $x(0) = 0.751566$, $\frac{dx(0)}{dt} = -0.550235$, $\frac{d^2x(0)}{dt^2} = -0.82637$, $\frac{d^3x(0)}{dx^3} = 2.101099$ and $\frac{d^4x(0)}{dt^4} = 3.655849$].

The perturbation results obtained by the solution (3.22) and the corresponding numerical results obtained by a fourth order Runge-Kutta method with a small time increment 0.5, are plotted (Fig. 3.1). The correlation between the results is 0.999037.

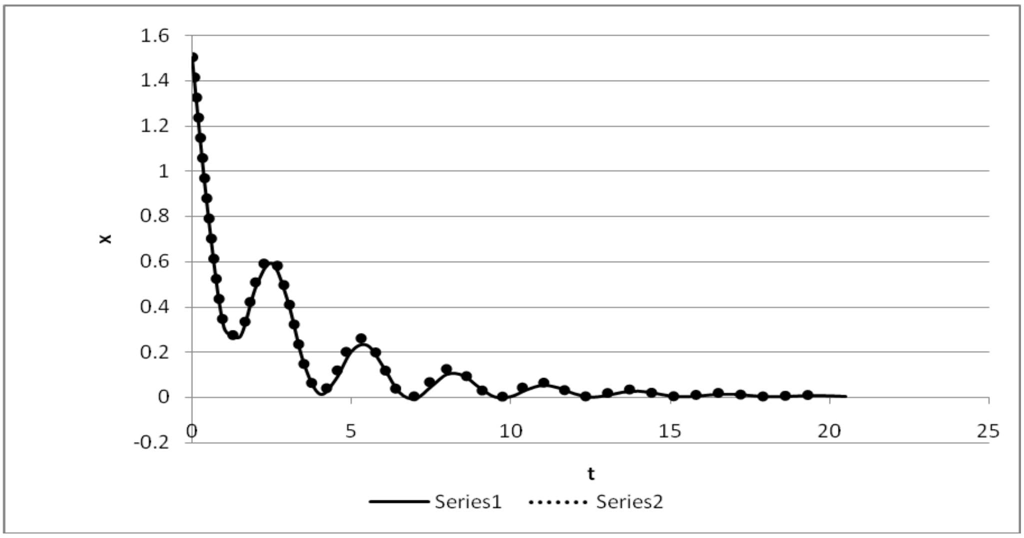

Fig. 3.1. Perturbation solution plotted by solid line and numerical solution plotted by dotted line.

Again, for $k_1 = 1/3$, $k_2 = 0.25$, $\omega_1 = \frac{\pi}{6}, \omega_2 = \frac{\pi}{6}$, $\xi = 0.5$ and $\varepsilon = 0.1$, $x(t,\varepsilon)$ has been computed by (3.22), in which a, b, c, ϕ_1 and ϕ_2 by the equation (3.21) with initial conditions $a_0 = 0.5$, $b_0 = 0.5$, $c_0 = 0.5$, $\phi_{1,0} = \frac{\pi}{6}$ and $\phi_{2,0} = \frac{\pi}{6}$ [or $x(0) = 1.503132$, $\frac{dx(0)}{dt} = -1.203219$, $\frac{d^2x(0)}{dt^2} = -1.556837$, $\frac{d^3x(0)}{dt^3} = 4.132355$ and $\frac{d^4x(0)}{dt^4} = 7.362476$].

The perturbation results obtained by the solution (3.22) and the corresponding numerical results obtained by a fourth order Runge-Kutta method with a small time increment 0.5, are plotted (Fig. 3.2). The correlation between the results is 0.998661.

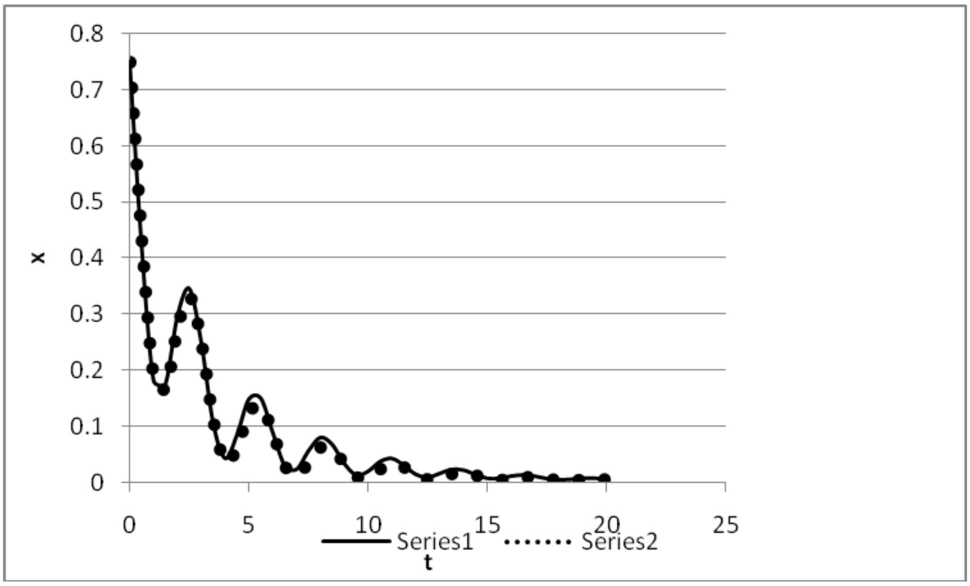

Fig. 3.2. Perturbation solution plotted by solid line and numerical solution plotted dotted line.

3.5. Conclusion

An analytical approximate solution based on the theory of KBM [22, 63] method for fifth order damped-oscillatory nonlinear differential systems is developed in this chapter. This study shows that the proposed method is quite efficient and practically well suited to be used in finding approximate solutions. The results obtained by the presented technique show good coincidence with those obtained by the fourth order Runge-Kutta method. The correlation between the results has also been calculated and it is seen that they are strongly correlated. The solution can also be used for over-damped systems replacing ω_2 by $-i\omega_2$. This is the importance of this technique.

Chapter-Four

Perturbation Solutions to Fifth Order Over-damped Nonlinear Systems

4.1. Introduction

Repetitive back-and-forth movement through a central, or equilibrium, position in which the maximum displacement on one side is equal to the maximum displacement on the other. Every entire vibration takes the same time, the period; the reciprocal of the period is the frequency of vibration. The force that causes the motion is always directed toward the equilibrium position and is directly proportional to the distance from it. A pendulum displays simple harmonic motion; other examples include the electrons in a wire carrying alternating current and the vibrating particles of a medium carrying sound waves. When the spring is first released, most likely it will fly upward with so much kinetic energy that it will, quite literally, bounce off the ceiling. But with each transit within the position of equilibrium, the friction produced by contact between the metal spring and the air, and by contact between molecules within the spring itself, will regularly reduce the energy that gives it movement. In time, it will come to a stop.

If the damping effect is small, the amplitude will gradually decrease, as the object continues to oscillate, until eventually oscillation ceases. On the other hand, the object may be "overdamped," such that it completes only a few cycles before ceasing to oscillate altogether. In the spring illustration, overdamping would occur if one were to grab the spring on a downward cycle, then slowly let it go, such that it no longer bounced.

Many researchers work on over-damp nonlinear differential systems for different order using different conditions. Murty , Deekshatulu and Krishna [75] established an asymptotic method following the Krylov-Bogoliubov [63] for overdamped nonlinear systems. Murty and Deekshatulu [76] has also offered method of variation of parameters for over-damped nonlinear systems. A unified KBM method to solve second order nonlinear systems which covers under-damped, over-damped and periodic system with constant coefficients was presented by Murty [74]. Sattar [99] studied a third order over damped nonlinear system. Akbar et al. [4] presented a method to solve fourth order over damped nonlinear systems which is easier, simple and less laborious than Murty et al. [75]. Shamsul [106] used special condition to find solution of third order over-damped nonlinear systems. Shamsul [113] has studied second order nonlinear systems both for over-damped and critically damped. Siddique and Akbar [115] has found an asymptotic solutions of fifth order over-damped nonlinear systems with cubic nonlinearity. In this chapter, we aim to obtain the analytical approximate solutions of fifth order over-damped nonlinear systems [52] extending the KBM method for obtaining the transient response in which the eigen values are in integral multiple. The results obtained by the presented technique show good coincidence with numerical results obtained by the fourth-order Runge-Kutta method. Figures are also provided to compare validation and usefulness of the solutions obtained between the results for different initial conditions.

4.2. The Method

Let us consider a fifth order nonlinear symmetrical over damped system governed by the fifth order differential equation

$$\frac{d^5x}{dt^5}+k_1\frac{d^4x}{dt^4}+k_2\frac{d^3x}{dt^3}+k_3\frac{d^2x}{dt^2}+k_4\frac{d\ x}{dt}+k_5x=-\varepsilon f(x,t) \tag{4.1}$$

where ε is a small parameter, $f(x,t)$ is such a nonlinear function that the system (4.1) becomes symmetrical, $k_i,\ i=1,2,..,4$ are the characteristic parameters of the system defined by $k_1=\sum_{i=1}^{5}\lambda_i$, $k_2=\sum_{\substack{i,j=1\\ i\neq j}}^{5}\lambda_i\lambda_j$, $k_3=\sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{5}\lambda_i\lambda_j\lambda_k$, $k_4=\sum_{\substack{i,j,k,l=1\\ i\neq j\neq k\neq l}}^{5}\lambda_i\lambda_j\lambda_k\lambda_l$ and $k_5=\prod_{i=1}^{5}\lambda_i$

where $-\lambda_1,-\lambda_2,-\lambda_3,-\lambda_4,-\lambda_5$ are the five eigenvalues of the equation (4.1).

When $\varepsilon = 0$, the equation (4.1) becomes linear, the above five eigenvalues for over damping forces are represented by the real and negative eigen values. In this case, the solution of the linear equation is: $x(t,0) = \sum_{j=1}^{5} a_{j,0} e^{-\lambda_j t}$ (4.2)

where $a_{j,0}$, $j = 1,2....,5$ are arbitrary constants.

When $\varepsilon \neq 0$, we seek a solution in accordance with Shamsul [108] or Murty and Deekshatulu [76] or the KBM [22, 63] method, an asymptotic expansion of the form:

$$x(t,\varepsilon) = \sum_{j=1}^{5} a_j(t) e^{-\lambda_j t} + \varepsilon u_1(a_1, a_2, \cdots, a_5, t) + \cdots \tag{4.3}$$

where each $a_j; j = 1,2,\cdots,5$, satisfies the first order equations

$$\frac{d}{dt}(a_j(t)) = \varepsilon A_j(a_1, a_2, \cdots, a_5, t) + \cdots \tag{4.4}$$

Keeping our concentration to some first terms $1,2,\cdots,m$ in the series expansions of equations (4.3) and (4.4), we calculate the functions u_1 and $A_j; j = 1,2,\cdots,5$ such that $a_j; j = 1,2,\cdots,5$, appearing in equation (4.3) and (4.4), satisfy the differential equation (4.1) with an correctness of ε^{m+1}. Though the solution can be obtained up to the accuracy of any order of approximation, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution in general confine to lower order [74]. In order to determine these unknown functions, it is assumed that the function u_1 exclude fundamental terms which are included in the series expansion (4.3) at order ε^0.

Differentiating $x(t,\varepsilon)$ five times with respect to t and substituting $x(t,\varepsilon)$ and their derivatives in eq. (4.1), using the relation in eq. (4.4) and finally extracting the coefficients of ε, we obtain

$$
\begin{aligned}
& e^{-\lambda_1 t}(\frac{d}{dt}-\lambda_1+\lambda_2)(\frac{d}{dt}-\lambda_1+\lambda_3)(\frac{d}{dt}-\lambda_1+\lambda_4)(\frac{d}{dt}-\lambda_1+\lambda_5)A_1 \\
& e^{-\lambda_2 t}(\frac{d}{dt}-\lambda_2+\lambda_1)(\frac{d}{dt}-\lambda_2+\lambda_3)(\frac{d}{dt}-\lambda_2+\lambda_4)(\frac{d}{dt}-\lambda_2+\lambda_5)A_2 \\
& e^{-\lambda_3 t}(\frac{d}{dt}-\lambda_3+\lambda_1)(\frac{d}{dt}-\lambda_3+\lambda_2)(\frac{d}{dt}-\lambda_3+\lambda_4)(\frac{d}{dt}-\lambda_3+\lambda_5)A_3 \\
& e^{-\lambda_4 t}(\frac{d}{dt}-\lambda_4+\lambda_1)(\frac{d}{dt}-\lambda_4+\lambda_2)(\frac{d}{dt}-\lambda_4+\lambda_3)(\frac{d}{dt}-\lambda_4+\lambda_5)A_4 \\
& e^{-\lambda_5 t}(\frac{d}{dt}-\lambda_5+\lambda_1)(\frac{d}{dt}-\lambda_5+\lambda_2)(\frac{d}{dt}-\lambda_5+\lambda_3)(\frac{d}{dt}-\lambda_5+\lambda_4)A_5 \\
& =-f^{(0)}(a_1,a_2,\cdots,a_5,t)
\end{aligned}
\tag{4.5}
$$

where $f^{(0)} = f(x_0)$ and $x_0 = \sum_{j=1}^{5} a_j(t)e^{-\lambda_j t}$

The function $f^{(0)}$ can be expanded in a Taylor series (see Murty and Deekshatulu [76] for details) as:

$$f^{(0)} = \sum_{m_1=-\infty\cdots m_5=-\infty}^{\infty.....\infty} F_{m_1,\cdots m_5} \sum_{i=1}^{5} a_i^{m_i} e^{-(m_1\lambda_1+m_2\lambda_2+\cdots+m_5\lambda_5)t}$$

Since the order of the equation (4.1) is finite, therefore, it is possible to choose, but in our method it is not necessary to keep any condition on λ_i, $(i=1,2,\cdots,5)$.

Therefore, in order to solve equation (4.5) for the unknown functions A_1, A_2, A_3, A_4, A_5 and u_1, it is assumed that u_1 does not contain terms fundamental terms. This is a significant assumption, since, under this assumption the coefficients of the terms of u_1 do not become large as well as u_1 does not contain secular type terms $t\,e^{-t}$.Thus, in accordance with this assumptions(see [76, 108] for details). Therefore, Eq. (4.5) can be separated into six equations for unknown functions u_1 and A_j; $j=1,2,.....,5$).

Substituting the functional value and equating the coefficients of $e^{-\lambda_j t}$; $j=1,2,...,5$, we obtain

$$e^{-\lambda_1 t}(\frac{d}{dt}-\lambda_1+\lambda_2)(\frac{d}{dt}-\lambda_1+\lambda_3)(\frac{d}{dt}-\lambda_1+\lambda_4)(\frac{d}{dt}-\lambda_1+\lambda_5)A_1 = -\sum_{\substack{m_1=-\infty,\cdots,m_5=-\infty \\ m_3=m_4\,,\,m_1=m_2+1}}^{\infty.....\infty} F_{m_1,\cdots m_5}\sum_{i=1}^{5} a_i^{m_i} e^{-(m_1\lambda_1+m_2\lambda_2+\cdots+m_5\lambda_5)t} \tag{4.6}$$

$$e^{-\lambda_2 t}(\frac{d}{dt}-\lambda_2+\lambda_1)(\frac{d}{dt}-\lambda_2+\lambda_3)(\frac{d}{dt}-\lambda_2+\lambda_4)(\frac{d}{dt}-\lambda_2+\lambda_5)A_2 = -\sum_{\substack{m_1=-\infty,\cdots,m_5=-\infty \\ m_3=m_4\,,\,m_1=m_2-1}}^{\infty.....\infty} F_{m_1,\cdots m_5}\sum_{i=1}^{5} a_i^{m_i} e^{-(m_1\lambda_1+m_2\lambda_2+\cdots+m_5\lambda_5)t} \tag{4.7}$$

$$e^{-\lambda_3 t}(\frac{d}{dt}-\lambda_3+\lambda_1)(\frac{d}{dt}-\lambda_3+\lambda_2)(\frac{d}{dt}-\lambda_3+\lambda_4)(\frac{d}{dt}-\lambda_3+\lambda_5)A_3 = -\sum_{\substack{m_1=-\infty,\cdots,m_5=-\infty \\ m_1=m_2\,,\,m_3=m_4+1}}^{\infty.....\infty} F_{m_1,\cdots m_5}\sum_{i=1}^{5} a_i^{m_i} e^{-(m_1\lambda_1+m_2\lambda_2+\cdots+m_5\lambda_5)t} \tag{4.8}$$

$$e^{-\lambda_4 t}(\frac{d}{dt}-\lambda_4+\lambda_1)(\frac{d}{dt}-\lambda_4+\lambda_2)(\frac{d}{dt}-\lambda_4+\lambda_3)(\frac{d}{dt}-\lambda_4+\lambda_5)A_4 = -\sum_{\substack{m_1=-\infty,m_5=-\infty \\ {}_1=m_2\,,\,m_3=m_4-1}}^{\infty.....\infty} F_{m_1,\cdots m_5}\sum_{i=1}^{5} a_i^{m_i} e^{-(m_1\lambda_1+m_2\lambda_2+.........+m_5\lambda_5)t} \tag{4.9}$$

$$e^{-\lambda_5 t}(\frac{d}{dt}-\lambda_5+\lambda_1)(\frac{d}{dt}-\lambda_5+\lambda_2)(\frac{d}{dt}-\lambda_5+\lambda_3)(\frac{d}{dt}-\lambda_5+\lambda_4)A_5 = -\sum_{\substack{m_1=-\infty\cdots m_5=-\infty \\ m_1=m_2\,,\,m_3=m_4}}^{\infty.....\infty} F_{m_1,\cdots m_5}\sum_{i=1}^{5} a_i^{m_i} e^{-(m_1\lambda_1+m_2\lambda_2+.........+m_5\lambda_5)t} \tag{4.10}$$

and

$$(\frac{d}{dt}+\lambda_1)(\frac{d}{dt}+\lambda_2)(\frac{d}{dt}+\lambda_3)(\frac{d}{dt}+\lambda_4)(\frac{d}{dt}+\lambda_5)u_1 = -\sum_{m_1=-\infty\cdots m_5=-\infty.}^{\infty.....\infty} F_{m_1,\cdots m_5}\sum_{i=1}^{5} a_i^{m_i} e^{-(m_1\lambda_1+m_2\lambda_2+.........+m_5\lambda_5)t} \tag{4.11}$$

where u_1 avoid the terms for $m_1 = m_2 \pm 1,\ m_3 = m_4 \pm 1,\ m_1 = m_2,\ m_3 = m_4$.

Solving Eqs. (4.6) to (4.11), we obtain the unknown functions $A_1, A_2, ..., A_5$ and u_1.

It is possible to transform solution Eq. (4.3) to the exact formal KBM [4, 6, 9, 108] solution by substituting $a_1 = \frac{a}{2}e^{\varphi_1}$, $a_2 = \frac{a}{2}e^{-\varphi_1}$, $a_3 = \frac{b}{2}e^{\varphi_2}$, $a_4 = \frac{b}{2}e^{-\varphi_2}$ and $a_5 = c$.

Herein a, b are amplitudes and φ_1, φ_2 are phase variables which are slowly varying function of time t.

4.3 Example

To demonstrate the applicability of the proposed method for solving the fifth order over damped nonlinear differential system type (4.1), we considered an example here. As an illustrating example, we consider the following Duffing type equation:

$$\frac{d^5x}{dt^5} + k_1\frac{d^4x}{dt^4} + k_2\frac{d^3x}{dt^3} + k_3\frac{d^2x}{dt^2} + k_4\frac{d\ x}{dt} + k_5 x = -\varepsilon\, x^3 \tag{4.12}$$

Here $f(x,t) = x^3$. Therefore, $f^{(0)} = (\sum_{i=1}^{5} a_i\, e^{-\lambda_i t})^3$

or

$$\begin{aligned} f^{(0)} &= a_1^3 e^{-3\lambda_1 t} + 3a_1^2 a_2 e^{-(2\lambda_1+\lambda_2)t} + 3a_1 a_2^2 e^{-(\lambda_1+2\lambda_2)t} + a_2^3 e^{-3\lambda_2 t} \\ &+ 3a_1^2 a_3\, e^{-(2\lambda_1+\lambda_3)t} + 3a_1^2 a_4 e^{-(2\lambda_1+\lambda_4)t} + 3a_1^2 a_5\, e^{-(2\lambda_1+\lambda_5)t} \\ &+ 6a_1 a_2 a_3 e^{-(\lambda_1+\lambda_2+\lambda_3)t} + 6a_1 a_2 a_4 e^{-(\lambda_1+\lambda_2+\lambda_4)t} + 6a_1 a_2 a_5 e^{-(\lambda_1+\lambda_2+\lambda_5)t} \\ &+ 3a_2^2 a_3 e^{-(2\lambda_2+\lambda_3)t} + 3a_2^2 a_4 e^{-(2\lambda_2+\lambda_4)t} + 3a_2^2\, a_5\, e^{-(2\lambda_2+\lambda_5)t} \\ &+ 3a_1 a_3^2 e^{-(\lambda_1+2\lambda_3)t} + 3a_1 a_4^2 e^{-(\lambda_1+2\lambda_4)t} + 3a_1 a_5^2\, e^{-(\lambda_1+2\lambda_5)t} \\ &+ 6a_1 a_3 a_4 e^{-(\lambda_1+\lambda_3+\lambda_4)t} + 6a_1 a_3 a_5 e^{-(\lambda_1+\lambda_3+\lambda_5)t} + 6a_1 a_4 a_5 e^{-(\lambda_1+\lambda_4+\lambda_5)t} \\ &+ 3a_2 a_3^2 e^{-(\lambda_2+2\lambda_3)t} + 3a_2 a_4^2 e^{-(\lambda_2+2\lambda_4)t} + 3a_2\, a_5^2\, e^{-(\lambda_2+2\lambda_5)t} \\ &+ 6a_2\, a_3\, a_4 e^{-(\lambda_2+\lambda_3+\lambda_4)t} + 6a_2 a_4 a_5 e^{-(\lambda_2+\lambda_4+\lambda_5)t} + 6a_2\, a_3\, a_5 e^{-(\lambda_2+\lambda_3+\lambda_5)t} \\ &+ a_3^3 e^{-3\lambda_3 t} + 3a_3^2 a_4 e^{-(2\lambda_3+\lambda_4)t} + 3a_3 a_4^2 e^{-(\lambda_3+2\lambda_4)t} + a_4^3 e^{-3\lambda_4 t} \\ &+ 3a_3^2 a_5 e^{-(2\lambda_3+\lambda_5)t} + 6a_3 a_4 a_5 e^{-(\lambda_3+\lambda_4+\lambda_5)t} + 3a_4^2 a_5 e^{-(2\lambda_4+\lambda_5)t} \\ &+ 3a_3 a_5^2 e^{-(\lambda_3+2\lambda_5)t} + 3a_4 a_5^2 e^{-(\lambda_4+2\lambda_5)t} + 3a_5^3\, e^{-3\lambda_5 t} \end{aligned} \tag{4.13}$$

Thus the equation (4.6) to (4.11) takes the outward appearance

$$\begin{aligned} &e^{-\lambda_1 t}(\frac{d}{dt} - \lambda_1 + \lambda_2)(\frac{d}{dt} - \lambda_1 + \lambda_3)(\frac{d}{dt} - \lambda_1 + \lambda_4)(\frac{d}{dt} - \lambda_1 + \lambda_5)A_1 \\ &\qquad = -3a_1^2 a_2 e^{-(2\lambda_1+\lambda_2)t} - 6a_1 a_3 a_4 e^{-(\lambda_1+\lambda_3+\lambda_4)t} \end{aligned} \tag{4.14}$$

$$e^{-\lambda_2 t}(\frac{d}{dt}-\lambda_2+\lambda_1)(\frac{d}{dt}-\lambda_2+\lambda_3)(\frac{d}{dt}-\lambda_2+\lambda_4)(\frac{d}{dt}-\lambda_2+\lambda_5)A_2 \\ =-3a_1a_2^2e^{-(\lambda_1+2\lambda_2)t}-6a_2a_3a_4e^{-(\lambda_2+\lambda_3+\lambda_4)t} \quad (4.15)$$

$$e^{-\lambda_3 t}(\frac{d}{dt}-\lambda_3+\lambda_1)(\frac{d}{dt}-\lambda_3+\lambda_2)(\frac{d}{dt}-\lambda_3+\lambda_4)(\frac{d}{dt}-\lambda_3+\lambda_5)A_3 \\ =-3a_3^2a_4e^{-(2\lambda_3+\lambda_4)t}-6a_1a_2a_3e^{-(\lambda_1+\lambda_2+\lambda_3)t} \quad (4.16)$$

$$e^{-\lambda_4 t}(\frac{d}{dt}-\lambda_4+\lambda_1)(\frac{d}{dt}-\lambda_4+\lambda_2)(\frac{d}{dt}-\lambda_4+\lambda_3)(\frac{d}{dt}-\lambda_4+\lambda_5)A_4 \\ =-3a_3a_4^2e^{-(\lambda_3+2\lambda_4)t}-6a_1a_2a_4e^{-(\lambda_1+\lambda_2+\lambda_4)t} \quad (4.17)$$

$$e^{-\lambda_5 t}(\frac{d}{dt}-\lambda_5+\lambda_1)(\frac{d}{dt}-\lambda_5+\lambda_2)(\frac{d}{dt}-\lambda_5+\lambda_3)(\frac{d}{dt}-\lambda_5+\lambda_4)A_5 \\ =-6a_1a_2a_5e^{-(\lambda_1+\lambda_2+\lambda_5)t}-6a_3a_4a_5e^{-(\lambda_3+\lambda_4+\lambda_5)t} \quad (4.18)$$

and

$$(\frac{d}{dt}+\lambda_1)(\frac{d}{dt}+\lambda_2)(\frac{d}{dt}+\lambda_3)(\frac{d}{dt}+\lambda_4)(\frac{d}{dt}+\lambda_5)u_1 = \\ -(a_1^3e^{-3\lambda_1 t}+a_2^3e^{-3\lambda_2 t}+3a_1^2a_3e^{-(2\lambda_1+\lambda_3)t}+3a_1^2a_4e^{-(2\lambda_1+\lambda_4)t}+3a_1^2a_5e^{-(2\lambda_1+\lambda_5)t} \\ +3a_2^2a_3e^{-(2\lambda_2+\lambda_3)t}+3a_2^2a_4e^{-(2\lambda_2+\lambda_4)t}+3a_2^2a_5e^{-(2\lambda_2+\lambda_5)t} \\ +3a_1a_3^2e^{-(\lambda_1+2\lambda_3)t}+3a_1a_4^2e^{-(\lambda_1+2\lambda_4)t}+3a_1a_5^2e^{-(\lambda_1+2\lambda_5)t} \\ +6a_1a_3a_5e^{-(\lambda_1+\lambda_3+\lambda_5)t}+6a_1a_4a_5e^{-(\lambda_1+\lambda_4+\lambda_5)t} \\ +3a_2a_3^2e^{-(\lambda_2+2\lambda_3)t}+3a_2a_4^2e^{-(\lambda_2+2\lambda_4)t}+3a_2a_5^2e^{-(\lambda_2+2\lambda_5)t} \\ +6a_2a_4a_5e^{-(\lambda_2+\lambda_4+\lambda_5)t}+6a_2a_3a_5e^{-(\lambda_2+\lambda_3+\lambda_5)t} \\ +a_3^3e^{-3\lambda_3 t}+a_4^3e^{-3\lambda_4 t}+3a_3^2a_5e^{-(2\lambda_3+\lambda_5)t}+3a_4^2a_5e^{-(2\lambda_4+\lambda_5)t} \\ +3a_3a_5^2e^{-(\lambda_3+2\lambda_5)t}+3a_4a_5^2e^{-(\lambda_4+2\lambda_5)t}+3a_5^3e^{-3\lambda_5 t}) \quad (4.19)$$

Now we have to solve equations (4.14)-(4.18) and inserting $\lambda_1 = k_1 - \omega_1$, $\lambda_2 = k_1 + \omega_1$, $\lambda_3 = k_2 - \omega_2$, $\lambda_4 = k_2 + \omega_2$ and $\lambda_5 = \xi$. To do this, using the symbolic computation software, like, Maple, Mathematica, Matlab etc. are utter simple. In this work, to solve equations (4.14)-(4.18), we have used Maple 13, we attain

$$A_1 = -\frac{3a_1^2 a_2 e^{-2k_1 t}}{2(k_1-\omega_1)\{(3k_1-k_2)-(\omega_1-\omega_2)\}\{(3k_1-k_2)-(\omega_1+\omega_2)\}\{(3k_1-\xi)-\omega_1\}}$$
$$-\frac{6a_1 a_3 a_4 e^{-2k_2 t}}{2(k_2-\omega_1)\{(k_1+k_2)-(\omega_1-\omega_2)\}\{(k_1+k_2)-(\omega_1+\omega_2)\}\{(k_1+2k_2-\xi)-\omega_1\}}$$

$$A_2 = -\frac{3a_1 a_2^2 e^{-2k_1 t}}{2(k_1+\omega_1)\{(3k_1-k_2)+(\omega_1+\omega_2)\}\{(3k_1-k_2)+(\omega_1-\omega_2)\}\{(3k_1-\xi)+\omega_1\}}$$
$$-\frac{6a_2 a_3 a_4 e^{-2k_2 t}}{2(k_2+\omega_1)\{(k_1+k_2)+(\omega_1+\omega_2)\}\{(k_1+k_2)+(\omega_1-\omega_2)\}\{(k_1+2k_2-\xi)+\omega_1\}}$$

$$A_3 = -\frac{3a_3^2 a_4 e^{-2k_2 t}}{2(k_2-\omega_2)\{(3k_2-k_1)+(\omega_1-\omega_2)\}\{(3k_2-k_1)-(\omega_1+\omega_2)\}\{(3k_2-\xi)-\omega_2\}}$$
$$-\frac{6a_1 a_3 a_4 e^{-2k_1 t}}{2(k_1-\omega_2)\{(k_1+k_2)+(\omega_1-\omega_2)\}\{(k_1+k_2)-(\omega_1+\omega_2)\}\{(2k_1+k_2-\xi)-\omega_2\}}$$

$$A_4 = -\frac{3a_3 a_4^2 e^{-2k_2 t}}{2(k_2+\omega_2)\{(3k_2-k_1)+(\omega_1+\omega_2)\}\{(3k_2-k_1)-(\omega_1-\omega_2)\}\{(3k_2-\xi)+\omega_2\}}$$
$$-\frac{6a_1 a_2 a_4 e^{-2k_1 t}}{2(k_1+\omega_2)\{(k_1+k_2)+(\omega_1+\omega_2)\}\{(k_1+k_2)-(\omega_1-\omega_2)\}\{(2k_1+k_2-\xi)+\omega_2\}}$$

$$A_5 = -\frac{6a_1 a_2 a_5 e^{-2k_1 t}}{\{(k_1+\xi)^2-\omega_1^2\}\{(2k_1-k_2+\xi)^2-\omega_2^2\}} - \frac{6a_3 a_4 a_5 e^{-2k_2 t}}{\{(k_2+\xi)^2-\omega_2^2\}\{(2k_2-k_1+\xi)^2-\omega_2^2\}}$$

Now inserting $A_j; j=1,2,..,5$ in the Eq. (4.4) and using $a_1 = ae^{\varphi_1}/2,\ \ a_2 = ae^{-\varphi_1}/2$ $a_3 = be^{\varphi_2}/2,\ \ a_4 = be^{-\varphi_2}/2$ and $a_5 = c$ we obtain

$$\dot{a} = \varepsilon(l_1 a^3 e^{-2k_1 t} + l_2 ab^2 e^{-2k_2 t}) \qquad \dot{b} = \varepsilon(n_1 b^3 e^{-2k_2 t} + n_2 a^2 b e^{-2k_1 t})$$

$$\dot{\varphi}_1 = \varepsilon(m_1 a^2 e^{-2k_1 t} + m_2 b^2 e^{-2k_2 t}) \qquad \dot{\varphi}_2 = \varepsilon(p_1 b^2 e^{-2k_2 t} + p_2 a^2 e^{-2k_1 t})$$

$$\dot{c} = \varepsilon(q_1 a^2 c e^{-2k_1 t} + q_2 b^2 c e^{-2k_2 t}) \tag{4.20}$$

where

$$l_1 = -\frac{3}{8}\left\{\frac{\begin{array}{c}(3k_1^2 - k_1k_2 + \omega_1^2 + \omega_1\omega_2)(9k_1^2 - 3k_1k_2 - 3k_1\xi + k_2\xi + \omega_1^2 - \omega_1\omega_2) + \\ (4k_1\omega_1 + k_1\omega_2 - k_2\omega_1)(6k_1\omega_1 - k_2\omega_1 - 3k_1\omega_2 - \omega_1\xi + \omega_2\xi)\end{array}}{(k_1^2-\omega_1^2)\{(3k_1-k_2)^2-(\omega_1+\omega_2)^2\}\{(3k_1-k_2)^2-(\omega_1-\omega_2)^2\}\{(3k_1-\xi)^2-\omega_1^2\}}\right\}$$

$$l_2 = -\frac{3}{4}\left\{\frac{\begin{array}{c}(k_1k_2 + k_2^2 + \omega_1^2 - \omega_1\omega_2)\{(k_1+k_2)(k_1+2k_2-\xi) + \omega_1^2 - \omega_1\omega_2\} + \\ (2k_2\omega_1 - k_2\omega_2 + k_1\omega_1)(2k_1\omega_1 + 3k_2\omega_1 + k_1\omega_2 + 2k_2\omega_2 - \omega_1\xi - \omega_2\xi)\end{array}}{(k_2^2-\omega_1^2)\{(k_1+k_2)^2-(\omega_1-\omega_2)^2\}\{(k_1+k_2)^2-(\omega_1+\omega_2)^2\}\{(k_1+2k_2-\xi)^2-\omega_1^2\}}\right\}$$

$$m_1 = -\frac{3}{8}\left\{\frac{\begin{array}{c}(3k_2^2 - k_1k_2 + \omega_2^2 + \omega_1\omega_2)(9k_2^2 - 3k_1k_2 - 3k_2\xi + k_1\xi + \omega_2^2 - \omega_1\omega_2) + \\ (4k_2\omega_2 - k_1\omega_2 + k_2\omega_1)(6k_2\omega_2 - k_1\omega_2 - 3k_2\omega_1 + \omega_1\xi - \omega_2\xi)\end{array}}{(k_2^2 - \omega_2^2)\{(3k_2 - k_1)^2 - (\omega_1 - \omega_2)^2\}\{(3k_2 - k_1)^2 - (\omega_1 + \omega_2)^2\}\{(3k_2 - \xi)^2 - \omega_2^2\}}\right\}$$

$$m_2 = -\frac{3}{4}\left\{\frac{\begin{array}{c}(k_1^2 + k_1k_2 + \omega_2^2 + \omega_1\omega_2)\{(k_1 + k_2)(2k_1 + k_2 - \xi) + \omega_2^2 - \omega_1\omega_2\} + \\ (k_1\omega_1 + 2k_1\omega_2 + k_2\omega_2)(2k_2\omega_2 + 3k_1\omega_2 - 2k_1\omega_1 - k_2\omega_1 + \omega_1\xi - \omega_2\xi)\end{array}}{(k_1^2 - \omega_2^2)\{(k_2 + k_1)^2 - (\omega_1 - \omega_2)^2\}\{(k_2 + k_1)^2 - (\omega_1 + \omega_2)^2\}\{(2k_1 + k_2 - \xi)^2 - \omega_2^2\}}\right\}$$

$$n_1 = -\frac{3}{8}\left\{\frac{\begin{array}{c}(3k_1^2 - k_1k_2 + \omega_1^2 + \omega_1\omega_2)(6k_1\omega_1 - k_2\omega_1 - 3k_1\omega_2 - \omega_1\xi + \omega_2\xi) + \\ (4k_1\omega_1 + k_1\omega_2 - k_2\omega_1)(9k_1^2 - 3k_1k_2 - 3k_1\xi + k_2\xi + \omega_1^2 - \omega_1\omega_2)\end{array}}{(k_1^2 - \omega_1^2)\{(3k_1 - k_2)^2 - (\omega_1 + \omega_2)^2\}\{(3k_1 - k_2)^2 - (\omega_1 - \omega_2)^2\}\{(3k_1 - \xi)^2 - \omega_1^2\}}\right\}$$

$$n_2 = -\frac{3}{4}\left\{\frac{\begin{array}{c}(k_1k_2 + k_2^2 + \omega_1^2 - \omega_1\omega_2)(2k_1\omega_1 + 3k_2\omega_1 + k_1\omega_2 + 2k_2\omega_2 - \omega_1\xi - \omega_2\xi) - \\ (2k_2\omega_1 - k_2\omega_2 + k_1\omega_1)\{(k_1 + k_2)(k_1 + 2k_2 - \xi) + \omega_1^2 + \omega_1\omega_2\}\end{array}}{(k_2^2 - \omega_1^2)\{(k_1 + k_2)^2 - (\omega_1 - \omega_2)^2\}\{(k_1 + k_2)^2 - (\omega_1 + \omega_2)^2\}\{(k_1 + 2k_2 - \xi)^2 - \omega_1^2\}}\right\}$$

$$p_1 = -\frac{3}{8}\left\{\frac{\begin{array}{c}(3k_2^2 - k_1k_2 + \omega_2^2 + \omega_1\omega_2)(6k_2\omega_2 - k_1\omega_2 - 3k_2\omega_1 + \omega_1\xi - \omega_2\xi) - \\ (4k_2\omega_2 - k_1\omega_2 + k_2\omega_1)(9k_2^2 - 3k_1k_2 - 3k_2\xi + k_1\xi + \omega_2^2 - \omega_1\omega_2)\end{array}}{(k_2^2 - \omega_2^2)\{(3k_2 - k_1)^2 - (\omega_1 - \omega_2)^2\}\{(3k_2 - k_1)^2 - (\omega_1 + \omega_2)^2\}\{(3k_2 - \xi)^2 - \omega_2^2\}}\right\}$$

$$p_2 = -\frac{3}{4}\left\{\frac{\begin{array}{c}(k_1^2 + k_1k_2 + \omega_2^2 + \omega_1\omega_2)(2k_2\omega_2 + 3k_1\omega_2 - 2k_1\omega_1 - k_2\omega_1 + \omega_1\xi - \omega_2\xi) \\ -(k_1\omega_1 + 2k_1\omega_2 + k_2\omega_2)\{(k_1 + k_2)(2k_1 + k_2 - \xi) + \omega_2^2 - \omega_1\omega_2\}\end{array}}{(k_1^2 - \omega_2^2)\{(k_2 + k_1)^2 - (\omega_1 - \omega_2)^2\}\{(k_2 + k_1)^2 - (\omega_1 + \omega_2)^2\}\{(2k_1 + k_2 - \xi)^2 - \omega_2^2\}}\right\}$$

$$q_1 = -\frac{3}{2}\frac{1}{\{(k_1 + \xi)^2 - \omega_1^2)\}\{(2k_1 - k_2 + \xi)^2 - \omega_2^2\}}$$

and

$$q_2 = -\frac{3}{2}\frac{1}{\{(k_2 + \xi)^2 - \omega_2^2)\}\{(2k_2 - k_1 + \xi)^2 - \omega_2^2\}}$$

Equations in (4.20) are nonlinear and have no exact solutions. We can solve (4.20) by considering a, b, c, φ_1 and φ_2 are constants in the right-hand sides of (4.20). Since ε is small, $\dot{a}, \dot{b}, \dot{c}, \dot{\varphi}_1$ and $\dot{\varphi}_2$ are slowly varying function of time, therefore, this consideration is applicable. This assumption was used by Murty et al. [75, 76] to solve the analogous nonlinear equations. The solution is thus

$$a(t) = a_0 + \varepsilon(l_1 a_0{}^3 \frac{(1 - e^{-2k_1 t})}{2k_1} + l_2 a_0 b_0{}^2 \frac{(1 - e^{-2k_2 t})}{2k_2})$$

$$b(t) = b_0 + \varepsilon(n_1 b_0{}^3 \frac{(1-e^{-2k_2 t})}{2k_2} + n_2 a_0{}^2 b_0 \frac{(1-e^{-2k_1 t})}{2k_1})$$

$$\varphi_1(t) = \varphi_{1,0} + \varepsilon(m_1 a_0{}^2 \frac{(1-e^{-2k_1 t})}{2k_1} + m_2 b_0{}^2 \frac{(1-e^{-2k_2 t})}{2k_2})$$

$$\varphi_2(t) = \varphi_{2,0} + \varepsilon(p_1 b_0{}^2 \frac{(1-e^{-2k_2 t})}{2k_2} + p_2 a_0{}^2 \frac{(1-e^{-2k_1 t})}{2k_1})$$

and

$$c(t) = c_0 + \varepsilon(q_1 a_0{}^2 c_0 \frac{(1-e^{-2k_1 t})}{2k_1} + q_2 b_0{}^2 c_0 \frac{(1-e^{-2k_2 t})}{2k_2}) \tag{4.21}$$

Therefore, the first order solution of Eq. (4.12) is

$$x(t) = a\cosh(\omega_1 t + \varphi_1) + b\cosh(\omega_2 t + \varphi_2) + ce^{-\xi t}. \tag{4.22}$$

where a, b, c, φ_1 and φ_2 are given in the Eq. (4.21).

4.4 Results and Discussions

In order to test the correctness of an approximate solution obtained by a certain perturbation method, we contrast the approximate solution to the numerical solution. With regard to such a comparison concerning the presented technique of this chapter, we refer to the work of Murty et. [75, 76]. Here, we have compared our obtained outcome to those obtained by the fourth order Runge-Kutta method for different sets of initial conditions as well as different sets of eigenvalues. Beside this, we have also computed the Pearson correlation between the perturbation results and the corresponding numerical results. From the figures we observed that our perturbation solution agree with numerical results suitably for different initial conditions.

At first, for $k_1 = 2.05$, $k_2 = 3.09$, $\omega_1 = 1.414, \omega_2 = 2.57$, $\xi = 0.03$ and $\varepsilon = 0.1$, $x(t,\varepsilon)$ has been computed by solution (4.22), in which a, b, c, φ_1 and φ_2 are computed by the equation (4.21) with initial conditions $a_0 = 0.63$, $b_0 = 0.52$, $c_0 = 0.3$, $\varphi_{1\,0} = 1.375$ and

$\varphi_{2\,0} = 0.5708$ [i. e, $x(0) = 2.23253$, $\frac{dx(0)}{dt} = -2.144589$, $\frac{d^2x(0)}{dt^2} = 6.275333$,

$\frac{d^3x(0)}{dt^3} = -30.274223$ and $\frac{d^4x(0)}{dt^4} = 162.2777256$.]

For the above mentioned initial conditions, the perturbation results obtained by the solution (4.22) and the corresponding numerical results obtained by a fourth order Runge-Kutta method with a small time increment $\Delta t = 0.05$, are plotted in Fig. 4.1. The correlation between the results is 0.999593.

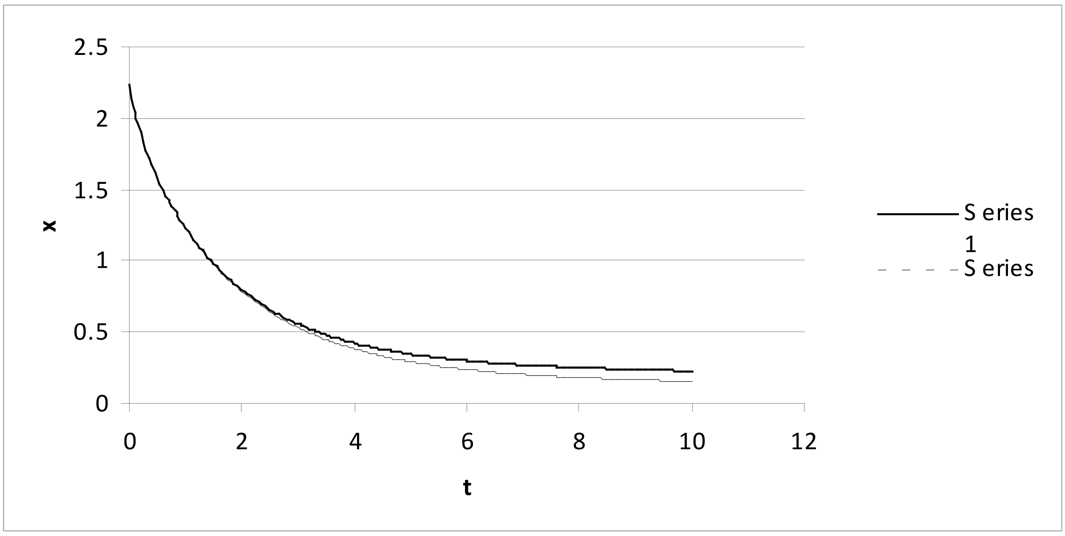

Fig. 4.1. Perturbation results are plotted by solid line and numerical results plotted dotted line.

Secondly, for $k_1 = 1$, $k_2 = 2$, $\omega_1 = 0.81$, $\omega_2 = 1.5704$, $\xi = 0.09$ and $\varepsilon = 0.1$, $x(t,\varepsilon)$ has been computed (4.22), in which a, b, c, φ_1 and φ_2 by the equation (4.21)with initial conditions $a_0 = 0.15, b_0 = 0.15, c_0 = 0.20, \varphi_{1,0} = 1.25664$ and $\varphi_{2,0} = 0.3927$ [i. e., $x(0) = 0.646581$, $\frac{dx(0)}{dt} = -0.334486$, $\frac{d^2x(0)}{dt^2} = 0.745605$, $\frac{d^3x(0)}{dt^3} = -2.439806$ and $\frac{d^4x(0)}{dt^4} = 8.461052$.]

In this section, the perturbation results obtained by the solution (4.22) and the corresponding numerical results obtained by a fourth order Runge-Kutta method with a small time increment $\Delta t = 0.05$, are plotted Fig. 4.2. The correlation between the results is 0.99965.

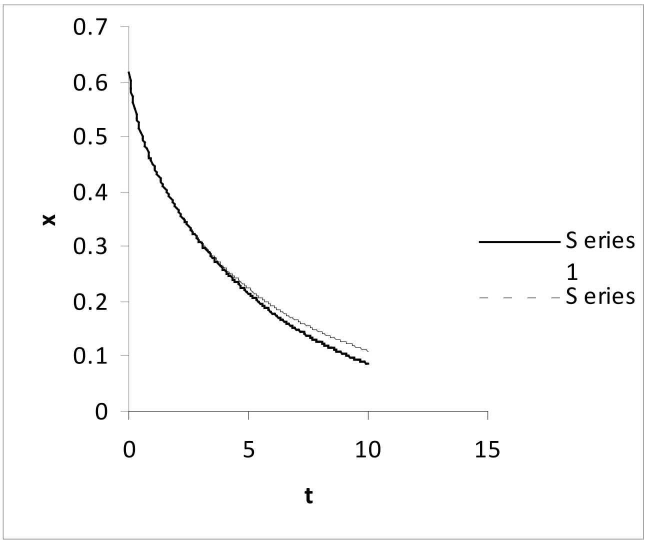

Fig. 4.2. Perturbation solution plotted by solid line and numerical solution plotted dotted line.

Finally, for $k_1 = 0.5$, $k_2 = 0.47$, $\omega_1 = 0.237$, $\omega_2 = 0.321$, $\xi = 0.003$ and $\varepsilon = 0.1$, $x(t,\varepsilon)$ has been computed (4.22), in which a, b, c, φ_1 and φ_2 by the equation (4.21) with initial conditions $a_0 = 0.02, b_0 = 0.02, c_0 = 0.008, \varphi_{1,0} = .\pi/2$ and $\varphi_{2,0} = 0.414$ [i. e. $x(0) = 0.079922$, $\frac{dx(0)}{dt} = -0.021774$, $\frac{d^2x(0)}{dt^2} = 0.009020$, $\frac{d^3x(0)}{dt^3} = -0.005127$ and $\frac{d^4x(0)}{dt^4} = 0.003545$.]

The perturbation results obtained by the solution (4.22) and the corresponding numerical results obtained by a fourth order Runge-Kutta method with a small time increment $\Delta t = 0.05$ are plotted Fig. 4.3. The correlation between the results is 0.999953.

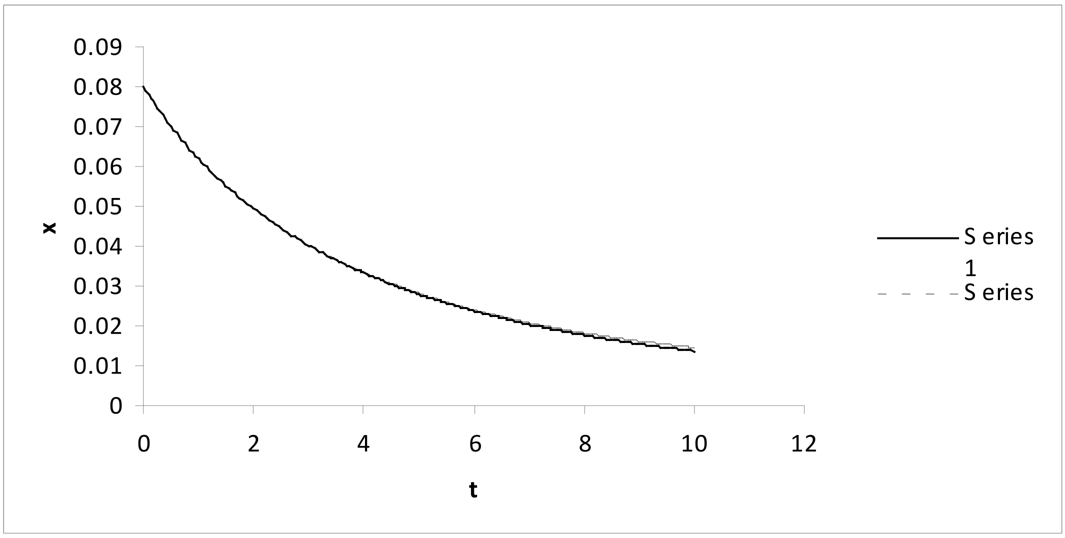

Fig. 4.3. Perturbation solution plotted by solid line and numerical solution plotted dotted line.

From Fig. 4.1 to fig.4.3 it is noteworthy to observe that perturbation results show a good agreement with those obtained by the fourth order Runge-Kutta method.

4.5. Conclusion

In this chapter, a procedure is formulated to find the first order analytical approximate solution of fifth order over damped nonlinear differential systems with small nonlinearities based on the KBM [22, 63] method. The correlation has been deliberated between the results acquired by the perturbation solution and the fourth order Runge-Kutta method of the same problem. The results obtained for different initial conditions, show a good fluke with corresponding numerical results and they are strongly correlated.

Chapter-Five

Perturbation solutions for fifth Order Critical-damped Nonlinear Systems

5.1. Introduction:

Springs are a most important part of our everyday life which can be found in everything from the shock-absorber legislative body of a motor vehicle to the supports of a trampoline fabric, and in both cases, springs blunt the force of impact. Spring produces vibration and vibration is sometimes used more closely to mean a mechanical oscillation but sometimes is used to be identical with oscillation. There is a type of damping less forceful than over-damping, but not so gradual as the slow dissipation of energy due to frictional forces alone. This is called critical damping. In a critically damped oscillator, the oscillating material is made to return to equilibrium as quickly as possible without oscillating.

Over time, obviously, the friction in the springs would wear down their energy and bring an end to their oscillation, but by then, the car would most likely have hit another bump. Therefore, it makes sense to apply critical damping to the oscillation of the springs by using shock absorbers.

The control of micro vibration has become a growing research field due to the demand of high-performance systems and the advent of micro and nanotechnology in various scientific and industrial fields, such as semiconductor manufacturing, biomedical engineering, aerospace-equipments, and high-precision measurements. In micro and nanotechnology, a small vibration may be the cause to make the product defective. So, in these fields vibration is not desirable. But vibration is unavoidable and arise in different ways, such as, earth quake, direct disturbance etc. Thus, vibration control in micro and nano-technological industries is very essential. In micro and nano-technological industries we keep watch that vibrations come to its equilibrium position in minimum time. The critically damped systems come to equilibrium position in minimum time. So, critically damped systems play an important role in micro and nano-technological industries.

The well-situated and widely used technique to obtain analytical approximate solutions to the nonlinear equations is the perturbation methods. To investigate the transient behavior of vibrating systems the Krylov-Bogoliubov-Mitropolskii (KBM) [22, 63] method is an extensively used method which was developed for obtaining the periodic solutions of second order nonlinear differential systems with small nonlinearities. Sattar [99] studied a third order over damped nonlinear system. Shamsul and Sattar [103] presented a method for critically damped and Islam and Akbar [55] for more critically damped third order nonlinear systems. Akbar et al. [4] presented a method to solve fourth order over damped nonlinear systems which is easier, simple and less laborious than Murty et al. [75]. Later, Islam et al. [56] investigated the solutions of fourth order more critically damped nonlinear systems where Akbar [5] examined a different type solution for the same. Akbar and Siddique [23] amplified the KBM method to obtain solutions of fifth order weakly nonlinear oscillatory systems.

The aim of this chapter is to obtain the analytical approximate solutions of a fifth order nonlinear differential system modeling a non-oscillatory process that characterized by critical damped. A perturbation technique based on the Krylov-Bogoliubov-Mitropolskii method [22, 63] is developed for obtaining the transient response of the systems. Here, we will consider fifth order differential nonlinear systems because the combination of a second and a third order dynamical systems lead to a fifth order dynamical system which occurs some complex nonlinear physical phenomena like as nano-technological tables. An example is solved to give the illustration of the method.

5.2. The Method

Consider a fifth order weakly nonlinear ordinary differential system :

$$\frac{d^5 x}{dt^5} + \sum_{i=1}^{4} c_i \frac{d^i x}{dt^i} + c_5 x = -\varepsilon f(x,t) \tag{5.1}$$

where ε is a small parameter, $f(x,t)$ is the nonlinear function, $c_i; i = 1,2,..,5$ are the characteristic parameters of the system defined by $c_1 = \sum_{i=1}^{5} \lambda_i$, $c_2 = \sum_{\substack{i,j=1\\ i\neq j}}^{5} \lambda_i \lambda_j$,

$c_3 = \sum_{\substack{i,j,k=1\\ i\neq j\neq k}}^{5} \lambda_i \lambda_j \lambda_k$, $c_4 = \sum_{\substack{i,j,k,l=1\\ i\neq j\neq k\neq l}}^{5} \lambda_i \lambda_j \lambda_k \lambda_l$ and $c_5 = \prod_{i=1}^{5} \lambda_i$ where $-\lambda_1, -\lambda_2, -\lambda_3, -\lambda_4, -\lambda_5$ are the eigenvalues of the unperturbed equation of (5.1). As the equation is of fifth order and we are considering a critical damped system , so there are five real negative eigenvalues and two pairs of the eigenvalues are equal (for critically damped). Suppose the roots are

$$\lambda_1 = \lambda_2 = \lambda,\ \lambda_3 = \lambda_4 = \mu,\ \lambda_5 = \xi$$

When $\varepsilon = 0$ the equation (5.1) becomes linear and the solution of the corresponding linear equation is

$$x(t,0) = (a_{1,0} + t a_{2,0}) e^{-\lambda t} + (a_{3,0} + t a_{4,0}) e^{-\mu t} + a_{5,0} e^{-\xi t} \tag{5.2}$$

where $a_{j,0},\ j = 1,2....,5$ are arbitrary constants.

But if $\varepsilon \neq 0$, following Shamsul [108] , an asymptotic solution of eq.(5.1) is of the form

$$x(t,\varepsilon) = (a_1 + t a_2) e^{-\lambda t} + (a_3 + t a_4) e^{-\mu t} + a_5 e^{-\xi t} + \varepsilon u_1(a_1, a_2, \cdots, a_5, t) + \cdots \tag{5.3}$$

where each $a_j; j = 1, 2, \cdots, 5$, satisfies the equations

$$\dot{a}_j(t) = \varepsilon A_j(a_1, a_2, \cdots, a_5, t) + \cdots \tag{5.4}$$

The Eq.(5.4) are known as variational equations, and KBM [22, 63] assumed that they are functions of amplitude only. But Akbar et al. [8] showed that if they are only functions of amplitude, sometimes the solution gives incorrect results and thus they are functions of both amplitude and phase. But in the case of non-oscillatory systems they are functions of amplitude only.

By considering only the first few terms in the series expansions of (5.3) and (5.4), we calculate the functions u_1 and A_j, where $j = 1,2,\cdots,5$, such that $a_j\,; j = 1,2,\cdots,5$ appearing in (5.3) and (5.4) satisfy the given differential Eq. (5.1) with an accuracy of order ε^{n+1}. In order to determine these unknown functions, it is customary in the KBM method that the correction terms u_1 must exclude secular terms, which make them large. Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However, owing to the rapidly growing algebraic complexity for the derivation of the formulae, the solution is in general confined to a lower order, usually the first-order because , is very small (Murty [74]).

In order to determine the unknown functions $A_j; j = 1,2,\cdots,5$, we differentiate the proposed solution (5.3), five times with respect to t. Substituting the values of x and its derivatives in the original Eq. (5.1), utilizing the relations presented in (5.4) and finally equating the coefficients of ε, we obtain:

$$\begin{aligned}
&e^{-\lambda t}\left(\frac{\partial}{\partial t}-\lambda+\mu\right)^2\left(\frac{\partial}{\partial t}-\lambda+\xi\right)\left(\frac{\partial A_1}{\partial t}+t\frac{\partial A_2}{\partial t}+2A_2\right)+\\
&e^{-\mu t}\left(\frac{\partial}{\partial t}-\mu+\lambda\right)^2\left(\frac{\partial}{\partial t}-\mu+\xi\right)\left(\frac{\partial A_3}{\partial t}+t\frac{\partial A_4}{\partial t}+2A_4\right)+\\
&e^{-\xi t}\left(\frac{\partial}{\partial t}-\xi+\lambda\right)^2\left(\frac{\partial}{\partial t}-\xi+\mu\right)\frac{\partial A_5}{\partial t}+\\
&\left(\frac{\partial}{\partial t}+\lambda\right)\left(\frac{\partial}{\partial t}+\mu\right)\left(\frac{\partial A_4}{\partial t}+\xi\right)u_1\\
&=-f^{(0)}(a_1,a_2,\cdots,a_5,t)
\end{aligned} \tag{5.5}$$

where $f^{(0)} = f(x_0)$ and $x_0 = (a_1 + ta_2)e^{-\lambda t} + (a_3 + ta_4)e^{-\mu t} + a_5 e^{-\xi t}$

The function $f^{(0)}$ can be expanded in a Taylor series (see Murty and Deekshatulu [76] for details) as:

$$f^{(0)} = \sum_{q=0}^{\infty} t^q \sum_{i,j,k,l=1}^{\infty} F_{q,k} e^{-(i\lambda + j\mu + l\xi)t} \tag{5.6}$$

Here the values of i, j, k, l have definite values for particular problem. Thus using eq.(5.6), Eq.(5.5) becomes:

$$\begin{aligned}
&e^{-\lambda t}\left(\frac{\partial}{\partial t} - \lambda + \mu\right)^2 \left(\frac{\partial}{\partial t} - \lambda + \xi\right)\left(\frac{\partial A_1}{\partial t} + t\frac{\partial A_2}{\partial t} + 2A_2\right) + \\
&e^{-\mu t}\left(\frac{\partial}{\partial t} - \mu + \lambda\right)^2 \left(\frac{\partial}{\partial t} - \mu + \xi\right)\left(\frac{\partial A_3}{\partial t} + t\frac{\partial A_4}{\partial t} + 2A_4\right) + \\
&e^{-\xi t}\left(\frac{\partial}{\partial t} - \xi + \lambda\right)^2 \left(\frac{\partial}{\partial t} - \xi + \mu\right)\frac{\partial A_5}{\partial t} + \\
&\left(\frac{\partial}{\partial t} + \lambda\right)\left(\frac{\partial}{\partial t} + \mu\right)\left(\frac{\partial A_4}{\partial t} + \xi\right)u_1 \\
&= -\sum_{q=0}^{\infty} t^q \sum_{i,j,k,l=1}^{\infty} F_{q,k}(a_1, a_2, \cdots, a_5) e^{-(i\lambda + j\mu + l\xi)t}
\end{aligned} \tag{5.7}$$

Following the KBM method, Murty and Deekshatulu [76], Shamsul [108], imposed the condition that u_1 does not contain the fundamental terms of $f^{(0)}$. Therefore,

To do this, eq. (5.7) can be separated for unknown functions $A_j; j = 1, 2, \cdots, 5$ and u_1 in the following way:

$$\begin{aligned}
&e^{-\lambda t}\left(\frac{\partial}{\partial t} - \lambda + \mu\right)^2 \left(\frac{\partial}{\partial t} - \lambda + \xi\right)\left(\frac{\partial A_1}{\partial t} + t\frac{\partial A_2}{\partial t} + 2A_2\right) + \\
&e^{-\mu t}\left(\frac{\partial}{\partial t} - \mu + \lambda\right)^2 \left(\frac{\partial}{\partial t} - \mu + \xi\right)\left(\frac{\partial A_3}{\partial t} + t\frac{\partial A_4}{\partial t} + 2A_4\right) + \\
&e^{-\xi t}\left(\frac{\partial}{\partial t} - \xi + \lambda\right)^2 \left(\frac{\partial}{\partial t} - \xi + \mu\right)\frac{\partial A_5}{\partial t} + \\
&= -\sum_{q=0}^{1} t^q \sum_{i,j,k,l=1}^{\infty} F_{q,k}(a_1, a_2, \cdots, a_5) e^{-(i\lambda + j\mu + l\xi)t}
\end{aligned} \tag{5.8}$$

and

$$\left(\frac{\partial}{\partial t}+\lambda\right)\left(\frac{\partial}{\partial t}+\mu\right)\left(\frac{\partial A_4}{\partial t}+\xi\right)u_1 = -\sum_{q=2}^{\infty} t^q \sum_{i,j,k,l=1}^{\infty} F_{q,k}(a_1,a_2,\cdots,a_5)e^{-(i\lambda+j\mu+l\xi)t} \tag{5.9}$$

Now, equating the coefficients of t^0 and t^1 from both sides of Eq. (5.8), we obtain

$$\begin{aligned}&e^{-\lambda t}\left(\frac{\partial}{\partial t}-\lambda+\mu\right)^2\left(\frac{\partial}{\partial t}-\lambda+\xi\right)\left(\frac{\partial A_1}{\partial t}+2A_2\right)+\\&e^{-\mu t}\left(\frac{\partial}{\partial t}-\mu+\lambda\right)^2\left(\frac{\partial}{\partial t}-\mu+\xi\right)\left(\frac{\partial A_3}{\partial t}+2A_4\right)+\\&e^{-\xi t}\left(\frac{\partial}{\partial t}-\xi+\lambda\right)^2\left(\frac{\partial}{\partial t}-\xi+\mu\right)\frac{\partial A_5}{\partial t}+\\&=-\sum_{i,j,k,l=1}^{\infty} F_{0,k}(a_1,a_2,\cdots,a_5)e^{-(i\lambda+j\mu+l\xi)t}\end{aligned} \tag{5.10}$$

$$\begin{aligned}&e^{-\lambda t}\left(\frac{\partial}{\partial t}-\lambda+\mu\right)^2\left(\frac{\partial}{\partial t}-\lambda+\xi\right)\frac{\partial A_2}{\partial t}+\\&e^{-\mu t}\left(\frac{\partial}{\partial t}-\mu+\lambda\right)^2\left(\frac{\partial}{\partial t}-\mu+\xi\right)\frac{\partial A_4}{\partial t}+\\&=-\sum_{i,j,k,l=1}^{\infty} F_{1,k}(a_1,a_2,\cdots,a_5)e^{-(i\lambda+j\mu+l\xi)t}\end{aligned} \tag{5.11}$$

Here, we have only two eq. (5.10) and (5.11) for determining the unknown functions $A_j; j=1,2,\cdots,5$. Thus, to obtain the unknown functions $A_j; j=1,2,\cdots,5$, we need to impose some conditions between the eigenvalues. Different authors imposed different conditions according to the behavior of the systems. In this study, we have investigated solutions for the case $\lambda \approx 3\mu,\ \mu \approx 3\xi$. Therefore, we shall be able to separate the eq. (5.11) for two unknown functions A_2 and A_4 and solving them for A_2 and A_4 substituting the values into the Eq. (5.10) and applying the condition $\lambda \approx 3\mu,\ \mu \approx 3\xi$. We can separate the eq. (5.10) for two unknown functions A_1 and A_3; and solving them for A_1 and A_3. Since $a_j; j=1,2,\cdots,5$ are proportional to small parameter , they are slowly varying functions of time t and for first approximate solution, we may consider them as constants in the right hand side. This assumption was first made by Murty and Deekshatulu [76]. Thus the solutions of the eq. (5.4) become:

$$a_j(t) = a_j(0) + \varepsilon \int_0^t A_j(a_1, a_2, \cdots, a_5, t)\, dt\,;\; j = 1,2,\cdots,5\,. \tag{5.12}$$

Now, solving eq.(5.9) for u_1 and substituting $a_j; j = 1,2,\cdots,5$ and u_1 in the eq.(5.3), we shall get the complete solution of (5.1).

5.3. Example

As an example of the above procedure, we are going to consider the Duffing type equation of fifth order

$$\frac{d^5 x}{dt^5} + \sum_{i=1}^{4} c_i \frac{d^i x}{dt^i} + c_5 x = -\varepsilon\, x^3 \tag{5.13}$$

Here $f(x,t) = x^3$.

We have $f^{(0)} = ((a_1 + t a_2)e^{-\lambda t} + (a_3 + t a_4)e^{-\mu t} + a_5 e^{-\xi t})^3$

or

$$\begin{aligned}
f^{(0)} &= a_1^3 e^{-3\lambda t} + a_3^3 e^{-3\mu t} + a_5^3 e^{-3\xi t} + 3a_1^2 a_3 e^{-(2\lambda+\mu)t} + 3a_1 a_3^2 e^{-(\lambda+2\mu)t} \\
&+ 3a_1^2 a_5 e^{-(2\lambda+\xi)t} + 6a_1 a_3 a_5 e^{-(\lambda+\mu+\xi)t} + 3a_3^2 a_5 e^{-(2\mu+\xi)t} \\
&+ 3a_1 a_5^2 e^{-(\lambda+2\xi)t} + 3a_3 a_5^2 e^{-(\mu+2\xi)t} + t(3a_1^2 a_2 e^{-3\lambda t} + 3a_2 a_3^2 e^{-(\lambda+2\mu)t} \\
&+ 3a_2 a_5^2 e^{-(\lambda+2\xi)t} + 6a_1 a_2 a_3 e^{-(2\lambda+\mu)t} + 6a_1 a_2 a_5 e^{-(2\lambda+\xi)t} \\
&+ 6a_2 a_3 a_5 e^{-(\lambda+\mu+\xi)t} + 3a_1^2 a_4 e^{-(2\lambda+\mu)t} + 3a_3^2 a_4 e^{-3\mu t} \\
&+ 3a_5^2 a_4 e^{-(\mu+2\xi)t} + 6a_1 a_3 a_4 e^{-(\lambda+2\mu)t} + 6a_1 a_4 a_5 e^{-(\lambda+\mu+\xi)t} \\
&+ 6a_3 a_4 a_5 e^{-(2\mu+\xi)t}) + t^2(3a_1 a_2^2 e^{-3\lambda t} + 3a_2^2 a_3 e^{-(2\lambda+\mu)t} \\
&+ 3a_2^2 a_5 e^{-(2\lambda+\xi)t} + 3a_1 a_4^2 e^{-(\lambda+2\mu)t} + 3a_3 a_4^2 e^{-3\mu t} + 3a_4^2 a_5 e^{-(2\mu+\xi)t} \\
&+ 6a_1 a_2 a_4 e^{-(2\lambda+\mu)t} + 6a_2 a_3 a_4 e^{-(\lambda+2\mu)t} + 6a_2 a_4 a_5 e^{-(\lambda+\mu+\xi)t}) \\
&+ t^3(a_2^3 e^{-3\lambda t} + 3a_2^2 a_4 e^{-(2\lambda+\mu)t} + a_4^3 e^{-3\mu t} + 3a_2 a_4^2 e^{-(\lambda+2\mu)t})
\end{aligned} \tag{5.14}$$

Thus the equations (5.9) to (5.11) takes the form

$$\left(\frac{\partial}{\partial t}+\lambda\right)\left(\frac{\partial}{\partial t}+\mu\right)\left(\frac{\partial A_4}{\partial t}+\xi\right)u_1$$
$$=-\{t^2(3a_1a_2^2e^{-3\lambda t}+3a_2^2a_3e^{-(2\lambda+\mu)t}$$
$$+3a_2^2a_5e^{-(2\lambda+\xi)t}+3a_1a_4^2e^{-(\lambda+2\mu)t}+3a_3a_4^2e^{-3\mu t}+3a_4^2a_5e^{-(2\mu+\xi)t} \tag{5.15}$$
$$+6a_1a_2a_4e^{-(2\lambda+\mu)t}+6a_2a_3a_4e^{-(\lambda+2\mu)t}+6a_2a_4a_5e^{-(\lambda+\mu+\xi)t})$$
$$+t^3(a_2^3e^{-3\lambda t}+3a_2^2a_4e^{-(2\lambda+\mu)t}+a_4^3e^{-3\mu t}+3a_2a_4^2e^{-(\lambda+2\mu)t})\}$$

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}-\lambda+\mu\right)^2\left(\frac{\partial}{\partial t}-\lambda+\xi\right)\frac{\partial A_2}{\partial t}+$$
$$e^{-\mu t}\left(\frac{\partial}{\partial t}-\mu+\lambda\right)^2\left(\frac{\partial}{\partial t}-\mu+\xi\right)\frac{\partial A_4}{\partial t}$$
$$=-\{3a_1^2a_2e^{-3\lambda t}+3a_2a_3^2e^{-(\lambda+2\mu)t}$$
$$+3a_2a_5^2e^{-(\lambda+2\xi)t}+6a_1a_2a_3e^{-(2\lambda+\mu)t}+6a_1a_2a_5e^{-(2\lambda+\xi)t} \tag{5.16}$$
$$+6a_2a_3a_5e^{-(\lambda+\mu+\xi)t}+3a_1^2a_4e^{-(2\lambda+\mu)t}+3a_3^2a_4e^{-3\mu t}$$
$$+3a_5^2a_4e^{-(\mu+2\xi)t}+6a_1a_3a_4e^{-(\lambda+2\mu)t}+6a_1a_4a_5e^{-(\lambda+\mu+\xi)t}$$
$$+6a_3a_4a_5e^{-(2\mu+\xi)t}\}$$

$$e^{-\lambda t}\left(\frac{\partial}{\partial t}-\lambda+\mu\right)^2\left(\frac{\partial}{\partial t}-\lambda+\xi\right)\left(\frac{\partial A_1}{\partial t}+2A_2\right)+$$
$$e^{-\mu t}\left(\frac{\partial}{\partial t}-\mu+\lambda\right)^2\left(\frac{\partial}{\partial t}-\mu+\xi\right)\left(\frac{\partial A_3}{\partial t}+2A_4\right)+$$
$$e^{-\xi t}\left(\frac{\partial}{\partial t}-\xi+\lambda\right)^2\left(\frac{\partial}{\partial t}-\xi+\mu\right)\frac{\partial A_5}{\partial t} \tag{5.17}$$
$$=-\{a_1^3e^{-3\lambda t}+a_3^3e^{-3\mu t}+a_5^3e^{-3\xi t}+3a_1^2a_3e^{-(2\lambda+\mu)t}+3a_1a_3^2e^{-(\lambda+2\mu)t}$$
$$+3a_1^2a_5e^{-(2\lambda+\xi)t}+6a_1a_3a_5e^{-(\lambda+\mu+\xi)t}+3a_3^2a_5e^{-(2\mu+\xi)t}$$
$$+3a_1a_5^2e^{-(\lambda+2\xi)t}+3a_3a_5^2e^{-(\mu+2\xi)t}\}$$

when $\lambda\approx 3\mu;\mu\approx 3\xi$, then from (5.16), we obtain:

$$\left(\frac{\partial}{\partial t}-\lambda+\mu\right)^2\left(\frac{\partial}{\partial t}-\lambda+\xi\right)\frac{\partial A_2}{\partial t}$$
$$=-3a_1^2a_2e^{-2\lambda t}-(6a_1a_2a_3+3a_1^2a_4)e^{-(\lambda+\mu)t} \tag{5.18}$$
$$-(3a_2a_3^2+6a_1a_3a_4)e^{-2\mu t}-6a_1a_2a_5e^{-(\lambda+\xi)t}$$
$$-(6a_1a_4a_5+6a_2a_3a_5)e^{-(\mu+\xi)t}-3a_2a_5^2e^{-2\xi t}-3a_3^2a_4e^{(\lambda-3\mu)t}$$

and

$$e^{-\mu t}\left(\frac{\partial}{\partial t}-\mu+\lambda\right)^2\left(\frac{\partial}{\partial t}-\mu+\xi\right)\frac{\partial A_4}{\partial t} = -3a_5^2a_4e^{-2\xi t}-6a_3a_4a_5e^{-(\mu+\xi)t} \tag{5.19}$$

The solution is thus

$$\begin{aligned} A_2 &= r_1a_1^2a_2e^{-2\lambda t}+r_2(2a_1a_2a_3+a_1^2a_4)e^{-(\lambda+\mu)t} \\ &+r_3(a_2a_3^2+2a_1a_3a_4e)e^{-2\mu t}+r_4a_1a_2a_5e^{-(\lambda+\xi)t} \\ &+r_5(a_1a_4a_5+a_2a_3a_5)e^{-(\mu+\xi)t}+r_6a_2a_5^2e^{-2\xi t}+r_7a_3^2a_4e^{(\lambda-3\mu)} \end{aligned} \tag{5.20}$$

where $r_1=\dfrac{3}{2\lambda(\mu-3\lambda)^2(\xi-3\lambda)}$, $\quad r_2=\dfrac{3}{4\lambda^2(\lambda+\mu)(\xi-2\lambda-\mu)}$,

$r_3=\dfrac{3}{2\mu(\lambda+\mu)^2(\xi-\lambda-2\mu)}$, $\quad r_4=\dfrac{-3}{\lambda(\lambda+\xi)(\mu-2\lambda-\xi)^2}$,

$r_5=\dfrac{-6}{(\mu+\xi)(\lambda+\xi)^2(\lambda+\mu)}$, $\quad r_6=\dfrac{-3}{2\xi(\mu-\lambda-2\xi)^2(\lambda+\xi)}$,

$r_7=\dfrac{-3}{4\mu^2(\lambda-3\mu)(\xi-3\mu)}$

and

$$A_4=S_1a_5^2a_4e^{-2\xi t}+S_2a_3a_4a_5e^{-(\mu+\xi)t} \tag{5.21}$$

where $S_1=\dfrac{-3}{2\xi(\lambda-\mu-3\xi)^2(\mu+\xi)}$, $\quad S_2=\dfrac{-3}{(\mu+\xi)\mu(\lambda-2\mu-\xi)^2}$

Putting the values of A_2 and A_4 in eq. (5.17) and performing some calculations and then eq.(5.17) can be separated in the following way(following the condition $\lambda\approx 3\mu;\mu\cong 3\xi$ exists among the eigen-values):

$$\begin{aligned} \left(\frac{\partial}{\partial t}-\lambda+\mu\right)^2\left(\frac{\partial}{\partial t}-\lambda+\xi\right)\left(\frac{\partial A_1}{\partial t}\right) &= -a_1^3e^{-2\lambda t}-3a_1^2a_3e^{-(\lambda+\mu)t}-3a_1a_3^2e^{-2\mu t} \\ &-3a_1^2a_5e^{-(\lambda+\xi)t}-3a_1a_5^2e^{-2\xi t}-6a_1a_3a_5e^{-(\mu+\xi)t}-\frac{3a_1^2a_2}{\lambda}e^{-2\lambda t} \\ &-\frac{6(2a_1a_2a_3+a_1^2a_4)}{\lambda+\mu}e^{-(\lambda+\mu)t}-\frac{(a_2a_3^2+2a_1a_3a_4)}{\lambda}e^{-2\lambda t}-\frac{12a_1a_2a_5}{\lambda+\xi}e^{-(\lambda+\xi)t} \\ &-\frac{12(a_1a_4a_5+a_2a_3a_5)}{\mu+\xi}e^{-(\mu+\xi)t}-\frac{3a_2a_5^2}{\xi}e^{-2\xi t}-a_3^3e^{(\lambda-3\mu)t}-\frac{6a_3^2a_4}{\lambda-3\mu}e^{(\lambda-3\mu)t} \end{aligned} \tag{5.22}$$

$$\left(\frac{\partial}{\partial t}-\mu+\lambda\right)^2\left(\frac{\partial}{\partial t}-\mu+\xi\right)\left(\frac{\partial A_3}{\partial t}\right)$$
$$=-a_5^3e^{(\mu-3\xi)t}-3a_3^2a_5e^{-(\mu+\xi)t}-3a_3a_5^2e^{-2\xi t} \tag{5.23}$$
$$-\frac{3a_5^2a_4}{\xi}e^{-2\xi t}-\frac{12a_3a_4a_5}{\mu+\xi}e^{-(\mu+\xi)t}$$

and $$\left(\frac{\partial}{\partial t}-\xi+\lambda\right)^2\left(\frac{\partial}{\partial t}-\xi+\mu\right)^2\left(\frac{\partial A_5}{\partial t}\right)=0 \tag{5.24}$$

Now, solving (5.22), (5.23) and (5.24), we get

$$\begin{aligned}A_1 &= p_1a_1^3e^{-2\lambda t}+p_2a_1^2a_3e^{-(\lambda+\mu)t}+p_3a_1a_3^2e^{-2\mu t}+p_4a_1^2a_5e^{-(\lambda+\xi)t}+p_5a_1a_5^2e^{-2\xi t}\\ &+p_6a_1a_3a_5e^{-(\mu+\xi)t}+p_8(2a_1a_2a_3+a_1^2a_4)e^{-(\lambda+\mu)t}+p_9(a_2a_3^2+2a_1a_3a_4)e^{-2\mu t}\\ &+p_7a_1^2a_2e^{-2\lambda t}+p_{10}a_1a_2a_5e^{-(\lambda+\xi)t}+p_{11}(a_1a_4a_5+a_2a_3a_5)e^{-(\mu+\xi)t}+p_{12}a_2a_5^2e^{-2\xi t}\\ &+p_{13}a_3^3e^{(\lambda-3\mu)t}+p_{14}a_3^2a_4e^{(\lambda-3\mu)t}\end{aligned} \tag{5.25}$$

where $p_1=\frac{1}{2\lambda(\mu-3\lambda)^2(\xi-3\lambda)}$, $p_2=\frac{3}{4\lambda^2(\lambda+\mu)(\xi-2\lambda-\mu)}$,

$p_3=\frac{3}{2\mu(\lambda+\mu)^2(\xi-\lambda-2\mu)}$, $p_4=\frac{-3}{2\lambda(\lambda+\xi)(\mu-2\lambda-\xi)^2}$,

$p_5=\frac{-3}{2\xi(\lambda+\xi)(\mu-\lambda-2\xi)^2}$, $p_6=\frac{-6}{(\mu+\xi)(\lambda+\xi)^2(\lambda+\mu)}$,

$p_7=\frac{3}{2\lambda^2(\xi-3\lambda)(\mu-3\lambda)^2}$, $p_8=\frac{3}{2\lambda^2(\lambda+\mu)^2(\xi-2\lambda-\mu)}$,

$p_9=\frac{3}{2\mu^2(\mu-3\lambda)^2(\xi-3\lambda)}$, $p_{10}=\frac{-6}{\lambda(\lambda+\xi)^2(\mu-2\lambda-\xi)^2}$,

$p_{11}=\frac{-12}{(\mu+\xi)^2(\lambda+\xi)^2(\lambda+\mu)}$, $p_{12}=\frac{-3}{2\xi^2(\lambda+\xi)(\mu-\lambda-2\xi)^2}$,

$p_{13}=\frac{-6}{\lambda(\lambda+\xi)^2(\mu-2\lambda-\xi)^2}$, $p_{14}=\frac{3}{(\lambda-3\mu)^2 2\mu^2(\xi-3\mu)}$,

$$\begin{aligned}A_3 &= l_1a_3^2a_5e^{-(\mu+\xi)t}+l_2a_3a_5^2e^{-2\xi t}\\ &+l_3a_5^2a_4e^{-2\xi t}+l_4a_3a_4a_5e^{-(\mu+\xi)t}+l_5a_5^3e^{-(\lambda_2+\lambda_3)t}\end{aligned} \tag{5.26}$$

$l_2=\frac{-3}{2\mu(\mu+\xi)(\lambda-2\mu-\xi)^2}$, $l_2=\frac{-3}{2\xi(\mu+\xi)(\lambda-\mu-2\xi)^2}$,

$$l_3 = \frac{-3}{2\xi^2(\mu+\xi)(\lambda-\mu-2\xi)^2}, \qquad l_4 = \frac{-6}{\mu(\mu+\xi)^2(\lambda-2\mu-\xi)^2},$$

$$l_5 = \frac{1}{2\xi(\mu-3\xi)(\lambda-3\xi)^2}$$

and $A_5 = 0$ (5.27)

and the solution of the equation

$$\begin{aligned} u_1 &= a_1 a_2^2 (q_1 t^2 + q_2 t + q_3) e^{-3\lambda t} + (a_1 a_4^2 + 2a_2 a_3 a_4)(q_4 t^2 + q_5 t + q_6) e^{-(\lambda+2\mu)t} \\ &+ (2a_1 a_2 a_4 + a_2^2 a_3)(q_7 t^2 + q_8 t + q_9) e^{-(2\lambda+\mu)t} + a_3 a_4^2 (q_{10} t^2 + q_{11} t + q_{12}) e^{-3\mu t} \\ &+ a_2^2 a_5 (q_{13} t^2 + q_{14} t + q_{15}) e^{-(2\lambda+\xi)t} + a_4^2 a_5 (q_{16} t^2 + q_{17} t + q_{18}) e^{-(2\mu+\xi)t} \\ &+ a_2 a_4 a_5 (q_{19} t^2 + q_{20} t + q_{21}) e^{-(\lambda+\mu+\xi)t} + a_2^3 (q_{22} t^3 + q_{23} t^2 + q_{24} t + q_{25}) e^{-3\lambda t} \\ &+ a_4^3 (q_{26} t^3 + q_{27} t^2 + q_{28} t + q_{29}) e^{-3\mu t} + a_2^2 a_4 (q_{30} t^3 + q_{31} t^2 + q_{32} t + q_{33}) e^{-(2\lambda+\mu)t} \\ &+ a_2 a_4^2 (q_{34} t^3 + q_{35} t^2 + q_{36} t + q_{37}) e^{-(\lambda+2\mu)t} \end{aligned} \quad (5.28)$$

where

$$q_1 = \frac{3}{4\lambda^2(3\lambda-\mu)^2(3\lambda-\xi)}, \quad q_2 = q_1\left(\frac{2}{\lambda} + \frac{2}{3\lambda-\xi} + \frac{4}{3\lambda-\mu}\right),$$

$$q_3 = q_1 \begin{pmatrix} \frac{3}{2\lambda^2} + \frac{2}{(3\lambda-\xi)^2} + \frac{2}{\mu(3\lambda-\xi)} + \frac{4}{(3\lambda-\mu)(3\lambda-\xi)} \\ + \frac{6}{(3\lambda-\mu)^2} + \frac{6}{\lambda(3\lambda-\mu)} \end{pmatrix},$$

$$q_4 = \frac{3}{4\mu^2(\lambda+\mu)^2(\lambda+2\mu-\xi)}, \qquad q_5 = q_4\left(\frac{2}{4\mu} + \frac{4}{\lambda+\mu} + \frac{2}{\lambda+2\mu-\xi}\right),$$

$$q_6 = q_4 \begin{pmatrix} \frac{6}{(\lambda+\mu)^2} + \frac{4}{\mu(\lambda+\mu)} + \frac{2}{\mu(\lambda+2\mu-\xi)} \\ + \frac{4}{(\lambda+\mu)(\lambda+2\mu-\xi)} + \frac{2}{(\lambda+2\mu-\xi)^2} + \frac{3}{2\mu^2)} \end{pmatrix},$$

$$q_7 = \frac{3}{4\lambda^2(\lambda+\mu)^2(2\lambda+\mu-\xi)}, \qquad q_8 = q_7\left(\frac{2}{\lambda} + \frac{4}{\lambda+\lambda} + \frac{2}{2\lambda+\mu-\xi}\right),$$

$$q_9 = q_7 \begin{pmatrix} \frac{3}{2\lambda^2} + \frac{4}{\lambda(\lambda+\mu)} + \frac{6}{(\lambda+\mu)^2} + \frac{2}{\lambda(2\lambda+\mu-\xi)} \\ + \frac{4}{(\lambda+\mu)(2\lambda+\mu-\xi)} + \frac{3}{(2\lambda+\mu-\xi)^2} \end{pmatrix},$$

$$q_{10}=\frac{3}{4\mu^2(3\mu-\lambda)^2(3\mu-\xi)^2}, \qquad q_{11}=q_{10}\left(\frac{2}{\mu}+\frac{4}{3\mu-\lambda}+\frac{2}{3\mu-\xi}\right),$$

$$q_{12}=q_{10}\left(\begin{array}{l}\frac{3}{2\mu^2}+\frac{4}{\mu(3\mu-\lambda)}+\frac{6}{(3\mu-\lambda)^2}+\frac{2}{\mu(3\mu-\xi)}\\ \quad+\frac{4}{(3\mu-\lambda)(3\mu-\xi)}+\frac{2}{(3\mu-\xi)^2}\end{array}\right),$$

$$q_{13}=\frac{3}{2\lambda(\lambda+\xi)^2(2\lambda-\mu+\xi)^2}, \qquad q_{14}=q_{13}\left(\frac{1}{\lambda}+\frac{4}{\lambda+\xi}+\frac{4}{2\lambda-\mu+\xi}\right),$$

$$q_{15}=q_{13}\left(\begin{array}{l}\frac{1}{2\lambda^2}+\frac{2}{\lambda(\lambda+\xi)}+\frac{2}{\lambda(2\lambda-\mu+\xi)^2}+\frac{6}{(\lambda+\xi)^2}\\ \quad+\frac{8}{(\lambda+\xi)(2\lambda-\mu+\xi)}+\frac{6}{(2\lambda-\mu+\xi)^2}\end{array}\right),$$

$$q_{16}=\frac{3}{2\mu(2\mu+\xi-\lambda)^2(\mu+\xi)^2}, \qquad q_{17}=q_{16}\left(\frac{1}{\mu}+\frac{4}{2\mu+\xi-\lambda}+\frac{4}{\mu+\xi}\right),$$

$$q_{18}=q_{16}\left(\begin{array}{l}\frac{1}{2\mu^2}+\frac{2}{\mu(2\mu+\xi-\lambda)}+\frac{2}{\mu(\mu+\xi)^2}+\frac{6}{(2\mu+\xi-\lambda)^2}\\ \quad+\frac{8}{(\mu+\xi)(2\mu+\xi-\lambda)}+\frac{6}{(\mu+\xi)^2}\end{array}\right),$$

$$q_{19}=\frac{6}{(\mu+\xi)^2(\lambda+\xi)^2(\lambda+\mu)}, \qquad q_{20}=q_{19}\left(\frac{4}{\mu+\xi}+\frac{4}{\lambda+\xi}+\frac{2}{\lambda+\mu}\right),$$

$$q_{21}=q_{19}\left(\begin{array}{l}\frac{6}{(\mu+\xi)^2}+\frac{8}{(\mu+\xi)(\lambda+\xi)}+\frac{6}{(\lambda+\xi)^2}+\frac{4}{(\lambda+\mu)(\mu+\xi)}\\ \quad+\frac{4}{(\lambda+\mu)(\lambda+\xi)}+\frac{2}{(\lambda+\mu)^2}\end{array}\right),$$

$$q_{22}=\frac{1}{4\lambda^2(3\lambda-\mu)^2(3\lambda-\xi)}, \quad q_{23}=q_{22}\left(\frac{3}{\lambda}+\frac{6}{3\lambda-\mu}+\frac{3}{3\lambda-\xi}\right),$$

$$q_{24}=q_{22}\left(\begin{array}{l}\frac{9}{2\lambda^2}+\frac{12}{\lambda(3\lambda-\mu)}+\frac{18}{(3\lambda-\mu)^2}+\frac{6}{\lambda(3\lambda-\xi)}\\ \quad+\frac{12}{(3\lambda-\mu)(3\lambda-\xi)}+\frac{6}{(3\lambda-\xi)^2}\end{array}\right),$$

$$q_{25}=q_{22}\begin{pmatrix}\frac{3}{\lambda^3}+\frac{9}{2\lambda^2(3\lambda-\mu)}+\frac{18}{\lambda(3\lambda-\mu)^2}+\frac{24}{(3\lambda-\mu)^3}\\+\frac{9}{2\lambda^2(3\lambda-\xi)}+\frac{12}{\lambda(3\lambda-\mu)}+\frac{18}{(3\lambda-\mu)^2(3\lambda-\xi)}\\+\frac{6}{\lambda(3\lambda-\xi)^2}+\frac{12}{(3\lambda-\mu)(3\lambda-\xi)^2}+\frac{6}{(3\lambda-\xi)^3}\end{pmatrix},$$

$$q_{26}=\frac{1}{4\mu^2(3\mu-\lambda)^2(3\mu-\xi)},\qquad q_{27}=q_{26}(\frac{3}{\mu}+\frac{6}{3\mu-\lambda}+\frac{3}{3\mu-\xi}),$$

$$q_{28}=q_{26}\begin{pmatrix}\frac{9}{2\mu^2}+\frac{12}{\mu(3\mu-\lambda)}+\frac{18}{(3\mu-\lambda)^2}+\frac{6}{\mu(3\mu-\xi)}\\+\frac{12}{(3\mu-\lambda)(3\mu-\xi)}+\frac{6}{(3\mu-\xi)^2}\end{pmatrix},$$

$$q_{29}=q_{26}\begin{pmatrix}\frac{3}{\mu^3}+\frac{9}{\mu^2(3\mu-\lambda)}+\frac{18}{\mu(3\mu-\lambda)^2}+\frac{24}{(3\mu-\lambda)^3}\\+\frac{9}{2\mu^2(3\mu-\xi)}+\frac{12}{\mu(3\mu-\lambda)(3\mu-\xi)}+\frac{18}{(3\mu-\lambda)^2(3\mu-\xi)}\\+\frac{6}{\mu(3\mu-\xi)^2}+\frac{12}{(3\mu-\lambda)(3\mu-\xi)^2}+\frac{6}{(3\mu-\xi)^3}\end{pmatrix},$$

$$q_{30}=\frac{3}{4\lambda^2(\lambda+\mu)^2(2\lambda+\mu-\xi)},\qquad q_{31}=q_{30}(\frac{3}{\lambda}+\frac{6}{\lambda+\mu}+\frac{3}{2\lambda+\mu-\xi}),$$

$$q_{32}=q_{30}\begin{pmatrix}\frac{9}{2\lambda^2}+\frac{12}{\lambda(\lambda+\mu)}+\frac{18}{(\lambda+\mu)^2}+\frac{6}{(2\lambda+\mu-\xi)^2}\\+\frac{12}{(\lambda+\mu)(2\lambda+\mu-\xi)}+\frac{6}{\lambda(2\lambda+\mu-\xi)}\end{pmatrix},$$

$$q_{33}=q_{30}\begin{pmatrix}\frac{3}{\lambda^3}+\frac{9}{\lambda^2(\lambda+\mu)}+\frac{18}{\lambda(\lambda+\mu)^2}+\frac{12}{(\lambda+\mu)(2\lambda+\mu-\xi)^2}\\+\frac{9}{2\lambda^2(3\lambda+\mu-\xi)}+\frac{18}{(\lambda+\mu)^2(2\lambda+\mu-\xi)}+\frac{6}{\lambda(2\lambda+\mu-\xi)^2}\\+\frac{12}{\lambda(\lambda+\mu)(2\lambda+\mu-\xi)}+\frac{24}{(\lambda+\mu)^3}+\frac{6}{(2\lambda+\mu-\xi)^3}\end{pmatrix},$$

$$q_{34}=\frac{3}{4\mu^2(\lambda+\mu)^2(\lambda+2\mu-\xi)},\qquad q_{35}=q_{34}(\frac{3}{\mu}+\frac{6}{\lambda+\mu}+\frac{3}{\lambda+2\mu-\xi}),$$

$$q_{36} = q_{34}\begin{pmatrix} \dfrac{9}{2\mu^2} + \dfrac{12}{\mu(\lambda+\mu)} + \dfrac{18}{(\lambda+\mu)^2} + \dfrac{6}{(\lambda+2\mu-\xi)^2} \\ + \dfrac{12}{(\lambda+\mu)(\lambda+2\mu-\xi)} + \dfrac{6}{\mu(\lambda+2\mu-\xi)} \end{pmatrix},$$

$$q_{37} = q_{34}\begin{pmatrix} \dfrac{3}{\mu^3} + \dfrac{9}{\mu^2(\lambda+\mu)} + \dfrac{18}{\mu(\lambda+\mu)^2} + \dfrac{12}{(\lambda+\mu)(\lambda+2\mu-\xi)^2} \\ + \dfrac{9}{2\mu^2(\lambda+2\mu-\xi)} + \dfrac{18}{(\lambda+\mu)^2(\lambda+2\mu-\xi)} + \dfrac{6}{\mu(\lambda+2\mu-\xi)^2} \\ + \dfrac{12}{\mu(\lambda+\mu)(\lambda+2\mu-\xi)} + \dfrac{24}{(\lambda+\mu)^3} + \dfrac{6}{(\lambda+2\mu-\xi)^3} \end{pmatrix}$$

When we Substituting the values of A_1, A_2, A_3, A_4 and A_5 from the eq. (5.20), (5.21), (5.25), (5.26) and (5.27) into eq. (5.4) , then eq. (5.4) reduces to

$$\begin{aligned} \dot{a}_1(t) = \varepsilon(& p_1 a_1^3 e^{-2\lambda t} + p_2 a_1^2 a_3 e^{-(\lambda+\mu)t} + p_3 a_1 a_3^2 e^{-2\mu t} + p_4 a_1^2 a_5 e^{-(\lambda+\xi)t} + p_5 a_1 a_5^2 e^{-2\xi t} \\ & + p_6 a_1 a_3 a_5 e^{-(\mu+\xi)t} + p_8(2a_1 a_2 a_3 + a_1^2 a_4) e^{-(\lambda+\mu)t} + p_9(a_2 a_3^2 + 2a_1 a_3 a_4) e^{-2\mu t} \\ & + p_7 a_1^2 a_2 e^{-2\lambda t} + p_{10} a_1 a_2 a_5 e^{-(\lambda+\xi)t} + p_{11}(a_1 a_4 a_5 + a_2 a_3 a_5) e^{-(\mu+\xi)t} + p_{12} a_2 a_5^2 e^{-2\xi t} \\ & + p_{13} a_3^3 e^{(\lambda-3\mu)t} + p_{14} a_3^2 a_4 e^{(\lambda-3\mu)t}) \end{aligned}$$

$$\begin{aligned} \dot{a}_2(t) = \varepsilon(& r_1 a_1^2 a_2 e^{-2\lambda t} + r_2(2a_1 a_2 a_3 + a_1^2 a_4) e^{-(\lambda+\mu)t} \\ & + r_3(a_2 a_3^2 + 2a_1 a_3 a_4 e) e^{-2\mu t} + r_4 a_1 a_2 a_5 e^{-(\lambda+\xi)t} \\ & + r_5(a_1 a_4 a_5 + a_2 a_3 a_5) e^{-(\mu+\xi)t} + r_6 a_2 a_5^2 e^{-2\xi t} + r_7 a_3^2 a_4 e^{(\lambda-3\mu)}) \end{aligned}$$

$$\begin{aligned} \dot{a}_3(t) = \varepsilon(& l_1 a_3^2 a_5 e^{-(\mu+\xi)t} + l_2 a_3 a_5^2 e^{-2\xi t} \\ & + l_3 a_5^2 a_4 e^{-2\xi t} + l_4 a_3 a_4 a_5 e^{-(\mu+\xi)t} + l_5 a_5^3 e^{-2\xi t}) \end{aligned}$$

$$\dot{a}_4(t) = \varepsilon(S_1 a_5^2 a_4 e^{-2\xi t} + S_2 a_3 a_4 a_5 e^{-(\mu+\xi)t})$$

and

$$\dot{a}_5(t) = 0 \tag{5.29}$$

These all of the eq. (5.29) are nonlinear and have no exact solutions. But since $\dot{a}_j;\ j = 1,2,\cdots,5$ are proportional to the small parameter ε, so they are slowly varying functions of time t. Thus, we can solve (5.29) by considering $a_j;\ j = 1,2,\cdots,5$ are constants in the right-hand sides of (5.29). This assumption was used by Murty et al. [75,76] to solve the similar nonlinear equations. The solution is thus

$$a_1(t) = a_{1,0} + \varepsilon \begin{pmatrix} p_1 a_1^3 \dfrac{1-e^{-2\lambda t}}{2\lambda} + p_2 a_1^2 a_3 \dfrac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + p_3 a_1 a_3^2 \dfrac{1-e^{-2\mu t}}{2\mu} + p_4 a_1^2 a_5 \dfrac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + p_5 a_1 a_5^2 \dfrac{1-e^{-2\xi t}}{2\xi} \\ + p_6 a_1 a_3 a_5 \dfrac{1-e^{-(\mu+\xi)t}}{\mu+\xi} + p_8 (2a_1 a_2 a_3 + a_1^2 a_4) \dfrac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} + p_9 (a_2 a_3^2 + 2a_1 a_3 a_4) \dfrac{1-e^{-2\mu t}}{2\mu} \\ + p_7 a_1^2 a_2 \dfrac{1-e^{-2\lambda t}}{2\lambda} + p_{10} a_1 a_2 a_5 \dfrac{1-e^{-(\lambda+\xi)t}}{\lambda+\xi} + p_{11} (a_1 a_4 a_5 + a_2 a_3 a_5) \dfrac{1-e^{-(\mu+\xi)t}}{\mu+\xi} \\ + p_{12} a_2 a_5^2 \dfrac{1-e^{-2\xi t}}{2\xi} + p_{13} a_3^3 \dfrac{1-e^{(\lambda-3\mu)t}}{3\mu-\lambda} + p_{14} a_3^2 a_4 \dfrac{1-e^{(\lambda-3\mu)t}}{3\mu-\lambda} \end{pmatrix}$$

$$a_2(t) = a_{2,0} + \varepsilon \begin{pmatrix} r_1 a_1^2 a_2 \dfrac{1-e^{-2\lambda t}}{2\lambda} + r_2 (2a_1 a_2 a_3 + a_1^2 a_4) \dfrac{1-e^{-(\lambda+\mu)t}}{\lambda+\mu} \\ + r_3 (a_2 a_3^2 + 2a_1 a_3 a_4 e) \dfrac{1-e^{-2\mu t}}{2\mu} + r_4 a_1 a_2 a_5 \dfrac{1-e^{-(\lambda+\xi)t}}{\lambda+\xi} \\ + r_5 (a_1 a_4 a_5 + a_2 a_3 a_5) \dfrac{1-e^{-(\mu+\xi)t}}{\mu+\xi} + r_6 a_2 a_5^2 \dfrac{1-e^{-2\xi t}}{2\xi} + r_7 a_3^2 a_4 \dfrac{1-e^{(\lambda-3\mu)}}{3\mu-\lambda} \end{pmatrix}$$

$$a_3(t) = a_{3,0} + \varepsilon \begin{pmatrix} \varepsilon (l_1 a_3^2 a_5 \dfrac{1-e^{-(\mu+\xi)t}}{\mu+\xi} + l_2 a_3 a_5^2 \dfrac{1-e^{-2\xi t}}{2\xi} \\ + l_3 a_5^2 a_4 \dfrac{1-e^{-2\xi t}}{2\xi} + l_4 a_3 a_4 a_5 \dfrac{1-e^{-(\mu+\xi)t}}{\mu+\xi} + l_5 a_5^3 \dfrac{1-e^{-2\xi t}}{2\xi}) \end{pmatrix}$$

$$a_4(t) = a_{4,0} + \varepsilon (S_1 a_5^2 a_4 \frac{1-e^{-2\xi t}}{2\xi} + S_2 a_3 a_4 a_5 \frac{1-e^{-(\mu+\xi)t}}{\mu+\xi})$$

and

$$a_5(t) = a_{5,0} \tag{5.30}$$

Finally, we obtain the solution in the form

$$x(t,\varepsilon) = (a_1(t) + t a_2(t)) e^{-\lambda t} + (a_3(t) + t a_4(t)) e^{-\mu t} + a_5(t) e^{-\xi t} + \varepsilon u_1(a_1(t), a_2(t), \cdots, a_5(t), t) \tag{5.31}$$

Here eq. (5.31) is the first order approximate solution of eq. (5.13), where $a_1(t)$, $a_2(t)$, $a_3(t)$, $a_4(t)$ and $a_5(t)$ are given by the eq. (5.30) and the value of u_1 is given by the eq. (5.28).

5.4. Results and Discussions

In order to check the accuracy of an analytical approximate solution obtained based on KBM method, we compare the approximate solution to the numerical solution. In this chapter, we have compared our obtained results (by perturbation) to those obtained by the fourth order Runge-Kutta method for different sets of initial conditions as well as different sets of eigenvalues.

Firstly, for $\lambda = 4.6,\ \mu = 1.6,\ \xi = 0.6$ and $\varepsilon = 0.1$, $x(t,\varepsilon)$ has been computed (5.31), in which $a_1(t), a_2(t), a_3(t), a_4(t)$ and $a_5(t)$ by the equation (5.30) with initial conditions

$$a_{1,0} = 0.20, a_{2,0} = 0.15, a_{3,0} = 0.20, a_{4,0} = 0.13,\ a_{5,0} = 0.15$$

i.e,

$$x(0) = 1.0107571,\ \frac{dx(0)}{dt} = -3.2072265,\ \frac{d^2x(0)}{dt^2} = 13.0002677,$$

$$\frac{d^3x(0)}{dt^3} = -56.34212365,\ \frac{d^4x(0)}{dt^4} = 246.9637919.$$

In this section, the perturbation results obtained by the solution (5.31) and the corresponding numerical results obtained by a fourth order Runge-Kutta method with a small time increment 0.5, are plotted (Fig. 5.1). The correlation between the results is 0.998762.

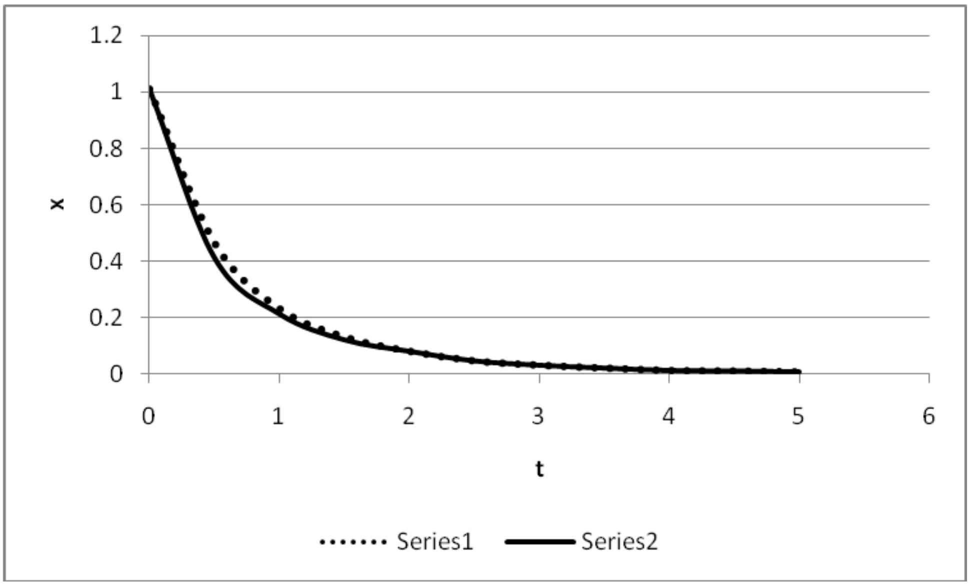

Fig. 5.1. Perturbation solution plotted by solid line and numerical solution plotted by dotted line.

Finally, for $\lambda = 3.8,\ \mu = 1.3,\ \xi = 0.5$ and $\varepsilon = 0.1$, $x(t,\varepsilon)$ has been computed (5.31), in which $a_1(t), a_2(t), a_3(t), a_4(t)$ and $a_5(t)$ by the equation (5.30) with initial conditions

$$a_{1,0} = 0.15, a_{2,0} = 0.04, a_{3,0} = 0.15, a_{4,0} = 0.04,\ a_{5,0} = 0.03$$

i.e,

$$x(0) = 1.113555, \ \frac{dx(0)}{dt} = -3.700838, \ \frac{d^2x(0)}{dt^2} = 13.5102566,$$

$$\frac{d^3x(0)}{dt^3} = -50.629624, \ \frac{d^4x(0)}{dt^4} = 191.0510405.$$

In this section, the perturbation results obtained by the solution (5.31) and the corresponding numerical results obtained by a fourth order Runge-Kutta method with a small time increment 0.5, are plotted (Fig. 5.2). The correlation between the results is 0.999469.

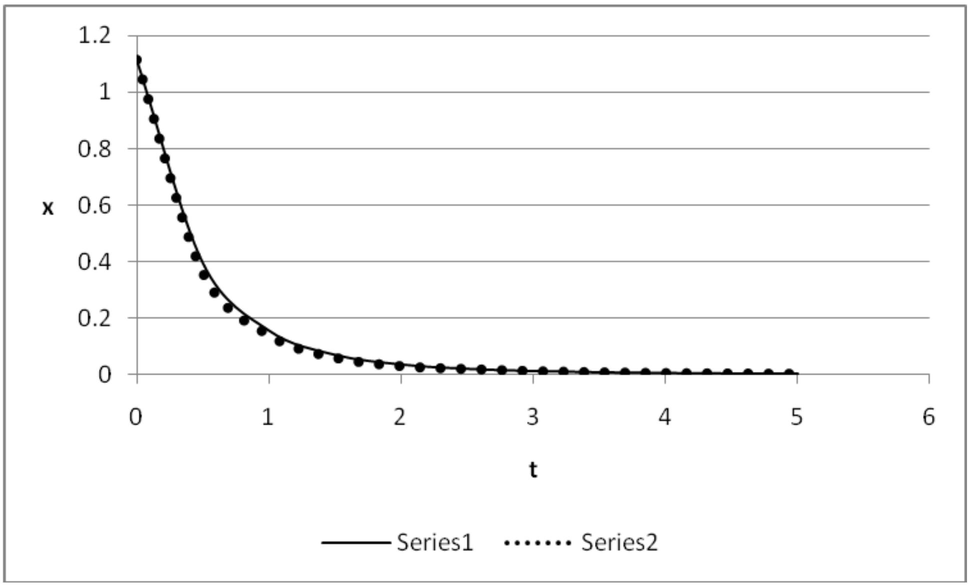

Fig. 5.2. Perturbation solution plotted by solid line and numerical solution plotted by dotted line.

Finally, for $\lambda = 2.2, \ \mu = 0.8, \ \xi = 0.3$ and $\varepsilon = 0.1$, $x(t,\varepsilon)$ has been computed (5.31), in which $a_1(t), a_2(t), a_3(t), a_4(t)$ and $a_5(t)$ by the equation (5.30) with initial conditions

$$a_{1,0} = 0.1, a_{2,0} = 0.07, a_{3,0} = 0.1, a_{4,0} = 0.07, \ a_{5,0} = 0.1$$

i.e,

$$x(0) = 1.020423, \ \frac{dx(0)}{dt} = -1.8230321, \ \frac{d^2x(0)}{dt^2} = 3.810971,$$

$$\frac{d^3x(0)}{dt^3} = -8.221826, \ \frac{d^4x(0)}{dt^4} = 17.655039.$$

In this section, the perturbation results obtained by the solution (5.31) and the corresponding numerical results obtained by a fourth order Runge-Kutta method with a small time increment 0.5, are plotted (Fig. 5.3). The correlation between the results is 0.999816.

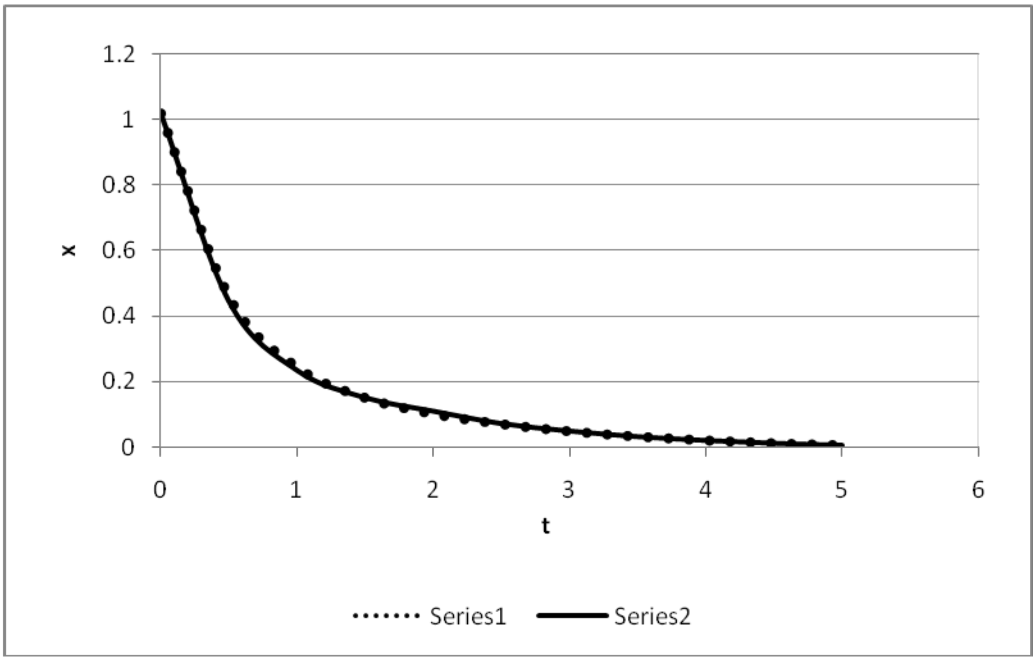

Fig. 5.3. Perturbation solution plotted by solid line and numerical solution plotted by dotted line.

3.5. Conclusion

A procedure is formulated to find the analytical first order approximate solution of fifth order critical damped nonlinear differential systems extending the KBM method to obtain transient response in this chapter. The results show good coincidence with numerical results for different sets of initial conditions as well as for different damping forces.

Chapter-Six

Asymptotic Solutions of Second Order Nonlinear Vibrating Systems with Slowly Varying Coefficients

6.1. Introduction:

Vibrations occur in almost all spring related things and their physical model are nonlinear Differential systems whose coefficients change slowly and periodically with time. The most common methods for constructing the analytical approximate solutions to the nonlinear oscillator equations are the perturbation methods. Most of the perturbation methods are based on an assumption that small parameter must exist in the equations. Krylov and Bogoliubov [63] originally developed a perturbation method to obtain an approximate solution of a second order nonlinear differential system. Then the method was amplified and justified by Bogoliubov and Mitropolskii [22]. Mitropolskii [73] has extended the method to nonlinear differential system with slowly varying coefficients. Following the extended KBM method , Arya and Bodadziev [15], Bojadziev and Edwards [34] studied some damped oscillatory and purely non-oscillatory systems with slowly varying coefficients. Murty [74] presented a unified KBM method for both under-damped and over-damped system with constant coefficients. Shamsul [110] presented a unified formula to obtain a general solution of an n-th order ordinary differential equation with constant and slowly varying coefficients. Hung and Wu [55] obtained an exact solution of a differential system in terms of Bessel's functions where the coefficients varying with time in an exponential order.

In the previous chapter, we established some procedure with classical KBM method but from this chapter we start with the differential equations with slowly varying coefficients. The aim of this chapter is to find an approximate solution of such nonlinear differential systems [49] based on the extended KBM (by Popov [91]) method in which the coefficients change slowly and periodically with time. Furthermore, a non-autonomous case also investigated in which an external force acts in this system.

6.2. The Method:

Let us consider the nonlinear differential system

$$\ddot{x} + (c_1 + c_2 \cos\tau + c_3 \sin\tau)x = -\varepsilon f(x,\tau), \qquad \tau = \varepsilon t \tag{6.1}$$

where the over-dots denote differentiation with respect to t, ε is a small parameter, c_1, c_2 and c_3 are constants, $c_2 = c_3 = O(\varepsilon)$, f is a given nonlinear function. Setting $\omega^2(\tau) = (c_1 + c_2 \cos\tau + c_3 \sin\tau)$, $\omega(\tau)$ is known as frequency.

For $\varepsilon = 0$ and $\tau = \tau_0 =$ constant, $\lambda_1(\tau_0) = i\omega(\tau_0)$, $\lambda_2(\tau_0) = -i\omega(\tau_0)$ are two eigen values of the unperturbed equation of (6.1) and has the solution

$$x(t,0) = \sum_{j=0}^{2} a_{j,0} e^{\lambda_j(\tau_0)t} . \tag{6.2}$$

When $\varepsilon \neq 0$ i. e, for unperturbed equations, we seek a solution in accordance with Shamsul [108] or Murty and Deekshatulu [76] or the KBM [22, 63] method, of the form

$$x(t,\varepsilon) = \sum_{j=1}^{2} a_{j,0}(t,\tau) + \varepsilon u_1(a_1,a_2,\tau) + \varepsilon^2 u_2(a_1,a_2,\tau) + ..., \tag{6.3}$$

where a_1 and a_2 satisfy the differential equations

$$\begin{aligned} \dot{a}_1 &= \lambda_1(\tau)a_1 + \varepsilon A_1(a_1,a_2,\tau) + \varepsilon^2 ..., \\ \dot{a}_2 &= \lambda_2(\tau)a_2 + \varepsilon A_2(a_1,a_2,\tau) + \varepsilon^2 ..., \end{aligned} \tag{6.4}$$

Taking our interest to the earliest few term $1, 2, \ldots, m$ in the series expansion of (6.3) and (6.4), we estimate functions $u_1, \ldots, A_1, A_2, \ldots,$ such that a_1 and a_2 appearing in (6.3) and

(6.4) gratify (6.1) with an accuracy of ε^{m+1}. In order to resolve these unknown functions it was early assumed that the functions $u_1,...$keep out all fundamental terms, since these are incorporated in the series expansion (6.3) at order ε^0.

Differentiating $x(t,\varepsilon)$ two times with respect to *t***,** substituting for the derivatives $\ddot{x}$ and x in the original equation (1) and equating the coefficient of ε, we get a hold

$$\begin{aligned}&\lambda_1' a_1 + \lambda_2' a_2 - \lambda_2 A_1 - \lambda_1 A_2 + \left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2}\right)(A_1 + A_2)\\ &+\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right)\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right)u_1\\ &= -f^{(0)}(a_1, a_2, \tau),\end{aligned} \tag{6.5}$$

where $\lambda_1' = \dfrac{d\lambda_1}{d\tau}$, $\lambda_2' = \dfrac{d\lambda_2}{d\tau}$, $f^{(0)} = f(x_0, \dot{x}_0, \tau)$ and $x_0 = a_1(t,\tau) + a_2(t,\tau)$.

It is assumed that both $f^{(0)}$ can be expanded in Taylor's series (Murty [74], Shamsul [110])

$$f^{(0)} = \sum_{r_1, r_2 = 0}^{\infty} F_{r_1, r_2}(\tau) a_1^{r_1} a_2^{r_2}\ , \tag{6.6}$$

To obtain this solution (6.1), it has been proposed in (Shamsul [110]) that u_1, u_2 eliminate the terms $a_1^{r_1} a_2^{r_2}$ of $f^{(0)}$, where $r_1 - r_2 = \pm 1$. This limitation guarantees that the solution always excludes *secular*-type terms or the first harmonic terms (see Shamsul [110] for details). According to our assumption, u_1 does not contain the fundamental terms, therefore equation (6.5) can be divided into three equations for unknown functions u_1 and A_1, A_2. we obtain

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right)A_1 + \lambda_1' a_1 = \sum_{r_1=0, r_2=0}^{\infty,\infty} F_{r_1, r_2}(a_1^{r_1}, a_2^{r_2}\ ),\ \text{ if } r_1 = r_2 + 1 \tag{6.7}$$

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right) A_2 + \lambda_2' a_2 = \sum_{r_1=0, r_2=0}^{\infty,\infty} F_{r_1, r_2}(a_1^{r_1}, a_2^{r_2}), \text{ if } r_2 = r_1 + 1 \tag{6.8}$$

and

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right)\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right) u_1 = \sum_{r_1=0, r_2=0}^{\infty,\infty} F_{r_1, r_2}(a_1^{r_1}, a_2^{r_2}), \tag{6.9}$$

where $\sum_{r_1=0, r_2=0}^{\infty,\infty} F_{r_1, r_2}(a_1^{r_1}, a_2^{r_2})$ exclude those terms for $r_1 = r_2 \pm 1$.

Thus the particular solutions of (6.7)-(6.9) give the values of the unknown functions A_1, A_2 and u_1. We have already mentioned that equation (6.1) is not a standard form of KBM method. We shall be able to transform (6.3) to the exact form of the KBM [4, 6, 9, 108] solution by substituting $a_1 = ae^{i\varphi}/2$ and $a_2 = ae^{-i\varphi}/2$. Herein, a and φ are respectively amplitude and phase variables (see Shamsul [110]). Under this assumption, we shall able to find the unknown functions u_1 and A_1, A_2 which completes the determination of the solution of a second order non-linear problem (6.1).

6.3. Examples:

6.3.1. A second order nonlinear problem without external force

We consider a second order nonlinear system with constant and slowly varying coefficient

$$\ddot{x} + (c_1^2 + c_2 \cos\tau + c_3 \sin\tau)x = -\varepsilon x^3, \tag{6.10}$$

Here over dots denote differentiation with respect to t, c_1, c_2 and c_3 are constants, $c_2 = c_3 = O(\varepsilon)$, $x_0 = a_1 + a_2$ and the function $f^{(0)}$ becomes,

$$f^{(0)} = -(a_1^3 + 3a_1^2 a_2 + 3a_1 a_2^2 + a_2^3). \tag{6.11}$$

Following the assumption (discussed in section 6.2) u_1 excludes the terms $3a_1^2 a_2$, $3a_1 a_2^2$.

We stand-in in (6.5) and break up it into two parts as

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right) A_1 + \lambda_1' a_1 + \left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right) A_2 + \lambda_2' a_2 \\ = -(3a_1^2 a_2 + 3a_1 a_2^2) \tag{6.12}$$

and

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right)\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right) u_1 = -(a_1^3 + a_2^3) \tag{6.13}$$

The particular solution of (6.13) is

$$u_1 = -\frac{a_1^3}{2\lambda_1(3\lambda_1 - \lambda_2)} - \frac{a_2^3}{2\lambda_2(3\lambda_2 - \lambda_1)} \tag{6.14}$$

Now, we have to solve (6.12) for two functions A_1 and A_2. According with the unified KBM method A_1 contains the term $3a_1^2 a_2$ and A_2 contains the term $3a_1 a_2^2$ (Shamsul [110]) obtain the following equations

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right) A_1 + \lambda_1' a_1 = -3a_1^2 a_2 \tag{6.15}$$

and

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right) A_2 + \lambda_2' a_2 = -3a_1 a_2^2 \tag{6.16}$$

The particular solutions of (6.15)-(6.16) are

and $$A_1 = -\frac{\lambda_1' a_1}{\lambda_1 - \lambda_2} - \frac{3a_1^2 a_2}{2\lambda_1}, \quad A_2 = \frac{\lambda_2' a_2}{\lambda_1 - \lambda_2} - \frac{3a_1 a_2^2}{2\lambda_2} \tag{6.17}$$

Substituting the functional values of A_1 and A_2 (6.17) into (6.4) and rearranging, we obtain

and

$$\dot{a}_1 = \lambda_1 a_1 + \varepsilon\left(-\frac{\lambda_1' a_1}{\lambda_1 - \lambda_2} - \frac{3a_1^2 a_2}{2\lambda_1} \right), \qquad \dot{a}_2 = \lambda_2 a_2 + \varepsilon\left(\frac{\lambda_2' a_2}{\lambda_1 - \lambda_2} - \frac{3a_1 a_2^2}{2\lambda_2} \right) \tag{6.18}$$

The variational equations of a and φ in the real form (a and φ are know as amplitude and phase) which transform (18) to

and $$\dot{a} = -\frac{\varepsilon a \omega'}{2\omega}, \quad \dot{\varphi} = \omega + \frac{3\varepsilon a^2}{8\omega}, \tag{6.19}$$

where $\omega = \sqrt{c_1 + c_2 \cos\tau + c_3 \sin\tau}$

The variational equation (6.19) is in the form of the KBM solution**.** The variational equations for amplitude and phase are usually appeared in a set of first order differential equations and solved by the numerical technique (see Shamsul [110]).

Therefore, the first order solution of the equation (6.10) is

$$x(t,\varepsilon) = a\cos\varphi + \varepsilon u_1 \tag{6.20}$$

where a and φ are the solution of the equation (6.19).

6.3.2. Let us consider another form of the nonlinear differential problem (6.10)

$$\ddot{x} + c_1 x = -c_2 \cos\tau\, x - c_3 \sin\tau\, x - \varepsilon x^3 = -\varepsilon\, c(\cos\tau + \sin\tau)\, x - \varepsilon x^3, \tag{6.21}$$

where $(c_2 + c_3) = \varepsilon c$ and $c_1 = \omega^2$. Here,

$$f^{(0)} = -(\alpha_1^3 + 3\alpha_1^2\alpha_2 + 3\alpha_1\alpha_2^2 + \alpha_2^3) - c(\cos\tau + \sin\tau)(\alpha_1 + \alpha_2)\ . \tag{6.22}$$

In our statement, u_1 excludes the terms $3\alpha_1^2\alpha_2, 3\alpha_1\alpha_2^2$ and $c(\cos\tau + \sin\tau)(\alpha_1 + \alpha_2)$. The equations of u_1, A_1 and A_2 become (discussed in Section 2)

$$\left(\lambda_1\alpha_1 \frac{\partial}{\partial\alpha_1} + \lambda_2\alpha_2 \frac{\partial}{\partial\alpha_2} - \lambda_1 \right)\left(\lambda_1\alpha_1 \frac{\partial}{\partial\alpha_1} + \lambda_2\alpha_2 \frac{\partial}{\partial\alpha_2} - \lambda_2 \right) u_1 = -(\alpha_1^3 + \alpha_2^3) \tag{6.23}$$

and

$$\left(\lambda_1\alpha_1\frac{\partial}{\partial\alpha_1}+\lambda_2\alpha_2\frac{\partial}{\partial\alpha_2}-\lambda_2\right)A_1=-3\alpha_1^2\alpha_2-c\alpha_1(\cos\tau+\sin\tau),$$
$$\left(\lambda_1\alpha_1\frac{\partial}{\partial\alpha_1}+\lambda_2\alpha_2\frac{\partial}{\partial\alpha_2}-\lambda_1\right)A_2=-3\alpha_1\alpha_2^2-c\alpha_2(\cos\tau+\sin\tau) \tag{6.24}$$

Solution of Eqs. (6.24)-(6.25) are

$$u_1=-\frac{\alpha_1^3}{2\lambda_1(3\lambda_1-\lambda_2)}-\frac{\alpha_2^3}{2\lambda_2(3\lambda_2-\lambda_1)} \tag{6.25}$$

and

$$A_1=-\frac{3\alpha_1^2\alpha_2}{2\lambda_1}-\frac{c\alpha_1(\cos\tau+\sin\tau)}{\lambda_1-\lambda_2},$$
$$A_2=-\frac{3\alpha_1\alpha_2^2}{2\lambda_2}+\frac{c\alpha_2(\cos\tau+\sin\tau)}{\lambda_1-\lambda_2} \tag{6.26}$$

Substituting the functional values of A_1 and A_2 (6.26) into (6.4) and rearranging, we obtain

$$\dot{\alpha}_1=\lambda_1a_1+\varepsilon\left(-\frac{3\alpha_1^2\alpha_2}{2\lambda_1}-\frac{c\alpha_1(\cos\tau+\sin\tau)}{\lambda_1-\lambda_2}\right),$$
$$\dot{\alpha}_2=\lambda_2a_2+\varepsilon\left(-\frac{3\alpha_1\alpha_2^2}{2\lambda_2}+\frac{c\alpha_2(\cos\tau+\sin\tau)}{\lambda_1-\lambda_2}\right) \tag{6.27}$$

The variational equation of α and φ in the real form (α and φ are know as amplitude and phase) which transform (6.27) to

$$\dot{\alpha}=0$$
$$\dot{\varphi}=\omega+\frac{3\varepsilon\alpha^2}{8\omega}+\frac{\varepsilon c(\cos\tau+\sin\tau)}{2\omega}, \tag{6.28}$$

where $\omega^2=c_1$.

Therefore, the first order solution of the equation (6.10) is

$$x(t,\varepsilon)=\alpha\cos\varphi+\varepsilon u_1, \tag{6.29}$$

where α and φ are the solution of the equation (6.27).

6.3.3. Let us consider a second order nonlinear differential system with an external force

$$\ddot{x} + (c_1 + c_2 \cos\tau + c_3 \cos\tau)x = -\varepsilon x^3 + \varepsilon E \cos\nu t \tag{6.30}$$

Here, over dots denote differentiation with respect to t; c_1, c_2 and c_3 are constants, $c_2 = c_3 = O(\varepsilon)$, $x_0 = a_1 + a_2$ and the function

$$f^{(0)} = -(a_1^3 + 3a_1^2 a_2 + 3a_1 a_2^2 + a_2^3) + \frac{\varepsilon E}{2}\left(e^{i\nu t} + e^{-i\nu t}\right). \tag{6.31}$$

Under the limitation (discussed in Section 6.2) u_1 excludes the terms $3a_1^2 a_2,\ 3a_1 a_2^2$. Moreover in our assumption u_1 excludes $\varepsilon E\left(e^{i\nu t} + e^{-i\nu t}\right)/(2)$. We substitute in (6.5) and break up it into two parts as

$$\begin{aligned}&\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right)A_1 + \lambda_1' a_1 + \left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right)A_2 + \lambda_2' a_2 \\ &= -(3a_1^2 a_2 + 3a_1 a_2^2) + \frac{\varepsilon E}{2}(e^{i\nu t} + e^{-i\nu t}),\end{aligned} \tag{6.32}$$

and

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right)\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right)u_1 = -(a_1^3 + a_2^3). \tag{6.33}$$

The particular solution of (6.33) is

$$u_1 = -\frac{a_1^3}{2\lambda_1(3\lambda_1 - \lambda_2)} - \frac{a_2^3}{2\lambda_2(3\lambda_2 - \lambda_1)} \tag{6.34}$$

Now, we have to solve (6.32) for two functions A_1 and A_2. According with unified KBM method A_1 contains the term $3a_1^2 a_2$, $Ee^{i\nu t}/2$ and A_2 contains the term $3a_1 a_2^2$, $Ee^{-i\nu t}/2$ obtain the following equation

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right)A_1 + \lambda_1' a_1 = -3a_1^2 a_2 + \frac{E}{2}e^{i\nu t}, \tag{6.35}$$

and

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right)A_2 + \lambda_2' a_2 = -3a_1 a_2^2 + \frac{E}{2}e^{-i\nu t} \tag{6.36}$$

The particular solutions of (6.35)-(6.36) are

$$A_1 = -\frac{\lambda_1' a_1}{\lambda_1 - \lambda_2} - \frac{3a_1^2 a_2}{2\lambda_1} + \frac{Ee^{i\nu t}}{2(i\nu - \lambda_2)}$$

and (6.37)

$$A_2 = \frac{\lambda_2' a_2}{\lambda_1 - \lambda_2} - \frac{3a_1 a_2^2}{2\lambda_2} + \frac{Ee^{-i\nu t}}{-2(i\nu + \lambda_1)}$$

Substituting the functional values of A_1 and A_2 (6.37) into (6.5) and rearranging, we obtain (see Sub-section 6.3.1)

$$\dot{a}_1 = \lambda_1 a_1 + \varepsilon\left(-\frac{\lambda_1' a_1}{\lambda_1 - \lambda_2} - \frac{3a_1^2 a_2}{2\lambda_1} + \frac{Ee^{i\nu t}}{2(i\nu - \lambda_2)}\right),$$

and (6.38)

$$\dot{a}_2 = \lambda_2 a_2 + \varepsilon\left(\frac{\lambda_2' a_2}{\lambda_1 - \lambda_2} - \frac{3a_1 a_2^2}{2\lambda_2} + \frac{Ee^{-i\nu t}}{-2(i\nu + \lambda_1)}\right).$$

The variational equation of a and φ in the real form (a and φ are know as amplitude and phase), which transform (6.38) to

$$\dot{a} = -\frac{\varepsilon a \omega'}{2\omega} - \frac{\varepsilon E \sin(\varphi - \nu t)}{\nu + \omega}$$

and (6.39)

$$\dot{\varphi} = \omega - \nu + \frac{3\varepsilon a^2}{8\omega} - \frac{\varepsilon E \cos(\varphi - \nu t)}{a(\nu + \omega)},$$

where $\omega = \sqrt{c_1 + c_2 \cos\tau + c_3 \sin\tau}$

Therefore, the first order solution of the equation (10) is

$$x(t, \varepsilon) = \alpha \cos\varphi + \varepsilon u_1 \qquad (6.40)$$

where α and φ are the solution of the equation (6.39).

6.4. Results and Discussions:

In this chapter, an analytic technique has been presented to obtain the first order analytical approximate solutions of a second-order time dependent nonlinear differential systems with constant and varying coefficients based on the extended KBM method (by Popov [91]). Theoretically, the solution can be obtained up to the accuracy of any order of approximation. However owing to the rapidly growing algebraic complexity for the derivation of the function, the solution is in general confined to a low order, usually the first. In order to test the accuracy of an approximate solution obtained by a certain perturbation method, one can easily compare the approximate solution to the numerical solution (considered to be exact). Due to such a comparison relating to the presented KBM method of this paper, we refer to the works of Murty [74], and Shamsul [110] have been compared to the corresponding numerical solution. In this chapter, we have also compared the perturbation solutions (6.20), (6.29) and (6.40) of Duffing's equation (6.10) and (6.30) to those obtained by Range-kutta (Fourth-order) procedure.

First of all, we plot in Fig. 6.1, the first approximate solution of Eq. (10) for $\varepsilon = .1$ with initial condition $[x(0) = 1,\ \dot{x}(0) = 0]$ or $a_1 = 1.00000$, $a_2 = -.000237$. The corresponding numerical solution has been computed by Runge-Kutta (fourth order) method. Seeing the figure it is clear that the asymptotic solution (6.20) shows a good agreement with the numerical solution of equation (6.10).

We have find the approximate solution of the same problem utilize the classical KBM method (see Sub-section 3.2) for $\varepsilon = .1$ with initial condition $[x(0) = 1,\ \dot{x}(0) = 0]$ or $\alpha_1 = 1.$, $\alpha_2 = 0$ presented in Fig.6.2. From the graph it is clear that the perturbation solution (6.29) does not agree with the numerical solution after a short time interval. Thus the extended KBM method is important.

In sub section 3.3, a perturbation solution (6.40) has been derived when an external force acts and the solution has been presented in Fig.6.3 for $\varepsilon = .1$ $\nu = .7,\ E = .5$ with initial condition$[x(0) = 1,\ \dot{x}(0) = 0]$, or, $a_1 = 1.003719$, $a_2 = .086112$. This solution also shows a good coincidence with the numerical solution.

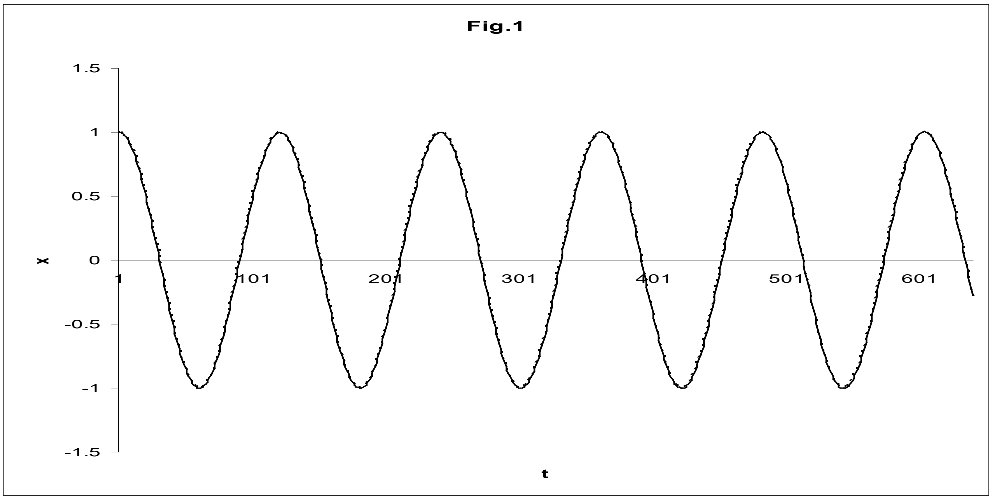

Fig 6.1: Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $a = 1.00000,\ \varphi = -.086112$ [$x(0) = 1.00000,\ \dot{x}(0) = 0.00000$] for $e = .1,\ h = .05$.

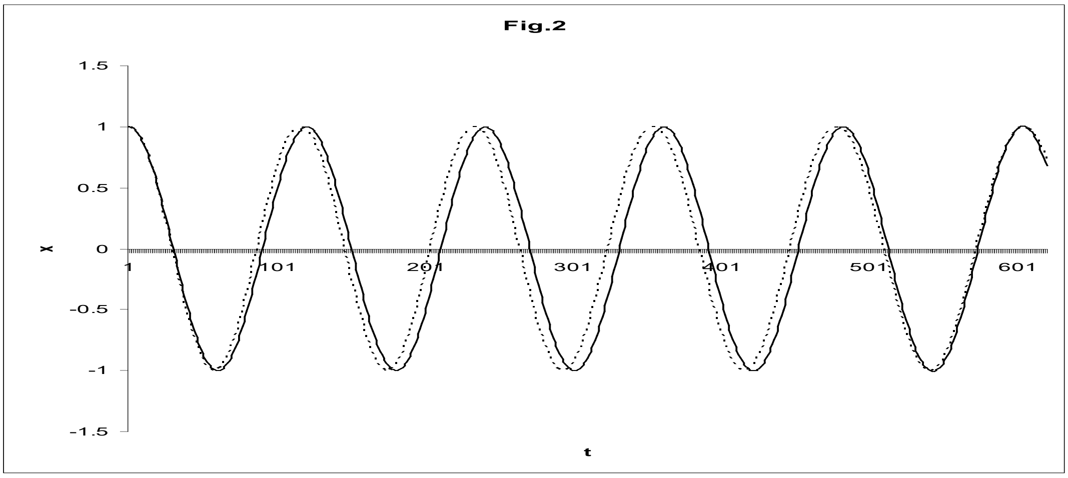

Fig 6.2: Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $a = 1.00000,\ \varphi = 0.00000$` [$x(0) = 1.00000,\ \dot{x}(0) = 0.00000$] for $e = .1,\ h = .05$.

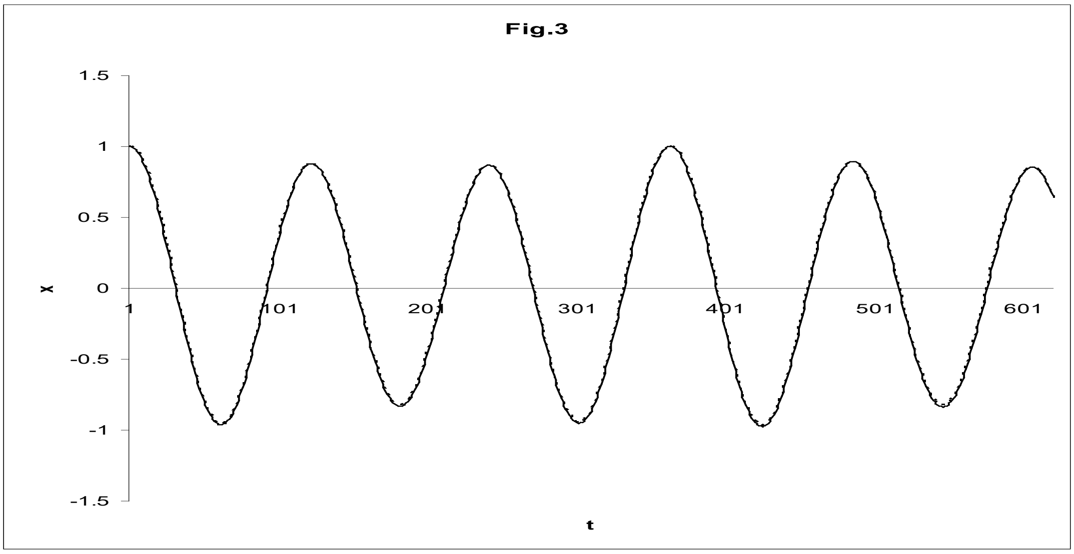

Fig.6.3: Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $a = 1.003719,\ \varphi = -.086112$ [$x(0) = 1.00000,\ \dot{x}(0) = 0.00000$] for $e = .1,\ \nu = 0.7,\ E = 0.5,\ h = .05.$.

6.5. Conclusion: An approximate solution of a second order nonlinear deferential system with slowly varying coefficients has been found. This improved method gives better results than classical KBM method. The solution for different initial condition shows good coincidence with corresponding numerical solution.

Chapter-Seven

Asymptotic Solutions of Second Order Damped-Oscillatory systems with Varying Coefficients

7.1 Introduction

Changes occur in every things, variational causes take place in every phenomena both in natural and artificial. Study in variation has been a hot tropics and the subject of active research. These problems generally arise in mathematical modeling of visco-elastic flows, physics, engineering, and other disciplines. Krylov and Bogoliubov [63] developed a perturbation method to obtain an approximate solution of a second order nonlinear differential system described by

$$\frac{d^2x}{dt^2}+\omega_0^2 x=-\varepsilon f(x,\frac{dx}{dt}), \qquad 0<\varepsilon\le 1 \tag{7.1}$$

where ω_0 is a positive constant and ε is a small parameter. This method was first appeared in published form in 1937. This method has been extended and justified mathematically by Bogoliubov and Mitropoisky. They are called the method asymptotic in the sense that $\varepsilon\to 0$. Then the method was amplified and justified by Bogoliubov and Mitropolskii [22]. Mitropolskii [73] has extended the method to nonlinear differential system with slowly varying coefficients

$$\frac{d^2x}{dt^2}+\omega_0^2(\tau).x=-\varepsilon\, f(x,\frac{dx}{dt},\tau), \qquad \tau=\varepsilon t \tag{7.2}$$

The advantage of the method is that it not only enables us to determine the steady-state periodic motions but also allows us to determine the transient process corresponding to perturbations of these oscillations. A closely related technique is that of van-der Pol. who proposed a method of slowly varying coefficients for the evaluation of periodic oscillations of certain nonlinear phenomena in electron tube oscillator. The Krylov-

Bogoliubov [63] method has been extended by Popov [91] and Mendelson [69] to the analysis of transsient nonlinear differential equation of the form

$$\frac{d^2x}{dt^2} + 2p\frac{dx}{dt} + q^2x = \varepsilon F\left(x, \frac{dx}{dt}\right), \qquad 0 < \varepsilon << 1 \tag{7.3}$$

where p and q are real constants and $F\left(x, \frac{dx}{dt}\right)$ is a nonlinear function.)

Following the extended Krylov-Bogoliubov-Mitropolskii (KBM) method [22, 63, 73]), Bojadziev and Edwards [34] studied some damped oscillatory and non-oscillatory systems modeled by

$$\frac{d^2x}{dt^2} + c(\tau).\frac{dx}{dt} + \omega^2(\tau).x = -\varepsilon\, f(x, \frac{dx}{dt}, \tau), \tag{7.4}$$

where $c(\tau)$ and $\omega(\tau)$ are positive.

In this chapter, we deliver a technique considering a nonlinear differential system of the form

$$\frac{d^2x}{dt^2} + 2k(\tau)\frac{dx}{dt} + \omega^2(\tau)x = -\varepsilon\, f(x, \frac{dx}{dt}, \tau), \qquad \tau = \varepsilon t \tag{7.5}$$

where ε is a small parameter, $\tau = \varepsilon t$ is the slowly varying time, $k(\tau) \geq 0$, f is a given nonlinear function and $\omega(\tau)$ is the frequency. In this system, the coefficients are slowly varying with their time derivatives are proportional to ε. The validity and advantages of the method is illustrated by an example in this chapter.

7.2 The Method

To achieve our goal of studying the mathematical behavior, let us consider the nonlinear differential system

$$\ddot{x} + 2k(\tau)\dot{x} + \omega^2(\tau)x = -\varepsilon\, f(x, \dot{x}, \tau), \qquad \tau = \varepsilon t \tag{7.4}$$

where the over-dots denote differentiation with respect to t, ε is a small parameter, $\tau = \varepsilon t$ is the slowly varying time, $k(\tau) \geq 0$, f is a given nonlinear function and $\omega(\tau)$ is the frequency. The coefficients in Eq. (7.4) are slowly varying in that their time derivatives are proportional to ε.

For $\varepsilon = 0$ and $\tau = \tau_0 =$ constant, $\lambda_1(\tau_0) = i\omega(\tau_0)$, $\lambda_2(\tau_0) = -i\omega(\tau_0)$ are two eigen values of the unperturbed equation of (7.4) and has the solution

$$x(t,0) = a_{1,0} e^{\lambda_1(\tau_0)t} + a_{2,0} e^{\lambda_2(\tau_0)t}, \tag{7.5}$$

When $\varepsilon \neq 0$, we seek a solution in accordance with Shamsul [108] or Murty and Deekshatulu [76] or the KBM [22, 63] method, of the form

$$x(t,\varepsilon) = a_1(t,\tau) + a_2(t,\tau) + \varepsilon u_1(a_1, a_2, t, \tau) + \varepsilon^2 ..., \tag{7.6}$$

where a_1 and a_2 satisfy the differential equations

$$\begin{aligned} \dot{a}_1 &= \lambda_1(\tau) a_1 + \varepsilon A_1(a_1, a_2, \tau) + \varepsilon^2 ..., \\ \dot{a}_2 &= \lambda_2(\tau) a_2 + \varepsilon A_2(a_1, a_2, \tau) + \varepsilon^2 ..., \end{aligned} \tag{7.7}$$

Keeping our attention to the first few term $1, 2, \ldots, m$ in the series expansion of eq.(7.6) and eq.(7.7), we evaluate functions $u_1, \ldots, A_1, A_2, \ldots,$ such that a_1 and a_2 appearing in eq.(7.6) and eq.(7.7) satisfy eq.(7.4) with an accuracy of ε^{m+1}. In order to determine these unknown functions it was early assumed by Murty [74], Shamsul [108] that the functions $u_1, \ldots$ exclude all fundamental terms, since these are included in the series expansion eq.(7.6) at order ε^0.

Differentiating $x(t, \varepsilon)$ twice with respect to t, substituting for the derivatives $\ddot{x}$ and x in the original equation eq.(7.4) and equating the coefficient of ε, we obtain

$$\lambda_1' a_1 + \lambda_2' a_2 - \lambda_2 A_1 - \lambda_1 A_2 + \left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} \right)(A_1 + A_2)$$
$$+ \left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1 \right)\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2 \right) u_1 \tag{7.8}$$
$$= -f^{(0)}(a_1, a_2, \tau),$$

where $\lambda_1' = \dfrac{d\lambda_1}{d\tau}$, $\lambda_2' = \dfrac{d\lambda_2}{d\tau}$, $f^{(0)} = f(x_0, \dot{x}_0, \tau)$ and $x_0 = a_1(t,\tau) + a_2(t,\tau)$.

It is assumed that both $f^{(0)}$ can be expanded in Taylor's series [74, 76, 108]

$$f^{(0)} = \sum_{r_1, r_2 = 0}^{\infty} F_{r_1, r_2}(\tau) a_1^{r_1} a_2^{r_2} \,, \tag{7.9}$$

To obtain this solution of eq.(7.4), it has been proposed in [110] that u_1, u_2 exclude the terms $a_1^{r_1} a_2^{r_2}$ of $f^{(0)}$, where $r_1 - r_2 = \pm 1$. This restriction guarantees that the solution always excludes *secular*-type terms or the first harmonic terms (see [110] for details). According to our assumption, u_1 does not contain the fundamental terms, therefore equation (7.8) can be separated into three equations for unknown functions u_1 and A_1, A_2 (see [110] for details). we obtain

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2 \right) A_1 + \lambda_1' a_1 = \sum_{r_1=0, r_2=0}^{\infty,\infty} F_{r_1, r_2}(a_1^{r_1}, a_2^{r_2}), \text{ if } r_1 = r_2 + 1 \tag{7.10}$$

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1 \right) A_2 + \lambda_2' a_2 = \sum_{r_1=0, r_2=0}^{\infty,\infty} F_{r_1, r_2}(a_1^{r_1}, a_2^{r_2}), \text{ if } r_2 = r_1 + 1 \tag{7.11}$$

and

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1 \right)\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2 \right) u_1 = \sum_{r_1=0, r_2=0}^{\infty,\infty} F_{r_1, r_2}(a_1^{r_1}, a_2^{r_2}), \tag{7.12}$$

where $\sum_{r_1=0, r_2=0}^{\infty,\infty} F_{r_1, r_2}(a_1^{r_1}, a_2^{r_2})$ exclude those terms for $r_1 = r_2 \pm 1$.

Thus the particular solutions of eq.(7.10)-(7.12) give the values of the unknown functions A_1, A_2 and u_1 which completes the determination of the solution of a second order non-linear problem eq.(7.4).

7.3 Example.

The simple procedure outlined above will be illustrated by discussing the following a example. To demonstrate the applicability of the proposed method for solving the second order damped nonlinear differential system type (7.1), we considered the example here. This example has been chosen because either analytical or approximate solutions are available in the literature and solutions obtained by the proposed method are compared with the solutions obtained by the methods available in the literature.

We consider a second order nonlinear system with slowly varying coefficients

$$\ddot{x} + 2k(\tau)\dot{x} + \omega^2(\tau)x = -\varepsilon x^3, \tag{7.13}$$

Here over dots denote differentiation with respect to t. In this case $x_0 = a_1 + a_2$ and the function $f^{(0)}$ becomes,

$$f^{(0)} = -(a_1^3 + 3a_1^2 a_2 + 3a_1 a_2^2 + a_2^3). \tag{7.14}$$

Following the assumption (discussed in section 7.2) u_1 excludes the terms $3a_1^2 a_2$ and $3a_1 a_2^2$.

We substitute in eq. (7.8) and separate it into two parts as

$$\begin{aligned} &\lambda_1' a_1 + \lambda_2' a_2 - \lambda_2 A_1 - \lambda_1 A_2 + \left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} \right)(A_1 + A_2) \\ &= -(3a_1^2 a_2 + 3a_1 a_2^2) \end{aligned} \tag{7.15}$$

and

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right)\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right) u_1 = -(a_1^3 + a_2^3) \tag{7.16}$$

The particular solution of eq. (7.16) is

$$u_1 = -\frac{a_1^3}{2\lambda_1(3\lambda_1 - \lambda_2)} - \frac{a_2^3}{2\lambda_2(3\lambda_2 - \lambda_1)} \tag{7.17}$$

Now, we have to solve eq.(7.15) for two functions A_1 and A_2. According with the unified KBM method A_1 contains the term $3a_1^2 a_2$ and A_2 contains the term $3a_1 a_2^2$ (Shamsul [110]) and thus we obtain the following equations

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_2\right) A_1 + \lambda_1' a_1 = -3a_1^2 a_2 \tag{7.18}$$

and

$$\left(\lambda_1 a_1 \frac{\partial}{\partial a_1} + \lambda_2 a_2 \frac{\partial}{\partial a_2} - \lambda_1\right) A_2 + \lambda_2' a_2 = -3a_1 a_2^2 \tag{7.19}$$

The particular solutions of eq.(7.18) and eq.(7.19) are

$$A_1 = -\frac{\lambda_1' a_1}{\lambda_1 - \lambda_2} - \frac{3a_1^2 a_2}{2\lambda_1} \tag{7.20}$$

and

$$A_2 = \frac{\lambda_2' a_2}{\lambda_1 - \lambda_2} - \frac{3a_1 a_2^2}{2\lambda_2} \tag{7.21}$$

Substituting the functional values of A_1, A_2 from eq.(7.20) and eq.(7.21) into eq.(7.7) and rearranging, we obtain

$$\dot{a}_1 = \lambda_1 a_1 + \varepsilon\left(-\frac{\lambda_1' a_1}{\lambda_1 - \lambda_2} - \frac{3a_1^2 a_2}{2\lambda_1}\right) \tag{7.22}$$

and

$$\dot{a}_2 = \lambda_2 a_2 + \varepsilon\left(\frac{\lambda_2' a_2}{\lambda_1 - \lambda_2} - \frac{3a_1 a_2^2}{2\lambda_2}\right) \tag{7.23}$$

Under the transformations, $a_1 = ae^{i\varphi}/2$ and $a_2 = ae^{-i\varphi}/2$ together with $\lambda_1 = -k + i\omega$, $\lambda_2 = -k - i\omega$ equations (7.22) and (7.23) reduce to

$$\dot{a} = \varepsilon \widetilde{A}_1(a) + \varepsilon^2 \ldots \text{ and } \dot{\varphi} = \omega + \varepsilon \widetilde{B}_1(a) + \varepsilon^2 \ldots \tag{7.24}$$

We shall obtain the variational equations of a and φ in the real form (a and φ are know as amplitude and phase respectively) which transform eq.(7.24) to

$$\dot{a} = -ka - \frac{\varepsilon a \omega'}{2\omega} + \frac{3\varepsilon a^3 k}{8(k^2 + \omega^2)} \tag{7.25}$$

and

$$\dot{\varphi} = \omega - \frac{\varepsilon k'}{2\omega} + \frac{3\varepsilon a^2 \omega}{8(k^2 + \omega^2)} \tag{7.26}$$

The variational equations (7.25) and (7.26) are in the form of the KBM solution**.** The variational equations for amplitude and phase are usually appeared in a set of first order differential equations and solved by the numerical technique (see Shamsul [110]).

Therefore, the first order solution of the equation (7.13) is

$$x(t,\varepsilon) = a\cos\varphi + \varepsilon u_1 \tag{7.27}$$

where a and φ are the solution of the equation (7.25) and (7.26) respectively.

7.4 Results and Discussions.

Based on the extended KBM method (by Popov [91]) an asymptotic solution of second order damped nonlinear systems has been found in this chapter. In order to test the accuracy of an approximate solution obtained by a certain perturbation method, one compares the approximate solution to the numerical solution (considered to be exact).

With regard to such a comparison concerning the presented KBM method of this chapter, we refer to the works of Murty [74], and Shamsul [110]. In our present paper, for different initial conditions, w*e* have compared the perturbation solutions (7.27) of Duffing's equations (7.13) to those obtained by Runge-Kutta Fourth-order procedure.

First of all, x is calculated by eq.(7.27) with initial conditions $[x(0)=1,\ \dot{x}(0)=0]$ or $a=1.000043$, $\varphi=-.009277$ for $\varepsilon=.1,\ \omega=\omega_0\sqrt{\cos\tau},\ k=.01\cos\tau$. Then corresponding numerical solutions is also computed by Runge-Kutta fourth-order method. All the results are shown in Fig.7.1. From Fig.7.1 it is clear that the asymptotic solution eq.(7.27) shows a good coincidence with the numerical solution of equation (7.13). We have find the approximate solutions of the same problem with initial conditions $[x(0)=1,\ \dot{x}(0)=0]$ or $a=1.001075$, $\varphi=-.046354$ for $\varepsilon=.1,\ \omega=\omega_0\sqrt{\cos\tau},\ k=.05\cos\tau$ and with initial conditions $[x(0)=1,\ \dot{x}(0)=0]$ or $a=1.004295$, $\varphi=-.092516$ for $\varepsilon=.1,\ \omega=\omega_0\sqrt{\cos\tau},\ k=.1\cos\tau$. The corresponding numerical solutions have also been computed by Runge-Kutta fourth-order method. From Fig. 7.2 and Fig. 7.3 we observe that the approximate solutions agree with numerical results nicely.

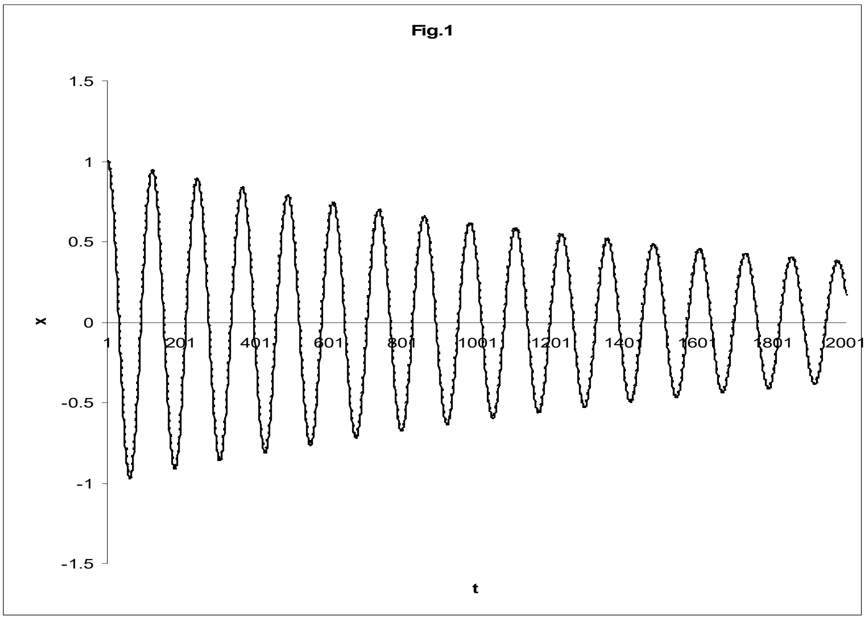

Fig 7.1: Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $a = 1.000043, \varphi = -.009277$ [$x(0) = 1.00000,\ \dot{x}(0) = 0.00000$] for $e = .1, \omega_0 = 1.\ h = .05.,\ k = .01\cos\tau$

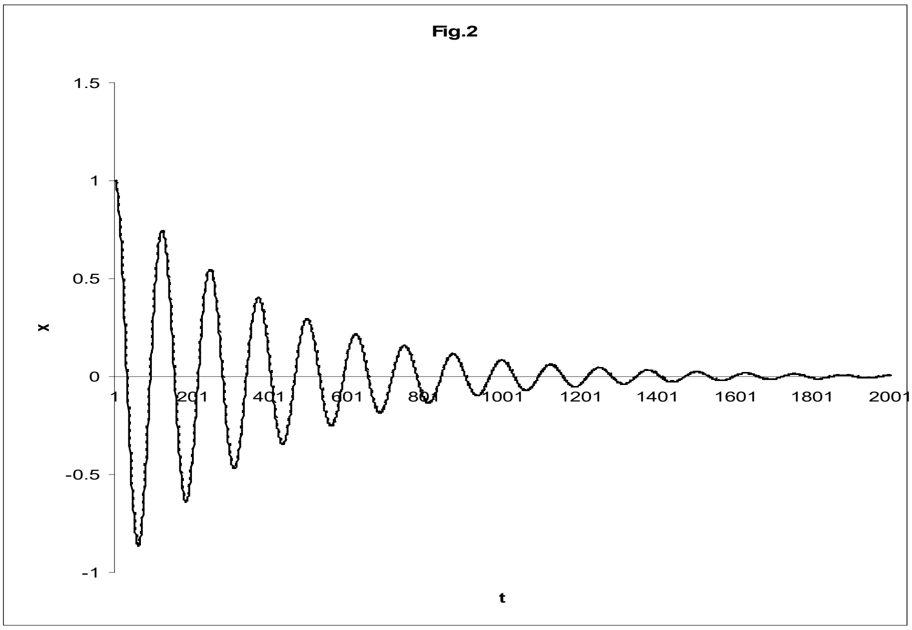

Fig 7.2: Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $a = 1.001075, \varphi = -.046354$ [$x(0) = 1.00000,\ \dot{x}(0) = 0.00000$] for $e = .1, \omega_0 = 1.\ h = .05,\ k = .05\cos\tau$

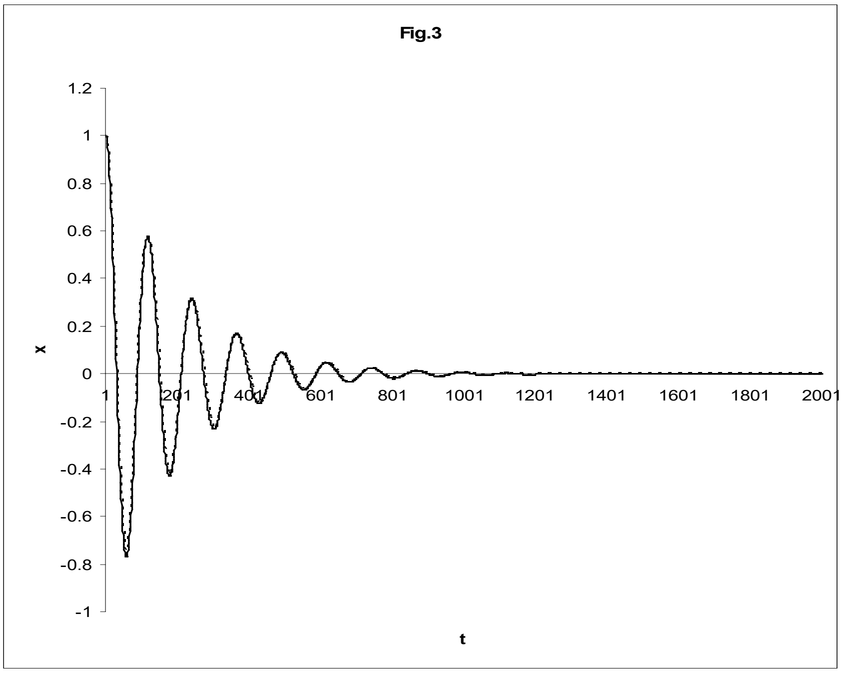

Fig 7.3: Perturbation solution (dotted line) with corresponding numerical solution (solid line) are plotted with initial conditions $a = 1.004295, \varphi = -.092516$ [$x(0) = 1.00000,\ \dot{x}(0) = 0.00000$] for $e = .1, \omega_0 = 1.\ h = .05,\ k = .1\cos\tau$

7.5 Conclusion.

In this chapter, we have modified the KBM method to find the approximate solution of a second order time dependent nonlinear deferential system with slowly varying coefficients under the action of damping forces. The preceding analysis has the virtue of utter simplicity, and the illustrating example shows that the suggested method is very effective and convenient in solving nonlinear equations. The objective of this chapter is to present a simple and direct technique to solve a second order special slowly varying coefficients problem. The solution is simpler than classical KBM method. Here it is found that if the damping force is significant, the solution is stable.

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