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On Connectedness Concepts in Fuzzy Topological Spaces

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A

Thesis submitted to the department of Mathematics, University of Rajshahi, Rajshahi – 6205, Bangladesh for the degree of Master of philosophy in Mathematics.

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Dedicated

To

My Parents

Acknowledgement

At first I would like to express my gratitude to the almighty Allah for giving me strength, patience and capability to complete this course of study.

I am highly indebted to my respected supervisor Dr. Dewan Muslim Ali, Professor, Department of Mathematics, University of Rajshahi, Rajshahi, Bangladesh. I wish to acknowledge my sincere thanks to him for his constant supervision, guidance, active help, cooperation, valuable suggestions and encouragement during the period of my work. I am also grateful to the officers and other staffs of the said department for their cordial cooperation.

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(Md. Abu Shayid)

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Certificate

*This is to certify that the M. Phil. thesis entitled "***On Connectedness Concepts in Fuzzy Topological Spaces***" which is being submitted by Md. Abu Shayid in fulfillment of the requirement for the degree of M. Phil. in Mathematics, University of Rajshahi, Rajshahi, Bangladesh has been completed under my supervision. I believe that the results embodied in the thesis are original and it has not been submitted elsewhere for any degree.*

To the best of my knowledge Md. Abu Shayid bears a good moral character and is mentally and physically fit to get the degree. I wish him a bright future and every success in his life.

Supervisor

Ali 29.06.13

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STATEMENT OF ORGINALITY

*I declare that the contents in my M. Phil. thesis entitled "**On Connectedness Concepts in Fuzzy Topological Spaces**" are original and accurate to the best of my knowledge. I also certify that the materials contained in my research work have not been previously published or written by any person for a degree or diploma.*

Department of Mathematics
June, 2013

(Md. Abu Shayid)

ABSTRACT

The goal of the thesis is to find out some new connectedness concepts in fuzzy topological spaces. Some concepts of connectedness in fuzzy topological spaces that already exist in the literature are recalled here also. In this work, various type of connectedness like C_i – connectedness (i = 1, 2, 3, 4), (C3) – connectedness, stronger forms of connectedness are studied in detail. Interrelations between various connectedness concepts in fuzzy topological spaces are discussed. In each chapter of this thesis, we give several possible definitions, both existing and new, of a concept and then compare the resulting concepts and determine thereby the interrelations among them. Some other properties of these concepts have also been discussed.

CONTENTS

INTRODUCTION

The concept of fuzzy sets was first introduced in 1965 by L. A. Zadeh in his classical paper [38] as an attempt to mathematically handle those phenomena which are inherently vague, imprecise or fuzzy in nature. Various merits and applications of fuzzy set theory have been extensively demonstrated by Zadeh and a large number of subsequence workers. It is impossible to illustrate these aspects of fuzzy set theory any further due to its enormity and diversity.

The advent of fuzzy set theory has also led to the development of some new areas of study in Mathematics. It has become a concern and a new tool for the mathematicians working in many different areas of Mathematics. These have been generally accomplished by replacing subsets, in various existing mathematical structures, by fuzzy sets. In 1968, C. L. Chang [5] did "fuzzification" of topology by replacing subsets in the definition of fuzzy topology by fuzzy sets. Since then a large body of concepts and results have been growing in this area which has come to be known as "fuzzy topology". In 1971, Goguen [8] defined fuzzy set by replacing the unit interval **I** by a completely distributive lattice **L** with an order reversing involution. A further development of L – fuzzy topology was made by Sarkar Mira [28, 29] and Hutton [12, 13]. Another significant approach to fuzzy topology was adopted by R. Lowen [17] in 1976. Seeing, what he regarded as some basic shortcomings in the Chang's concept of fuzzy topology, he redefined fuzzy topology as being families of fuzzy sets which are closed under arbitrary suprema and finite infima and which contain all constant fuzzy sets. This equips Lowen's fuzzy topologies with several advantages not enjoyed by Chang's fuzzy topologies, some of which are

(i) Constant maps are always continuous.

(ii) Projection maps of a product fuzzy topological space are always open.

(iii) In product spaces, slices are always homeomorphic to corresponding factor spaces.

(iv) The category of fuzzy topological spaces, like that of topological spaces, becomes a topological category. (For further details see Lowen and Wuyts [21]).

The present state of ongoing research in fuzzy topology can be divided in two separate sections, one of which is exclusively using the unit interval **I** to describe fuzziness (Chang's fuzzy topologies) and the other using L – fuzzy topologies. In our investigation, we have preferred the concepts of fuzzy topology developed by C. L. Chang [5].

Fuzzy sets have become a concern and tool for persons working in a variety of fields. Particularly, mathematicians working in many different areas such as logic, topology, algebra, probability theory, category theory etc. They have shown keen interest and have done a lot of research in the area of fuzzy sets. In the same way, researchers are applying fuzzy sets in the areas of general systems, intelligent systems, artificial intelligence, decisions theory and optimization, approximate reasoning, sociology and behaviors science etc. Also the theory of fuzzy sets has already affected a wide variety of disciplines such as control theory, information processing, cluster analysis, genetics, electrical engineering and operation research etc. However, a few workers have made some effort to point out the possibilities of applying fuzzy topology in some system – theoretic problems and more recently, in computer graphics. Without elaborating any further, we refer to the paper by Nazaroff [23], Warren [35] and klein [14] in this context.

In 1998 a definition of fuzzy connectedness was given by Gunther Jager [10] and proved to be equivalent to Pu and Liu's definition (1980) (c. f. e.g. P. M Pu and Y. M. Liu [27]). In 2007 Q. E. Hassan [11] introduced new results in fuzzy connectedness spaces. Among the results obtained mentioned fuzzy super connectedness which was introduced by Fatteh and Bassan [7].

The present thesis entitled "**On Connectedness Concepts in Fuzzy Topological Spaces**" is mainly a review work and devoted to the study of some connectedness for fuzzy topological spaces. Actually we collected several materials on fuzzy connectedness from different published papers and accumulated them so that one can find them at a glace in one place.

However, the materials of this thesis have been divided into five chapters and a brief discussion of this is given bellow:

The first chapter is corporate with some basic concepts, definitions and known results on fuzzy sets, fuzzy topological spaces and different mapping on fuzzy topological spaces which are necessary for the subsequent chapters. Results are provided without proof and can be seen in papers referred to.

In chapter two, we recall various concepts of connected fuzzy topological spaces, comparison of various concepts of fuzzy connectedness and we also discuss some other type of fuzzy connectedness like $\alpha - C4$, $2_\alpha - C$ and $D - C$.

In chapter three, we discuss C_i – Connectedness (i = 1, 2, 3, 4) in fuzzy topological spaces. We study their interrelations and preservation. Some other theorems also added regarding to this concepts.

In chapter four, we recall some existing propositions regarding (C3) – connectedness for fuzzy topological spaces. We compare this connectedness with other connectedness introduced in chapter two.

In chapter five, stronger forms of fuzzy connectedness have been discussed. Characteristics of fuzzy super connectedness and fuzzy strong connectedness are included in this chapter.

CHAPTER – 1

Preliminaries

1. Introduction: L. A introduced the concept of fuzzy set. Zadeh in his classical paper [38] described those situations where the defining property of a "subset" of a set X is imprecise or fuzzy (e.g. those real number which are much larger than 5). In such situations it is generally impossible to say that an element either belongs to, or does not belong to, that "subset". One possible way to overcome this difficulty is to assign to each element of a set X a "grade of membership", say for convenience between 0 and 1. This leads to function $\alpha : X \to [0,1]$ which Zadeh called a fuzzy set in X.

In 1968, C.L. Chang [5] defined a fuzzy topology on a set X in the expected way, viz. as a collection of fuzzy sets in X, closed under finite infimum and arbitrary suprema, and containing 0 and 1. The most significant approach to fuzzy topology was adopted by R. Lowen [17] in 1976.

2. Preliminaries: Now we recall several definitions and basic results that will be used in our work.

Definition 2.1: Let X be non – empty set and $I = [0,1]$. A fuzzy set in X is a function $\lambda : X \to I$ which assigns to each element $x \in X$, a degree or grade of membership $\lambda(x) \in I$. Thus a (usual) subset is a special type of fuzzy set in which the range of the function is $\{0,1\}$. Therefore, one can consider subsets of X as fuzzy sets in X.

Example 2.1: Suppose $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and $u(x) = \dfrac{2}{5x}$ is a fuzzy set. Then we can construct the following table:

x	1	2	3	4	5	6	7	8	9
u(x)	0.4	0.2	0.13	0.1	0.08	0.06	0.057	0.05	0.04

The graph of a fuzzy set which is defined above is given bellow:

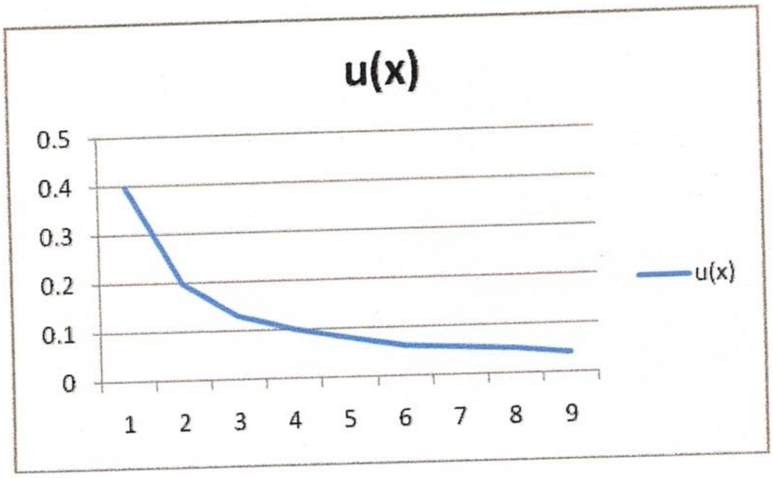


Figure 1: Graph of a fuzzy set

Remarks 2.1: If $a, b, c, d \in X$, then the grade of membership is 1.
If $e, f, g, \dots \dots \dots, z \notin X$, then the grade of membership is 0.

Definition 2.2: Let X be a non – empty set and $A \subseteq X$. Consider $\lambda : X \to I$ be a fuzzy set in X. Then we say that A is a fuzzy set if for every $x \in A$ such that $\lambda(x) = 1$. Similarly, X is a fuzzy set if every element of X goes to 1. Subsets of X will also be called crisp subsets of X.

Definition 2.3: If λ is a fuzzy set in X then the set $\{x \in X : \lambda(x) > 0\}$ is called the support of λ and is denoted by λ or λ_0 . i.e. every element of X mapped into I except 0.

Remarks 2.2: If $A \subseteq X$ then 1_A will denote the characteristic function of A and 1_X for the characteristic function of {x} and $\alpha 1_A$ for $\alpha \wedge 1_A$. We also write I_0, I_1 and $I_{0,1}$ respectively for the intervals (0,1], [0,1) and (0,1).

Generally, we shall use A, B, C ………, etc. to denote subsets of X and $\lambda, \mu, u, v, w, \dots\dots\dots$ to denote fuzzy sets in X. For a fuzzy set λ in X, $1-\lambda$ is called the complement of λ in X. We also denote $1-\lambda$ by λ^c sometimes. For most purpose supremum, infimum and complementation have turned out to be appropriate translations of the usual set – theoretic operations of union, intersection and complementation respectively.

Definition 2.4: Let α and β be two fuzzy sets in X. Then we define

(i) $\alpha = \beta$ iff $\alpha(x) = \beta(x), \forall x \in X$.

(ii) $\alpha \subseteq \beta$ iff $\alpha(x) \leq \beta(x), \forall x \in X$.

(iii) $\lambda = \alpha \vee \beta$ iff $\lambda(x) = \max\{\alpha(x), \beta(x)\}\ \forall x \in X$.

(iv) $\lambda = \alpha \wedge \beta$ iff $\lambda(x) = \min\{\alpha(x), \beta(x)\}\ \forall x \in X$.

(v) $\eta = \alpha'$ iff $\eta(x) = 1 - \alpha(x), \forall x \in X$.

Definition 2.5: Let A be the collection of fuzzy sets in X. i.e. $A = \{\alpha_j\}$ where $j \in J$. Then $\cup \alpha_j(x) = \sup\ \{\alpha_j(x)\}, \forall\ x \in X$ and $\cap \alpha_j(x) = \inf\ \{\alpha_j(x)\}, \forall\ x \in X$.

Definition 2.6: If $\alpha \in I$ and λ is a fuzzy set in X defined by $\lambda(x) = \alpha,\ for\ all\ x \in X$, then we refer to λ as a constant fuzzy set and denote it by α itself. In particular, we have the constant fuzzy sets 0, 1.

Example 2.2: Suppose $X = \{1, 2, 3, 4, 5\}$ and $u, v \in t$, where

$$u(x) = \frac{1}{x^2} \text{ and } v(x) = \frac{1}{x^3}$$

Then we construct the following table:

x	1	2	3	4	5
u(x)	1	0.25	0.11	0.0625	0.04
v(x)	1	0.125	0.037	0.015	0.008

The graph of the two fuzzy sets which are defined above is given bellow:

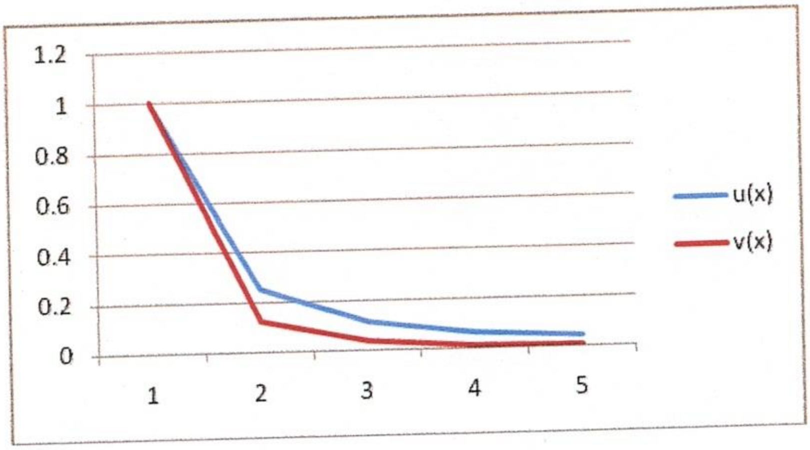


Figure 2: Graph of fuzzy subset

Clearly for all $x \in X, v(x) \le u(x)$. Therefore $u \subseteq v$.

Example 2.3: Suppose $X = \{a, b, c, d, e\}$, $u = \{0.2, 0.4, 1, 0.8, 0.3\}$ and $v = \{0.3, 0.2, 0.8, 0.9, 0.4\}$. Then $w = u \vee v = \{0.3, 0.4, 1, 0.9, 0.4\}$ and the graph of the union of two fuzzy sets is given bellow:

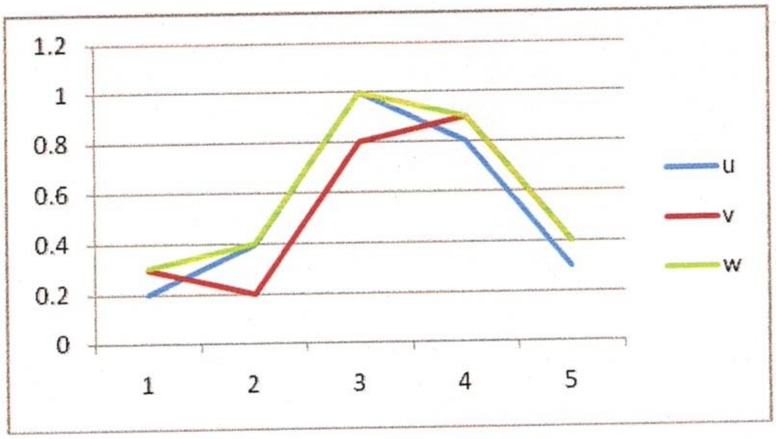


Figure 3: Union of two fuzzy sets

Example 2.4: Suppose $X = \{a, b, c, d, e\}$, $u = \{0.2, 0.4, 1, 0.8, 0.3\}$ and $v = \{0.3, 0.2, 0.8, 0.9, 0.4\}$. Then $w = u \wedge v = \{0.2, 0.2, 0.8, 0.8, 0.3\}$ and the graph of the intersection of two fuzzy sets is given bellow:

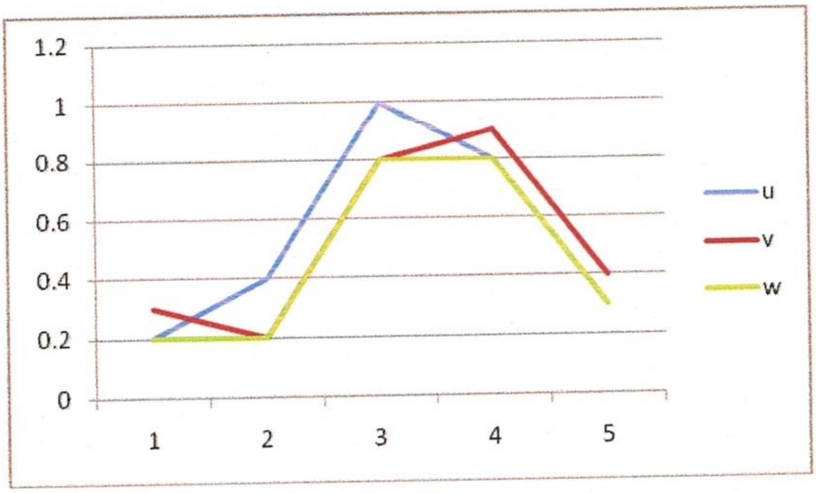


Figure 4: Intersection of two sets

Example 2.5: Let $X = \{1, 2, 3, 4, 5, 6, 7\}$ and $u(x) = \frac{1}{x}$ be a fuzzy set, then we can construct the following table:

X	1	2	3	4	5	6	7
u(x)	1	0.5	0.33	0.25	0.2	0.16	0.14
$u^c(x)=1-u(x)$	0	0.5	0.67	0.75	0.8	0.84	0.86

The graph of the complement of a fuzzy set which is defined above is given bellow:

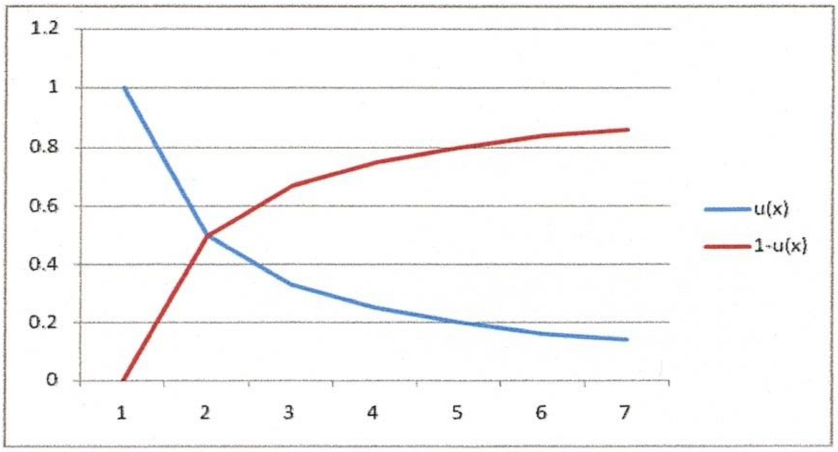


Figure 5: Complement of a fuzzy set

We can say that fo a fuzzy set u, $u \wedge u' = 0.5$ but in general set theory $A \cap A' = \phi$ always.

Definition 2.7: Let $f : X \to Y$ be a mapping and α be a fuzzy set in X. The image $f(\alpha)$ is a fuzzy set in Y whose membership function is defined by

$$f(\alpha)(y) = \begin{cases} \sup \alpha(x) & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi \end{cases}$$

Definition 2.8: Let $f : X \to Y$ be a mapping and β be a fuzzy set in Y. The inverse image $f^{-1}(\beta)$ is a fuzzy set in X defined by

$$f^{-1}(\beta)(x) = \beta f(x).$$

Definition 2.9: A fuzzy point p in X is a special type of fuzzy set in X with membership function is defined by

$$p(x) = r, \text{ where } 0 < r < 1$$
$$p(y) = 0, \text{ for all } x \neq y$$

P is said to have support x and value r. We also write this as x_r or $r1_x$.

A fuzzy point p is said to belong to a fuzzy set α in X denoted $p \in \alpha$ iff $p(x) < \alpha(x)$ and $p(y) \leq \alpha(y)$ if $y \neq x$. i.e. $x_r \in \alpha \ \Rightarrow r < \alpha(x)$.

De – Morgan's law in fuzzy mathematics: For any two fuzzy sets α and β we have

(i) $1-(\alpha \wedge \beta) = (1-\alpha) \vee (1-\beta)$

(ii) $1-(\alpha \vee \beta) = (1-\alpha) \wedge (1-\beta)$

Remarks 2.3: For a fuzzy set λ in X, $\lambda \cap (1-\lambda)$ need not be zero and for two fuzzy sets α, β in X

$$f(\alpha \cap \beta) \neq f(\alpha) \cap f(\beta).$$

Definition 2.10: Let X be a non – empty set and $t \subseteq I^X$ i.e. t is the collection of fuzzy sets in X. Then t is called a fuzzy topology on X if it satisfies the following conditions:

(i) 0 and 1 belong to t.

(ii) $\alpha, \beta \in t \quad \Rightarrow \alpha \wedge \beta \in t$

(iii) $\alpha_j \in t \quad \Rightarrow \vee_{j \in J} \alpha \in t$

The topology on X is in the sense of C. L. Chang. The ordered pair (X, t) is called the fuzzy topological space.

Definition 2.11: Let X be a non – empty set and $t \subseteq I^X$ i.e. t is the collection of fuzzy sets in X. Then t is called a fuzzy topology on X if it satisfies the following conditions:

(i) 0 and 1 belong to t.

(ii) $\alpha, \beta \in t \quad \Rightarrow \alpha \wedge \beta \in t$

(iii) $\alpha_j \in t \quad \Rightarrow \vee_{j \in J} \alpha \in t$

(iv) t contains all constant fuzzy set.

The topology on X is in the sense of R. Lowen. The ordered pair (X,t) is called the fuzzy topological space.

Remarks 2.4: Members of t are called t – open or simply open fuzzy sets and their complements are called t – closed or simply closed fuzzy sets.

Definition 2.12: Let λ be a fuzzy subset in an fts (X, t), then the closure of λ is denoted by $\bar{\lambda}$ or cl λ, is defined by $\bar{\lambda} = \cap\{\mu : \lambda \subseteq \mu \ and \ \mu \in t'\}$.

Definition 2.13: Let λ be a fuzzy subset in an fts (X, t), then the interior of λ is denoted by λ^o or int λ and is defined by $\lambda^o = \cup\{\mu : \mu \subseteq \lambda \ and \ \mu \in t\}$.

Definition 2.14: If (X, t) is an fts and $A \subseteq X$. Then $t_A = \{u / A : u \in t\} = \{u \cap A : u \in t\}$ is a fuzzy topology on A. This fuzzy topology on A is called the subspace fuzzy topology on A and the pair (A, t_A) is called the fuzzy subspace of (X, t).

Definition 2.15: Let (X, t) be an fts . Then

(i) a subfamily B of t is called a base for t iff each member of t can be expressed as a suppremum (union) of members of B.

(ii) a subfamily S of t is called a sub - base for t iff the family of all the finite infima of members of S is a base for t.

Definition 2.16: A fuzzy singleton p in X is a special type of fuzzy set in X with membership function is defined by

$$p(x) = r, \ where \ 0 < r \leq 1$$
$$p(y) = 0, \ for \ all \ x \neq y$$

We write this as x_r. A fuzzy singleton p is said to belong to a fuzzy set α in X i.e. $x_r \in \alpha \ \Rightarrow r \leq \alpha(x)$ where $0 < r \leq 1$.

CHAPTER - 2

Connectedness in Fuzzy Topological Spaces

1. Introduction

Several concepts of fuzzy topological spaces have been introduced and studied by many authors. In [2] Ali and Srivastava gave a comparison of some fuzzy connectedness concepts. Another similar comparison was later given by Ali [1]. In this chapter we will give some definition of fuzzy connectedness and upgrade the comparison by including some more connectedness concepts due to Fatteh and Bassan [6], Ajmal and Kokli [3], Ali [1], and Lowen and Srivastava [19]. We use fuzzy topology in the sense of Lowen [17]. We put $I=[0,1]$ and $I_0=(0,1]$. The $\alpha-$valued constant fuzzy set will be denoted by α itself. By 1_A will be denoted by the characteristic function of a subset A of X where (X,t) is a fuzzy topological space (in short fts).

2. Connectedness in Fuzzy Topological Spaces

To discuss connectedness in fts, we will discuss some definition related to connectedness in fts.

Definition 2.1: Two fuzzy sets β,γ in an fts (X,t) are said to be Q – separated if there are fuzzy sets λ,μ such that $\lambda>\beta,\mu>\gamma$ and $\lambda\wedge\gamma=0=\mu\wedge\beta$.

Definition 2.2: Two fuzzy sets β,γ in a fts (X,t) are said to be separated if there exist fuzzy sets $\lambda,\mu\in t$ such that

(i) $\lambda>\beta,\mu>\gamma$ and

(ii) $\lambda\wedge\gamma=0=\mu\wedge\beta$.

Definition 2.3: A fuzzy set μ in a fuzzy topological space (X,t) is said to be disconnected if there are non-empty fuzzy sets β,γ in the subspace (X_1,t_1) such that β,γ are Q – separation and $\beta \vee \gamma = \mu$. A fuzzy set μ is called connected if it is not disconnected.

Definition 2.4: Let (X,t) be an fts and μ be a fuzzy set in X such that $\mu >> 0$. The pair (u,v), $u,v \in t$ is said to be (C1) – separation iff

(i) $u \neq \mu \neq v$

(ii) $u \vee v = \mu$ and

(iii) $u \wedge v = 0$.

Definition 2.5: Suppose for some $\varepsilon > 0$ we have $\mu >> \varepsilon$. The pair (u,v), $u,v \in t$ is said to be (C2) – separation iff there exists some $\varepsilon_1 \in (0,\varepsilon]$ such that

(i) $u \neq \mu \neq v$

(ii) $u \vee v = \mu$ and

(iii) $u \wedge v = \mu - \varepsilon_1$.

We first recall the following definition of various fuzzy connectedness concepts that will be later compared in this chapter.

Definition 2.6: We call an fts (X,t)

(a) FC (i) iff (X,t) has no clopen fuzzy set except 0, 1. [12]

(b) FC (ii) iff no clopen fuzzy set $\mu >> 0$ (i.e. $\mu(x) >> 0$ for all x) can be (C1) - separated. [18]

(c) FC (iii) iff no clopen fuzzy set $\mu >> r > 0$ $(r \in (0,1])$ can be (C2) - separated. [18]

(d) FC (iv) iff there do not exist non – zero Q – separated fuzzy sets β,γ in X such that $\beta \vee \gamma = 1$. [27]

(e) FC (v) iff there do not exist non – zero separated fuzzy sets β,γ in X such that $\beta \vee \gamma = 1$. [39]

(f) FC (vi) iff there do not exist $\beta,\gamma \in t$ other than 0,1 such that $\beta \vee \gamma > 0$ and $\beta \wedge \gamma = 0$. [30]

(g) FC (vii) iff (X,t) has no non – constant clopen fuzzy set. [31]

(h) FC (viii) iff (X,t) has no non – constant clopen fuzzy sets $\beta,\gamma \in t$ with $\beta + \gamma \leq 1$. [7]

(i) FC (ix) iff there do not exist non – constant clopen fuzzy sets $\beta,\gamma \in t$ with $\beta + \gamma \geq 1$. [7]

(j) FC (x) iff there do not exist non – zero fuzzy sets $\beta,\gamma \in t$ with $\beta + \gamma = 1$ and $\beta \wedge \gamma = 0$. [3]

(k) FC (xi) iff for no $\beta \in I_0$ there exist $u,v \in t$ with $u \vee v > 1-\beta$, $u \wedge v \leq 1-\beta$ and $u^{-1}(1-\beta,1] \neq \phi \neq v^{-1}(1-\beta,1]$. [1]

(l) FC (xii) iff for no $\beta \in I_0$ there exist a proper non – empty subset $A \subseteq X$ such that $\beta 1_A, \beta 1_{1-A} \in t$. [1]

3. Comparison of various concepts of fuzzy connectedness

Some of the above fuzzy connectedness concepts have already been compared in [2] and [1] as is conveyed by the following theorems:

Theorem 3.1: [2] Regarding FC(i) to FC(vii), we have the following implications

$$\begin{array}{ccccccc}
 & & FC(vi) & & FC(iii) & & \\
 & & \Downarrow & & \Downarrow & & \\
FC(i) & \Rightarrow & FC(ii) & \Rightarrow & FC(iv) & \Leftrightarrow & FC(v) \\
 & \searrow & & & \Uparrow & & \\
 & & & & FC(vii) & &
\end{array}$$

Moreover, none of the other implications holds.

Theorem 3.2: [1] Regarding FC(ii) to FC(iv), FC(vi), FC(vii), FC(xi) and FC(xii), we have the following implications

$$\begin{array}{ccccccccc} & & & & FC(iii) & & FC(vii) \\ & & & & \Downarrow & & \Downarrow \\ FC(xi) & \Rightarrow & FC(vi) & \Rightarrow & FC(ii) \Leftrightarrow FC(xii) & \Rightarrow & FC(iv) \end{array}$$

Moreover, none of the other implications hold good.

Proof: For $FC(vii) \Rightarrow FC(iv)$, see [31]. The other remaining implications are obvious. We now give some examples.

Example 3.1: Let $X = (\frac{1}{3},1)$ and $u, v, w \in I^X$ be defined by

$$u(x) = \begin{cases} x, & if\ x \in \left(\frac{1}{3},\frac{2}{3}\right] \\ x-\delta, & if\ x \in \left(\frac{2}{3},1\right] \end{cases}$$

$$v(x) = \begin{cases} x-\delta, & if\ x \in \left(\frac{1}{3},\frac{2}{3}\right] \\ x, & if\ x \in \left(\frac{2}{3},1\right] \end{cases}$$

$$w(x) = x \text{ for all } x \in X \text{ and } \delta = \frac{1}{3}.$$

Let t be the fuzzy topology on X generated by $\{u, v, w, 1-w\} \cup \{Conosants\}$. Then it is obvious that (X,t) is FC (xi). However (X,t) is not FC (iii). For $w \geq \frac{1}{3} = \delta > 0$ is clopen. $u \vee v = w$, $u \neq w \neq v$, and $u \wedge v = w - \delta$. Now choose $\varepsilon_1 \in (0,\varepsilon]$ with $\varepsilon_1 \leq \delta$. Then $x - \varepsilon_1 \geq x - \delta$. Therefore $u \wedge v = w - \delta \leq w - \varepsilon_1$.

This Example shows that $FC(xi) \not\Rightarrow FC(iii)$

Example 3.2: Let $X = I$ and t be a fuzzy topology on X generated by $\{id, 1-id\} \cup \{Consants\}$. Then it is easily seen that (X,t) is FC (xi). However, (X,t) is not FC (vi), since id is clopen and non – constant.

From this example we see that $FC(xi) \not\Rightarrow FC(vii)$

Example 3.3: Let $X = \{x, y\}$ and $u, v \in I^X$, where $u = \frac{2}{3}1_x \vee \frac{1}{3}1_y$ and $v = \frac{1}{3}1_x \vee \frac{2}{3}1_y$. Let t be the fuzzy topology on X generated by $\{u, v\} \cup \{Conosants\}$. Then we see that (X,t) is FC (vi). However (X,t) is not FC (xi). Since for $\delta = \frac{2}{3}$, $u \vee v = \frac{2}{3} > 1-\delta$ and $u \wedge v = \frac{1}{3} \leq 1-\delta$. So it is seen that $FC(vi) \not\Rightarrow FC(xi)$.

Example 3.4: Let $X = I_0$ and $u, v \in I^X$ be given by

$$u(x) = \begin{cases} x, & if\ x \in \left(0, \frac{1}{2}\right] \\ 0, & if\ x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

$$v(x) = \begin{cases} 0, & if\ x \in \left(0, \frac{1}{2}\right] \\ x, & if\ x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

Let t be the fuzzy topology on X generated by $\{u, v\} \cup \{Conosants\}$. Then clearly (X,t) is FC (ii). However (X,t) is not FC (vi). Since $u \vee v > 0$ and $u \wedge v = 0$.

From this example one observes that $FC(ii) \not\Rightarrow FC(vi)$.

Example 3.5: Let $X = I_0$. Define $\mu : X \to I$ by $\mu(x) = \frac{1}{2}x$, for all $x \in X$. Also consider $\mu_Q = \mu \wedge 1_{Q \cap X}$ and $\mu_J = \mu \wedge 1_{J \cap X}$ where $J = R / Q$. Let t be the fuzzy topology on X generated by $\{\mu, 1-\mu, \mu_Q, \mu_J\} \cup \{Conosants\}$. Then as observed by Lowen [18], (X,t) is FC (iii), however, (X,t) is not FC (vi) since μ is clopen and non – constants.

One can observe, from this example, that $FC(iii) \not\Rightarrow FC(vii)$.

Example 3.6: If $X = I$, take $u, v \in I^X$ with $u = \frac{1}{2}1_X$, $v = \frac{1}{2}1_X - A$ and $A = [0, \frac{1}{2}]$. Let t be the fuzzy topology on X generated by $\{u, v\} \cup \{Conosants\}$. Then it is clear that (X,t) is FC (vii). However (X,t) is not FC (xii) since for $\alpha = \frac{1}{2}$ we have $\alpha 1_A, \alpha 1_{X-A} \in t$ were $A \subseteq X$. This example therefore shows that $FC(vii) \not\Rightarrow FC(xii)$. Also, one can see from Lowen [18], counter example B, that $FC(iii) \not\Rightarrow FC(ii)$.

According to the implication diagram, the proof is now complete.

Remarks 3.1: Since the fuzzy connectedness concept $FC(vi)^*$ defines by Hutton [12] does not make sense for Lowen type fuzzy topologies, we have not considered it in the above Theorem 3.2. Note, however, that the concepts FC (xi) through FC (vii) make perfect sense even if we work with Chang type fuzzy topologies. This and the fact that Hutton's concept is historically the first fuzzy connectedness concept possessing several desirable features, that us to upgrade Theorem 3.2 to the following.

Theorem 3.3: The following implications exist:

$$\begin{array}{ccccc}
FC(vi)^* & & \Longrightarrow & & FC(vii) \\
\Downarrow & & & & \Downarrow \\
FC(ii) & \Rightarrow & FC(xii) & \Rightarrow & FC(iv) \\
\Uparrow & & \Uparrow & & \\
FC(vi) & & FC(iii) & & \\
\Uparrow & & & & \\
FC(xi) & & & &
\end{array}$$

Proof: We prove only $FC(vi)^* \Rightarrow FC(ii)$. Let (X,t) be Chang type fts that is $FC(vi)^*$. If it is not FC (ii), then there exists a (C1) – separation (u,v) of some t – clopen $\mu >> 0$. But μ must be 1 (as (X,t) is $FC(vi)^*$) and so u is t – clopen with $u \neq 0,1$. Which is a contradiction.

We now give few examples.

$FC(vi)^* \not\Rightarrow FC(vi)$.

Example 3.4 works here also if in that example t is the Chang type fuzzy topology generated by $\{u,v\}$.

$FC(vi)^* \not\Rightarrow FC(iii)$.

Consider the following example for this.

Example 3.7: Let $X = I$ and $u, v \in I^X$, where $u = 1_{[0,\frac{1}{2}]} \vee \left(\frac{1}{2}\right) 1_{(\frac{1}{2},1]}$ and $v = \left(\frac{1}{2}\right) 1_{[0,\frac{1}{2}]} \vee 1_{(\frac{1}{2},1]}$.

Let t be the fuzzy topology on X generated by $\{u,v\}$. Clearly (X,t) is $FC(vi)^*$. However, (X,t) is not FC (v) since 1 is clopen in (X,t) and 1 can be (C2) – separated.

$FC(xi) \not\Rightarrow FC(vi)^*$.

Example 3.2 works here also if in that example t is the Chang type fuzzy topology generated by $\{id, 1-id\}$.

$FC(iii) \not\Rightarrow FC(vi)^{*}$.

The following example will serve the purpose.

Example 3.8: Let $X = I$ and $t = \{0, u, 1\}$, where u is a constant fuzzy set in X with value $\frac{1}{2}$. Then (X,t) is FC (iii). However, (X,t) is not $FC(vi)^{*}$, for u is clopen in (X,t) with $u \neq 0,1$.

$FC(vii) \not\Rightarrow FC(vi)^{*}$

Example 3.8 works here also.

We now give a somewhat more exhaustive comparison. We leave out FC (i) completely this concept does not make sense in fuzzy topologies owing to the presence of all constants as open fuzzy sets.

Theorem 3.4: We have the following implications:

$$
\begin{array}{ccccccc}
 & & & & FC(x) & & \\
 & & & & \Updownarrow & & \\
 & & & & FC(v) & & \\
 & & & & \Updownarrow & & \\
FC(ix) & \Rightarrow & FC(vii) & \Rightarrow & FC(iv) & \Leftarrow & FC(iii) \\
 & & \Uparrow & & \Uparrow & & \\
 & & \Uparrow & & FC(xii) & & \\
 & & \Uparrow & & \Uparrow & & \\
 & & \Uparrow & & FC(ii) & & \\
 & & \Uparrow & & \Uparrow & & \\
 & & FC(viii) & \Rightarrow & FC(vi) & & \\
 & & & & \Uparrow & & \\
 & & & & FC(xi) & &
\end{array}
$$

Moreover, none of the other implications hold good.

Proof: In view of Theorem 3.1 and 3.2 in addition to proving $FC(x) \Leftrightarrow FC(vi)$, the only remaining implications to be proved are $FC(ix) \Rightarrow FC(vii)$, $FC(viii) \Rightarrow FC(vii)$ and $FC(viii) \Rightarrow FC(vi)$.

$FC(x) \Leftrightarrow FC(vi)$: By definition on fts (X,t) is $FC(x)$ iff there do not exist any non – zero $\lambda, \delta \in t$ with $\lambda + \delta = 1$ and $\lambda \wedge \delta = 0$. But $\lambda + \delta = 1$ and $\lambda \wedge \delta = 0$ iff $\lambda = 1 - \delta$ and $\lambda \wedge 1 - \lambda = 0$. This is equivalent to requiring that $\lambda = 1_A$ where $A = \lambda^{-1}\{1\}$. Thus (X,t) is FC (x) iff there do not exist any proper subset A of X such that both $1_A, 1_{X-A}$ are in t. In view of [2] (Proposition 2.3) there, (X,t) is FC (x) iff it is FC (iv). This is view of Theorem 3.1, shows (X,t) is FC (x) iff it is FC (v).

$FC(ix) \Rightarrow FC(vii)$, $FC(viii) \Rightarrow FC(vii)$: If an fts (X,t) is not FC (vii) then there exist a non – constant clopen fuzzy set λ. But then $\delta = 1 - \lambda$ is also non – constant and clopen with $\lambda + \delta = 1$. Thus (X,t) can neither be FC (vii) nor FC (ix).

$FC(viii) \Rightarrow FC(vi)$: If an fts (X,t) is not FC (vi) then there exist $\lambda, \delta \in t - \{0,1\}$ with $\lambda \vee \delta > 0$ and $\lambda \wedge \delta = 0$. But then λ and δ are non – constants and $\lambda + \delta \leq 1$. Hence, (X,t) is not FC (viii).

Now we show the needed non – implications $FC(vii) \not\Rightarrow FC(ix)$ and $FC(iv) \not\Rightarrow FC(ix)$: Let (I, t_S) be the "fuzzy sierpinski Space" as introduced in [32], where t_S is the fuzzy topology on I generated by $\{id\}$. Then (I, t_S) is obviously FC (vii). Choose any $\beta, \lambda \in \left(\frac{1}{2}, 1\right]$ and put $\beta = \alpha \vee id$ and $\alpha = \lambda \vee id$. Then $\lambda, \delta \in t_S$ and these are non – constants with $\lambda + \delta > 1$. Hence, (I, t_S) is not FC (ix). The above counter example also shows that $FC(iv) \not\Rightarrow FC(ix)$.

$FC(ix) \not\Rightarrow FC(vi)$, $FC(ix) \not\Rightarrow FC(viii)$ and $FC(vii) \not\Rightarrow FC(viii)$: On I let t_1 be the topology generated by $\{\mu \in I^I : \mu < 1\}$. Then (I, t_1) is evidently FC (ix). Define $\lambda, \delta \in I^I$ by

$$\lambda(x) = \begin{cases} 0, & \text{if } x \in \left[0, \frac{1}{2}\right) \\ \frac{1}{3}, & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

$$\delta(x) = \begin{cases} \frac{1}{3}, & \text{if } x \in \left[0, \frac{1}{2}\right) \\ 0, & \text{if } x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

Then $\lambda, \delta \in t_1$ with $\lambda \vee \delta > 0$ and $\lambda \wedge \delta = 0$. So (I, t_1) is not FC (vi).

The above example also shows that $FC(ii) \not\Rightarrow FC(viii)$ or FC (ix) and that $FC(iv) \not\Rightarrow FC(viii)$ or FC (ix).

$FC(iii) \Rightarrow FC(viii)$ or FC (ix): Let $X = I_0$. Define $\mu : X \to I$ by $\mu(x) = \frac{x}{2}, x \in X$ and let $\mu_Q = \mu \wedge 1_{Q \cap X}$ and $\mu_J = \mu \wedge 1_{J \cap X}$ where $J = R - Q$. Let t be the fuzzy topology on X generated by $\{\mu, 1 - \mu, \mu_Q, \mu_J\}$. Then as observe by Lowen [18], (X, t) is FC (viii). However, it is neither FC (viii) nor FC (ix).

By Theorem 3.1 as $FC(ii) \not\Rightarrow FC(vii)$, we get that $FC(ii) \not\Rightarrow FC(viii)$. as a consequence therefore, $FC(iv) \not\Rightarrow FC(viii)$. Similarly, by Theorem 3.1, $FC(iii) \not\Rightarrow FC(vi)$. So $FC(iii) \not\Rightarrow FC(viii)$ and $FC(iv) \not\Rightarrow FC(viii)$.

$FC(ix) \not\Rightarrow FC(ii)$: On, I, Let t_4 be the fuzzy topology generated by $\{\alpha 1_{A_1}, \alpha 1_{A_2}\}$ where $\alpha \in (0,1)$ and A_1, A_2 are non – empty disjoint subsets of I with $A_1 \cup A_2 = I$. Then (I, t_4)

cannot be FC (ii) as $\alpha 1_{A_1}$ and $\alpha 1_{A_2}$ provided a (C1) – separation of the clopen constant fuzzy set α. However (I, t_4) is clearly FC (ix).

$FC(viii) \not\Rightarrow FC(ix)$: On, I, consider the fuzzy topology t_4 generated by $\left\{\mu \in I^I : \mu > \frac{1}{2}\right\}$. Then, (I, t_5) is clearly FC (viii) but not FC (ix).

$FC(ix) \not\Rightarrow FC(xii)$: On, I, consider the fuzzy topology t_1 introduced earlier so that (I, t_1) is FC (ix). However, for any proper subset A of I. For any $\alpha \in I$ with $\alpha < \frac{1}{2}; \alpha 1_A$ and $\alpha 1_{1-A} \in t_1$. Thus (I, t_1) is not FC (xii).

$FC(viii) \not\Rightarrow FC(xii)$: On, I, consider the fuzzy topology t_5 generated by $\left\{\mu \in I^I : \mu > \frac{1}{2}\right\}$. Then (I, t_5) is FC (viii). Now let $\alpha = 0.4$; then $1 - \alpha = 0.6$. Take $\beta = 0.7$ and choose a proper subset A of I. Let $\gamma = 0.6$ and put $u = \beta 1_A \vee \gamma$ and $v = \beta 1_{I-A} \vee \gamma$. Then $u, v \in t_5$ with $u \vee v > 1 - \alpha$ and $u \wedge v \leq 1 - \alpha$. Hence (I, t_5) is not FC (xii). Therefore $FC(viii) \not\Rightarrow FC(xii)$.

The proof of the theorem is now complete.

4. Some other type of fuzzy connectedness:

Here we discuss a connectedness concept $\alpha - C3$ of which (C1) – separation of Lowen [18] becomes a particular case. We also introduce other connectedness concepts, namely $\alpha - C4$ and $S - C4$ which look similar to the recently introduced connectedness concepts $2_\alpha - C$ and $D - C$ of Lowen and Srivastava [20]. We observe that all these concepts are good extension of connectedness and have several pleasing properties.

Definition 4.1: Let (X,t) be an fts and $\alpha \in I_0$

(a) Let $\mu > 1-\alpha$ be a fuzzy set in X. Then $(u,v), u,v \in t$ is called $\alpha - separation$ of μ iff $u \neq \mu \neq v$, $u \vee v = \mu$ and $u \wedge v \leq 1-\alpha$. (X,t) is called $\alpha - C3$ iff no clopen $\mu > 1-\alpha$ can be $\alpha - separated$.

(b) (X,t) is called strongly $\alpha - C4$ ($S-C4$, in short) iff (X,t) is FC (xi).

(c) (X,t) is called $D-C$ iff (X,t) is FC (xii) for all $\alpha \in I_0$ [20].

(d) (X,t) is called $\alpha - C$ iff there do not exist $u,v \in t - t - \{0,1\}$ such that $u \vee v > 1-\alpha$ and $u \wedge v = 0$ [30].

Remarks 4.1: For $\alpha \leq \beta$, $\alpha,\beta \in I_0$, we have $2_\alpha - C \Rightarrow 2_\beta - C$.

Theorem 4.1: The following are equivalent:

(i) (X,t) is $\alpha - C4$.

(ii) $(X, i_{1-\alpha}(t))$ is connected.

Proof: $(i) \Rightarrow (ii)$: If $(X, i_{1-\alpha}(t))$ is not connected then there exist $A,B \in i_{1-\alpha}(t)$, $A \neq \phi \neq B$ such that $A \cup B = X$ and $A \cap B = \phi$. Since $A,B \in i_{1-\alpha}(t)$, there exist $u,v \in t$ such that $A = u^{-1}(1-\alpha,1]$ and $B = v^{-1}(1-\alpha,1]$. Clearly, $u \vee v > 1-\alpha$, $u \wedge v \leq 1-\alpha$ and $u^{-1}(1-\alpha,1] \neq \phi \neq v^{-1}(1-\alpha,1]$.

By setting $A = u^{-1}(1-\alpha,1]$ and $B = v^{-1}(1-\alpha,1]$, we see that $A,B \in i_{1-\alpha}(t)$, $A \neq \phi \neq B$, $X = A \cup B$ and $A \cap B = \phi$. Hence $(X, i_{1-\alpha}(t))$ is not connected. Which is a contradiction.

Theorem 4.2: $\alpha - C3$, $\alpha - C4$ and $S - C4$ are preserved under continuous functions.

Proof: Let $f:(X,t) \to (Y,s)$ be continuous and on – to function. (X,t) be $\alpha - C3$. If (Y,s) is not $\alpha - C3$, then there exist $\mu > 1-\alpha$ clopen in Y and its $\alpha - separation$ (u,v). It follows trivially that $(f^{-1}(u), f^{-1}(v))$ is an $\alpha - separation$ of the clopen $(f^{-1}(\mu))$.

The proof for $\alpha - C4$ and $S - C4$ are similar.

Remarks 4.2: $\alpha-C3$, $\alpha-C4$, $S-C4$ and $\alpha-C$ are not preserved under almost and weakly continuous function (for definition of almost and weakly continuous functions, c.f. [4] and [34]).

For these, we consider the following example. Let X denotes the unit interval I with the fuzzy topology consisting of all constant fuzzy sets. Then X is clearly $\alpha-C3$, $\alpha-C4$, $S-C4$ and $\alpha-C$.

Let 2_r, where $0<r<1$, denote the two point set $\{a,b\}$ with the fuzzy topology generated by $\{r1_a, r1_b\} \cup \{Consants\}$.

Now consider the function $f: X \to 2_r$ defined by (see also [2])

$$f(x) = \begin{cases} a, if x \in \left[0, \frac{1}{2}\right] \\ b, if x \in \left(\frac{1}{2}, 1\right] \end{cases}$$

For $\alpha > \frac{1}{2}$ and $r = \frac{1}{2}$, f is found almost and weakly continuous, while 2_r is not $\alpha-C3$, $\alpha-C4$, $S-C4$ and $\alpha-C$.

For $0<r\leq\frac{1}{2}$ and $0<r<1$ with $r>1-\alpha$, f again is almost and weakly continuous, while 2_r is not $\alpha-C3$, $\alpha-C4$, $S-C4$ and $\alpha-C$.

Remarks 4.3: In this example, just on putting $\alpha = r$. We see that $2_\alpha - C$ for $\alpha<1$ is not preserved under almost and weakly continuous function.

Theorem 4.3: $\alpha-C3$, $\alpha-C4$, $S-C4$ and $\alpha-C$ are good extensions.

Proof: Let (X,T) be connected. If $(X,\omega(T))$ is not $\alpha-C3$, then there exist $\mu > 1-\alpha$ clopen in $(X,\omega(T))$, $u, v \in \omega(T)$ such that $u \neq \mu \neq v$, $u \vee v = \mu$ and $u \wedge v = 1-\alpha$. Put

$A = u^{-1}(1-\alpha,1]$ and $B = v^{-1}(1-\alpha,1]$. Then $A, B \in T$, $A \cup B = X$ and $A \cap B = \phi$. Moreover $A \neq \phi \neq B$. Thus (X,T) is not connected. Which is a contradiction.

Conversely, let $(X,\omega(T))$ be $\alpha - C3$. If (X,T) is not connected, then there exist $A, B \in T$ such that $A \cup B = X$, $A \cap B = \phi$ and $A \neq \phi \neq B$. Now $1_A, 1_B$ being to $\omega(T)$ with $1_A \neq 1 \neq 1_B$, $1_A \vee 1_B = 1 > 1-\alpha$ and $1_A \wedge 1_B = 0 \leq 1-\alpha$. So 1 can be $\alpha - separated$ and therefore $(X,\omega(T))$ is not $\alpha - C3$, a contradiction.

Proofs are on similar lines for $\alpha - C4$, $S - C4$ and $\alpha - C$.

Definition 4.2: A subset A of a fts (X,t) is fuzzy dense in X for every $u \in t$ $\sup u(x)_{x\in X} = \sup u(a)_{a\in A}$, (see also [22]).

In the following theorem $\overline{A}$ denotes the closure of A in $(X,i(t))$.

Theorem 4.4: Let (X,t) be an fts and (A,t_A) be a subspace of (X,t). Then for any B with $A \subseteq B \subseteq \overline{A}$, we have

(i) (A,t_A) is $\alpha - C3 \Rightarrow (B,t_B)$ is $\alpha - C3$.

(ii) (A,t_A) is $\alpha - C4 \Rightarrow (B,t_B)$ is $\alpha - C4$.

(iii) (A,t_A) is $S - C4 \Rightarrow (B,t_B)$ is $S - C4$.

Proof: (i) Suppose (B,t_B) is not $\alpha - C3$. Then there exist $\mu > 1-\alpha$ clopen in B and $u, v \in t_B$ such that $u \neq \mu \neq v$, $u \vee v = \mu$ and $u \wedge v \leq 1-\alpha$ (on B). Then Obviously μ / A clopen in A, $u/A, v/A \in t_A$, $u/A \vee v/A = \mu/A$ and $u/A \wedge v/A \leq 1-\alpha$ (on A). Moreover $\mu/A \neq u/A$ and $\mu/A \neq v/A$. If $\mu = u$ on A, then $v \leq 1-\alpha$ on A. So $\sup v(a)(x)_{a\in A} \leq 1-\alpha$. Again we have $\sup v(b)_{b\in B} > 1-\alpha$. Thus $\sup v(a)_{a\in A} \neq \sup v(b)_{b\in B}$. This, however, is not possible, since by denseness of A in B (which follows from $B \subseteq \overline{A}$) $\sup v(a)_{a\in A} = \sup v(b)_{b\in B}$. Hence $(u/A, v/A)$ is an $\alpha - separation$ of μ/A which is a contradiction with the fact that (A,t_A) is $\alpha - C3$.

(ii) Let (B,t_B) be not $\alpha-C4$. Then there exist $u,v\in t_B$ such that $u\vee v>1-\alpha$, $u\wedge v\leq 1-\alpha$ (on B) and $u^{-1}(1-\alpha,1]\neq\phi\neq v^{-1}(1-\alpha,1]$.

Clearly, then $u/A, v/A\in t_A$, $u/A\vee v/A>1-\alpha$ and $u/A\wedge v/A\leq 1-\alpha$. Moreover $(u/A)^{-1}(1-\alpha,1]\neq\phi\neq(v/A)^{-1}(1-\alpha,1]$, as A is fuzzy dense in B. So, we see that (A,t_A) is not $\alpha-C4$, a contradiction.

The proof for $S-C4$ is similar.

Theorem 4.5: If $\{A_j,t_{A_j}: j\in J\}$ is a family of $\alpha-C3$ (resp $\alpha-C4$ or $S-C4$) subspaces of (X,t) with $\cap_{j\in J}A_j\neq\phi$, then if $A=\cup_{j\in J}A_j$, (A,t_A) is $\alpha-C3$ (resp $\alpha-C4$ or $S-C4$).

Proof: First suppose that (A_j,t_{A_j}), for each $j\in J$, is $\alpha-C3$.

Let $a_0\in\cap_{j\in J}A_j$. If (A,t_A) is not $\alpha-C3$, then there exists $\mu>1-\alpha$ clopen in A, $u,v\in t_A$ such that $u\neq\mu\neq v$, $u\vee v=\mu$ and $u\wedge v\leq 1-\alpha$ (on A). Since $\mu(a_0)>1-\alpha$, we have, e.g. $v(a_0)=\mu(a_0)>1-\alpha$ and since $v\neq\mu$ on A, there exists some $j\in J$ such that $v/A_j\neq\mu/A_j$, since $a_0\in A$ and $v(a_0)>1-\alpha$, we have $u(a_0)\leq 1-\alpha$. Whereby $v/A_j\neq\mu/A_j$. Clearly, then $(u/A_j, v/A_j)$ is on $\alpha-$*separation* of μ/A_j, which is a contradiction the fact that (A_j,t_{A_j}) is $\alpha-C3$.

Next, suppose that (A_j,t_{A_j}), for each $j\in J$, is $\alpha-C4$.

Let $a_0\in\cap_{j\in J}A_j$. If (A,t_A) is not $\alpha-C4$, then there exist $u,v\in t_A$ such that $u\vee v>1-\alpha$, $u\wedge v\leq 1-\alpha$ (on A) and $u^{-1}(1-\alpha,1]\neq\phi\neq v^{-1}(1-\alpha,1]$. Since $u\vee v>1-\alpha$ on A, we have, e.g. $v(a_0)\not>1-\alpha$. But $v\not>1-\alpha$ everywhere on A, so there exists some $j\in J$ such that $v>1-\alpha$ on A_j. Let $u_j=u/A_j$ and $v_j=v/A_j$. Then $u_j,v_j\in t_{A_j}$. Let $a_j\in A_j$ with $v_j(a_j)\leq 1-\alpha$, we then see that $u_j(a_j)>1-\alpha$. Hence it is clear that $u_j\vee v_j>1-\alpha$,

$u_j \wedge v_j \leq 1-\alpha$ and $u_j^{-1}(1-\alpha,1] \neq \phi \neq v_j^{-1}(1-\alpha,1]$. Thus (A_j, t_{A_j}) is not $\alpha - C4$ which is a contradiction. The case $S - C4$ can similarly be handled.

Theorem 4.6: A non – empty product space is $\alpha - C3$ (resp $\alpha - C4$ or $S - C4$) iff each factor space is $\alpha - C3$ (resp $\alpha - C4$ or $S - C4$).

The proof is similar to that of theorem 3.1 of Lowen [18].

Theorem 4.7: The following are true:

(a) $\alpha - C3 \Leftarrow \alpha - C4 \Rightarrow \alpha - C$

(b) For $\alpha > \beta, \alpha \in I_0, \beta \in I_1, \alpha - C3 \Rightarrow 2_{1-\beta} - C \Leftarrow \alpha - C$

(c) For $\alpha > \frac{1}{2}, \alpha - C3 \Rightarrow 2_\alpha - C \Leftarrow \alpha - C$

(d) $1 - C3 \Leftrightarrow FC(vi)$ and $1 - C4 \Leftrightarrow FC(vi)$

Proof: Let us first note that the implication in (a) are obvious. $\alpha - C3 \not\Rightarrow \alpha - C4$ or $\alpha - C$ For this consider the following example.

Example 4.1: Let $X = \left(\frac{1}{2,1}\right]$ and $u, v \in 1^X$ where

$$u(x) = \begin{cases} x, if x \in \left[\frac{1}{2}, \frac{3}{5}\right] \\ 0, if x \in \left(\frac{3}{5}, 1\right] \end{cases}$$

$$v(x) = \begin{cases} 0, if x \in \left[\frac{1}{2}, \frac{3}{5}\right] \\ x, if x \in \left(\frac{3}{5}, 1\right] \end{cases}$$

Let t be the fuzzy topology on X generated by $\{u,v\}\cup\{Consant\}$. It can be shown that (X,t) is $\alpha - C3$ for $\alpha = \frac{1}{2}$, since $u \vee v > \frac{1}{2}$, $u \wedge v \leq \frac{1}{2}$ and $u^{-1}(\left(\frac{1}{2},1\right] \neq \phi \neq v^{-1}\left(\frac{1}{2},1\right]$.

It is also clear that (X,t) is not $\alpha - C$.

$\alpha - C3 \not\Rightarrow \alpha - C4$ or $\alpha - C3$

The following example will serve the purpose.

Example 4.2: Let $X = \{x,y\}$ and $u,v \in I^X$ be given by $u = \left(\frac{2}{3}\right)1_x \vee \left(\frac{1}{3}\right)1_y$ and $v = \left(\frac{1}{3}\right)1_x \vee \left(\frac{2}{3}\right)1_y$. Let t be the fuzzy topology on X generated by $\{u,v\}\cup\{Consant\}$. It is then clear that (X,t) is $\alpha - C$ for any α, in particular $\alpha = \frac{1}{2}$. However, (X,t) is neither $\alpha - C4$ nor $\alpha - C3$ for $\alpha = \frac{1}{2}$.

(b) Let $\alpha \in I_0$, $\beta \in I_1$, $\alpha > \beta$ and (X,t) be $\alpha - C3$. If (X,t) is not $2_{1-\beta} - C$, then there exists a non – empty proper subset $A \subseteq X$ such that $(1-\beta)1_A, (1-\beta)1_{X-A} \in t$. Therefore, $(1-\beta)1_A \vee (1-\beta)1_{X-A} = 1-\beta > 1-\alpha$ and $(1-\beta)1_A \wedge (1-\beta)1_{X-A} = 0 < 1-\alpha$. Hence (X,t) is not $\alpha - C3$ since the constant clopen fuzzy set with value $1-\beta$ can be $\alpha - separated$. The remaining case can be handled similarly.

(c) If $\alpha > \beta$ and $\beta = \frac{1}{2}$, then from (b), we have $\alpha - C3 \Rightarrow 2_\beta - C$ and have $2_\beta - C \Rightarrow 2_\alpha - C$ by Remarks 4.1. The remaining case can similarly be handled. Again Example 4.1 and 4.2 show that the arrows in (c) are not reversible.

(d) These are obvious from the relevant definitions.

CHAPTER – 3

C_i – Connectedness in Fuzzy topological spaces

1. Introduction

Lowen [18] defined an extension of connectedness in a restricted family of fuzzy topologies, for fuzzy set which is everywhere strictly positive. Fatteh and Bassan [7] studied further the nations of fuzzy supper connected spaces and fuzzy strongly connected spaces. However, they defined connectedness only for a crisp set of a fuzzy topological space. In this chapter we discuss four types of connectedness for a fuzzy set, we study the implications that exist between them. These conditions are called C_i – connectedness (i = 1, 2, 3, 4). In this chapter we have given proofs only for the results on C_1-connectedness and C_3 – connectedness; the corresponding proofs for the results on C_2 – connectedness and C_4 – connectedness respectively, essentially being the same except for obvious modifications are omitted.

2. C_i – Connectedness

It is well known that for fuzzy sets u and v, the following implication is valid: $u \wedge v = 0 \Rightarrow u \leq 1 - v$. However, the reverse implication is not true in general. This departure of fuzzy set theory from ordinary set theory allows us to have the following variations in the fuzzy setting of the traditional notion of disconnection of a subset in a topological space.

Definition 2.1: A fuzzy set μ has a C_i – disconnection (i = 1, 2, 3, 4), if there exist fuzzy open sets $u, v \in t$, such that

$C_1 : \mu \leq u \vee v, u \wedge v \leq 1 - \mu, \mu \wedge v \neq 0 and \mu \wedge u \neq 0$,

$C_2 : \mu \leq u \vee v, \mu \wedge u \wedge v = 0, \mu \wedge v \neq 0 and \mu \wedge u \neq 0$,

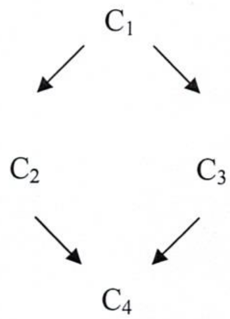
$C_3 : \mu \leq u \vee v, u \wedge v \leq 1 - \mu, u \nleq 1 - \mu and v \nleq 1 - \mu$ and

$C_4 : \mu \leq u \vee v, \mu \wedge u \wedge v \leq 0, u \nleq 1 - \mu and v \nleq 1 - \mu$ respectively.

Definition 2.2: A fuzzy set μ in fts (X,t) is said to be C_i – connected (i = 1, 2, 3, 4) if there exists no C_i – disconnection (i = 1, 2, 3, 4) of μ in X.

3. Comparison of C_i – connectedness (i = 1, 2, 3, 4)

In a fts (X,t) the classes of C_i – connected (i = 1, 2, 3, 4) fuzzy sets can be described by the following lattice diagram:



C_1-connectedness being the strongest condition gives us the smallest class whereas C_4 – connected fuzzy sets form the largest class. We demonstrate through examples that the inclusions are proper; moreover, the intersection of the classes of C_2 – connected and C_3 – connected fuzzy sets may not be empty. In that case, there exist fuzzy sets in fuzzy topological spaces which are C_2 – connected as well as C_3 – connected but not C_1 – connected. So C_2 – connectedness and C_3 – connectedness even together do not imply C_1 – connectedness.

Implications in the above diagram are immediate from the definitions. Here, we illustrate all the reverse implications by counter examples.

Example 3.1: $C_4 \not\Rightarrow C_3$

Let $X = [0,1]$ and define fuzzy sets u and u as follows:

$$u(x) = \begin{cases} \frac{1}{3} \; if \; \frac{1}{3} < x \leq 1 \\ 1 \; if \; 0 \leq x \leq \frac{1}{3} \end{cases}$$

$$\text{and } v(x) = \begin{cases} 1 \; if \; \frac{1}{3} < x \leq 1 \\ \frac{1}{3} \; if \; 0 \leq x \leq \frac{1}{3} \end{cases}$$

Then $t = \{0, u, v, u \wedge v, 1_X\}$ is a fuzzy topology on X. Define μ by $\mu(x) = \frac{2}{3}$ if $0 \leq x \leq 1$.

Here, μ is C_4 – connected but not C_3 – connected. i.e. $C_4 \not\Rightarrow C_3$

Example 3.2: $C_4 \not\Rightarrow C_2$

Let $X = [0,1]$ and define fuzzy sets u and u as follows:

$$u(x) = \begin{cases} 0 \; if \; \frac{1}{3} < x \leq 1 \\ \frac{1}{3} \; if \; 0 \leq x \leq \frac{1}{3} \end{cases}$$

$$\text{and } v(x) = \begin{cases} \frac{1}{3} \; if \; \frac{1}{3} < x \leq 1 \\ 0 \; if \; 0 \leq x \leq \frac{1}{3} \end{cases}$$

Then $t = \{0, u, v, u \wedge v, 1_X\}$ is a fuzzy topology on X. Consider the fuzzy set μ to be $u \vee v$. Then μ is C_4 – connected but not C_2 – connected. i.e. $C_4 \not\Rightarrow C_3$. From above examples, we see that $C_4 \not\Rightarrow C_1$ also.

Example 3.3: $C_3 \not\Rightarrow C_1$ and $C_2 \not\Rightarrow C_1$

Let $X = [0,1]$ and define fuzzy sets u and u as follows:

$$u(x) = \begin{cases} \frac{1}{3} \; if \; \frac{1}{3} < x \leq 1 \\ \frac{2}{3} \; if \; 0 \leq x \leq \frac{1}{3} \end{cases}$$

$$\text{and } v(x) = \begin{cases} \frac{2}{3} \; if \; \frac{1}{3} < x \leq 1 \\ \frac{1}{3} \; if \; 0 \leq x \leq \frac{1}{3} \end{cases}$$

Then $t = \{0, u, v, u \wedge v, 1_X\}$ is a fuzzy topology on X. Let us take the fuzzy set μ to be the constant function $\mu(x) = \frac{1}{3}$ for all $x \in X$. One can verify that μ is C_3 – connected as well as C_2 – connected but not C_1 – connected. i.e. $C_3 \not\Rightarrow C_1$ and $C_2 \not\Rightarrow C_1$.

Remarks 3.1: Counter example 3.3 also establishes the fact that the intersection of the classes of C_3 – connected fuzzy sets and C_2 – connected fuzzy sets in an fts may not empty.

Example 3.4: $C_3 \not\Rightarrow C_2$

Let $X = [0,1]$ and define fuzzy sets u and u as follows:

$$u(x) = \begin{cases} 0 \; if \; \frac{2}{3} < x \leq 1 \\ \frac{2}{3} \; if \; 0 \leq x \leq \frac{2}{3} \end{cases}$$

$$\text{and } v(x) = \begin{cases} \frac{2}{3} \; if \; \frac{2}{3} < x \leq 1 \\ 0 \; if \; 0 \leq x \leq \frac{2}{3} \end{cases}$$

Then $t = \{0, u, v, u \wedge v, 1_X\}$ is a fuzzy topology on X. Let us define μ as follows:

$$\mu(x) = \begin{cases} \frac{1}{3} \; if \; \frac{2}{3} < x \leq 1 \\ \frac{2}{3} \quad if \; 0 \leq x \leq \frac{2}{3} \end{cases}$$

Here, μ is C_3 – connected, but not C_2 – connected (not C_1 – connected).

Example 3.5: $C_2 \not\Rightarrow C_3$

Let (X, t) be the fts considered in counter example 3.3, and let μ be the fuzzy constant function $\mu(x) = \frac{1}{3}$ for all $x \in X$. Then μ is C_2 – connected but not C_3 – connected.

Remark 3.2: Counter examples 3.4 and 3.5 establish the facts that the classes of C_2 – connected fuzzy sets and C_3 – connected fuzzy sets in fts may not be comparable.

4. Preservation of C_i – connectedness (i = 1, 2, 3, 4)

In the following theorems, we discuss the preservation of C_i – connectedness (i = 1, 2, 3, 4) under fuzzy continuity.

Theorem 4.1: If $F : X \to Y$ is a fuzzy continuous function and μ is a C_1 – connected (C_2 – connected) fuzzy set in X, then $F(\mu)$ is a C_1 – connected (C_2 – Connected) fuzzy set in Y.

Proof: Proof of the above theorem is straight forward.

Theorem 4.2: It $F : X \to Y$ is a fuzzy continuous surjection and if μ is a C_3 – connected (C_4 – connected) fuzzy set in X, then $F(\mu)$ is a C_3 – connected (C_4 – connected) fuzzy set in Y.

Proof: Suppose $F(\mu)$ is not C_3 – connected. Thus there exist fuzzy open sets u and v in Y, such that $F(\mu) \leq u \vee v, u \wedge v \leq 1 - F(\mu), u \not\leq 1 - F(\mu)$ and $v \not\leq 1 - F(\mu)$. Therefore,

$\mu \le F^{-1}(u) \vee F^{-1}(v)$ and $F^{-1}(u) \wedge F^{-1}(v) \le 1-\mu$ as $\mu \le F^{-1}(F(\mu)) \Rightarrow$ $1-F^{-1}(F(\mu)) \le 1-\mu$, where $F^{-1}(u)$ and $F^{-1}(v)$ are fuzzy open sets in X, since F is fuzzy continuous.

Again, there exist $y_1, y_2 \in Y$, such that

(i) $u(y_1) > 1-F(\mu)(y_1)$

(ii) $u(y_2) > 1-F(\mu)(y_2)$

As F is onto, $F^{-1}(y_1)$ and $F^{-1}(y_2)$ are non-empty subsets of X. By definitions, we have $F^{-1}(u)(x_i) = u(y_1)$, for every $x_i \in F^{-1}(y_1)$ and $F(\mu)(y_1) = \sup\{\mu(x_i)\}$, where $x_i \in F^{-1}(y_1)$.

We claim that $F^{-1}(u) \nleq 1-\mu$ and $F^{-1}(v) \le 1-\mu$. Suppose

$$F^{-1}(u) \le 1-\mu$$

$$\Rightarrow F^{-1}(u)(x_i) \le 1-\mu(x_i), \text{ for every } x_i \in F^{-1}(y_1)$$

$$\Rightarrow \mu(x_i) \le 1-u(y_1), \text{ for every } x_i \in F^{-1}(y_1)$$

$$\Rightarrow \sup\{\mu(x_i)\} \le 1-u(y_1), \text{ for every } x_i \in F^{-1}(y_1)$$

$$\Rightarrow F(\mu)(y_1) \le 1-u(y_1)$$

This contradicts (i) similarly by considering $F^{-1}(v) \le 1-\mu$ we have a contradiction with (ii).

In the reverse direction, we have the following theorem.

Theorem 4.3: Let $F: X \to Y$ be a fuzzy open injection and let μ be a fuzzy set in Y. Then μ is C_i – connected implies that $F^{-1}(\mu)$ is C_i – connected (i = 1, 2, 3, 4).

Proof: Proof of the above theorem is easy.

Definition 4.1: A fuzzy point x_α is said to be contained in a fuzzy set u or u is said to captain x_α if $u(x) \ge \alpha$. We denote it by $u \ge x_\alpha$.

We observe that a fuzzy point x_α is C_i – connected (i = 2, 3) hence C_4 – connected, but not necessarily C_1 – connected which is a departure from general topology where points are connected sets.

Example 4.1: Let $X = \{x, y\}$ and define fuzzy sets u and v, as follows:

$$u(x) = \frac{1}{3},\ u(y) = \frac{2}{3} \text{ and}$$

$$v(x) = \frac{2}{3},\ v(y) = \frac{1}{3}.$$

Then $t = \{0, u, v, u \wedge v, 1_X\}$ is a fuzzy topology on X. Here, the fuzzy point $x_{1/3}$ is not C_1– connected. Moreover, we observe that the fuzzy set 0 is C_1 – connected and hence C_i – connected (i = 1, 2, 3, 4).

Definition 4.2: Two fuzzy sets u and v in X are said to be non-overlapping if $u \le 1 - v$. u and v are overlapping if there exists a point $x \in X$ such that $u(x) > 1 - v(x)$. In this case u and v are said to overlap at x.

Theorem 4.4: If u and v are intersecting C_1 – connected (C_2 – connected) fuzzy sets is X, then $u \vee v$ is a C_1 – connected (C_2 – connected) fuzzy set in X.

Proof: Let η_1 and η_2 be fuzzy open sets in X such that

$$u \vee v \le \eta_1 \vee \eta_2 \text{ and } \eta_1 \wedge \eta_2 \le 1 - u \vee v \text{ --------------- (1).}$$

Then, as u and v are C_1 – connected, (1) gives us

$$(u \wedge \eta_1 = 0 \text{ or } v \wedge \eta_2 = 0) \text{ and } (v \wedge \eta_1 = 0 \text{ or } u \wedge \eta_2 = 0).$$

Now we consider the following cases:

Case I: Suppose $u \wedge \eta_1 = 0$. As u and v are intersecting there exists $x_1 \in X$ such as $u(x_1) \ne 0 \ne v(x_1)$. We claim that $v \wedge \eta_2 \ne 0$. Suppose, if possible, $v \wedge \eta_2 = 0$. Then

$(v \wedge \eta_2)(x_1) = 0$, but $v(x_1) \neq 0$; therefore $\eta_2(x_1) = 0$. Now as $u \wedge \eta_1 = 0$, we have $\eta_1(x_1) = 0$, since $u(x_1) \neq 0$. Hence $(\eta_1 \vee \eta_2)(x_1) = 0$ which contradicts (1) as $(u \vee v)(x_1) \neq 0$. Therefore, we have $v \wedge \eta_1 = 0$, which implies that $(u \vee v) \wedge \eta_1 = 0$.

Case II: Suppose $u \wedge \eta_2 = 0$. As in case I, we can show here that $v \wedge \eta_1 = 0$ is not possible, hence $v \wedge \eta_2 = 0$. Therefore $(u \vee v) \wedge \eta_2 = 0$. So, $(u \vee v)$ is C_1 – connected.

The following example illustrates that the above theorem is not valid for disjoint (non – intersecting) fuzzy sets. Moreover, if also reflects the divergence in fuzzy setting from that of general topology. For, the theorem also fails for two fuzzy sets u and v which are not separated in the since

$$\overline{u} \wedge v \neq 0 \text{ or } v \wedge \overline{u} \neq 0$$

Example 4.2: Let $X = [0,1]$ and define fuzzy sets μ and γ as follows:

$$\mu(\mathrm{x}) = \begin{cases} 0 \text{ if } \frac{2}{3} < \mathrm{x} \leq 1 \\ \frac{2}{3} \text{ if } 0 \leq \mathrm{x} \leq \frac{2}{3} \end{cases}$$

$$\gamma(\mathrm{x}) = \begin{cases} \frac{2}{3} \text{ if } \frac{2}{3} < \mathrm{x} \leq 1 \\ 0 \text{ if } 0 \leq \mathrm{n} \leq \frac{2}{3} \end{cases}$$

Then $t = \{0, \mu, \gamma, \mu \vee \gamma, 1_X\}$ is a fuzzy topology on X. Let η_1 and η_2 be defined as

$$\eta_1(\mathrm{x}) = \begin{cases} \frac{1}{3} \text{ if } \frac{2}{3} < \mathrm{x} \leq 1 \\ 0 \text{ if } 0 \leq \mathrm{x} \leq \frac{2}{3} \end{cases}$$

$$\eta_2(\mathrm{x}) = \begin{cases} 0 \text{ if } \frac{2}{3} < \mathrm{x} \leq 1 \\ \frac{1}{3} \text{ if } 0 \leq \mathrm{x} \leq \frac{2}{3} \end{cases}$$

Here, $\eta_1 \wedge \eta_2 = 0$, and it can be verified that η_1 and η_2 are C_2 – connected fuzzy sets but $\eta_1 \vee \eta_2$ has a C_2 – disconnection. Also η_1 and η_2 are not separated in the sense of general topology. For $\overline{\eta_1} \wedge \eta_2 \neq 0$.

The following example shows that theorem 4.4 fails for C_3 – connectedness (C_4 – connectedness).

Example 4.3: Let $X = [0,1]$ and define fuzzy sets η_1 and η_2 as follows:

$$\eta_1(x) = \begin{cases} \frac{4}{5} \text{ if } \frac{4}{5} < x \leq 1 \\ \frac{2}{5} \text{ if } 0 \leq x \leq \frac{4}{5} \end{cases}$$

$$\eta_2(x) = \begin{cases} \frac{2}{5} \text{ if } \frac{4}{5} < x \leq 1 \\ \frac{4}{5} \text{ if } 0 \leq x \leq \frac{4}{5} \end{cases}$$

Then $t = \{0, \eta_1, \eta_2, \eta_1 \vee \eta_2, \eta_1 \wedge \eta_2, 1_X\}$ is a fuzzy topology on X. Let us define u and v as

$$u(x) = \begin{cases} \frac{1}{5} \text{ if } \frac{4}{5} < x \leq 1 \\ \frac{2}{5} \text{ if } 0 \leq x \leq \frac{4}{5} \end{cases}$$

$$v(x) = \begin{cases} \frac{2}{5} \text{ if } \frac{4}{5} < x \leq 1 \\ \frac{1}{5} \text{ if } 0 \leq x \leq \frac{4}{5} \end{cases}$$

Here $u \wedge v \neq 0$ and u, v are C_3 – connected fuzzy sets in X, but $u \vee v$ has a C_3 – disconnection.

Theorem 4.5: Let $\{\eta_i\}_{i\in I}$ be a family of C_1 – connected (C_2 – connected) fuzzy sets in X. Such that for $i, j \in I, i \neq j$, the fuzzy sets η_i and η_j are interesting. Then $\vee_{i\in I}\eta_i$ is a C_1 – connected (C_2 – connected) fuzzy set in X.

Proof: Let u and v be fuzzy open sets in X such that $\eta \le u \vee v$, $u \wedge v \le 1-\eta$, where $\eta = \vee_{i\in I}\eta_i$. Now, let η_{i_0} be any fuzzy set of the given family. Then

$$\eta_{i_0} \le u \vee v,\ u \wedge v \le 1-\eta_{i_0}$$

as $u \wedge v \le 1-\eta$ implies $u \wedge v \le \wedge_{i\in I}(1-\eta_i)$ and therefore $u \wedge v \le 1-\eta_{i_0}$. But η_{i_0} is C_1 – connected. Hence

$$\eta_{i_0} \wedge u = 0 \text{ or } \eta_{i_0} \wedge v = 0$$

Now, the result follows in view of the facts that $\eta_{i_0} \wedge u = 0$, then as in theorem 4.4, we can prove $\eta_i \wedge u = 0$ for each $i \in I \approx \{i_0\}$ and $\vee_{i\in I}(\eta_i \wedge u) = 0$ implies $\eta \wedge u = 0$.

Corollary 4.1: If $\{\eta_i\}_{i\in I}$ is a family of C_1 – connected (C_2 – connected) fuzzy sets in X and $\wedge_{i\in I}\eta_i \neq 0$, then $\vee_{i\in I}\eta_i$ is a C_1 – connected (C_2 – connected) fuzzy set in X.

Proof: Straightforward in view of theorem 4.5.

Corollary 4.2: Let $\{\eta_n\}$ be a sequence of C_1 – connected (C_2 – connected) fuzzy sets in X, such that η_n and η_{n+1} are intersecting for each n,. Then $\vee_{n=1}^{\infty}\eta_n$ is a C_1 – connected (C_2 – connected) fuzzy set in X.

Proof: Follows by using theorem 4.4 and induction.

Theorem 4.6: If u and v are overlapping C_3 – connected (C_4 – connected) fuzzy sets in X, then $u \wedge v$ is a C_3 – connected (C_4 – connected) fuzzy set in X.

Proof: Let η_1 and η_2 be fuzzy open sets in X such that

$$u \vee v \leq \eta_1 \vee \eta_2,\ \eta_1 \wedge \eta_2 \leq 1 - u \vee v \text{ ---------------- (1)}$$

Now, as u and v are C_3 – connected, (1) gives us

$$(\eta_1 \leq 1-u \text{ or } \eta_2 \leq 1-u) \text{ and } (\eta_1 \leq 1-v \text{ or } \eta_2 \leq 1-v).$$

Moreover, as u and v are overlapping fuzzy sets, there exists $x_1 \in X$ such that

$$u(x_1) > 1 - v(x_1) \text{ ------------------------------------- (2)}$$

Now, consider the following cases:

Case I: Suppose $\eta_1 \leq 1-u$, then by (2), we have

$$\eta_1(x_1) \leq v(x_1) \text{ -- (3)}$$

We claim that, $\eta_2 \nleq 1-v$. For, if not then

$$\eta_2(x_1) \leq 1 - v(x_1) < u(x_1) \text{ --------------------------- (4)}$$

Now, by (3) and (4) we have

$$(\eta_1 \vee \eta_2)(x_1) < (u \vee v)(x_1)$$

Which implies that $u \vee v < \eta_1 \vee \eta_2$, this contradicts (1). Hence $\eta_1 \leq 1-v$. Therefore,

$$\eta_1 \leq (1-u) \wedge (1-v) = 1 - u \vee v .$$

Case II: Suppose $\eta_2 \leq 1-u$. Here, we can show as in case I that $\eta_2 \nleq 1-v$. Therefore, $\eta_2 \leq 1-v$. Hence $\eta_2 \leq 1-u \vee v$.

Theorem 4.7: Let $\{\eta_i\}_{i\in I}$ be a family of C_3 – connected (C_4 – connected) fuzzy sets in X. Such that for $i, j \in I, i \neq j$, the fuzzy sets η_i and η_j are overlapping. Then $\vee_{i\in I} \eta_i$ is a C_3 – connected (C_4 – connected) fuzzy set in X.

Proof: Let u and v be fuzzy open sets in X such that $\eta \leq u \vee v$, $u \wedge v \leq 1-\eta$, where $\eta = \vee_{i\in I} \eta_i$. Now, let η_{i_0} be any fuzzy set of the given family. Then as η_{i_0} is a C_3 – connected fuzzy set, so we have

$$u \leq 1-\eta_{i_0} \text{ or } v \leq 1-\eta_{i_0}$$

Now, the result follows in view of the facts that if $\eta_{i_0} \leq 1-u$, then $\eta_i \leq 1-u$ for each $i \in I \approx \{i_0\}$, since η_{i_0} and η_i are overlapping C_3 – connected fuzzy sets, and

$$u \leq \wedge_{i\in I} (1-\eta_i) = 1-\eta .$$

Corollary 4.3: Let $\{\eta_i\}_{i\in I}$ be a family of C_3 – connected (C_4 – connected) fuzzy sets in X and x_α be a fuzzy point such that $\alpha > \frac{1}{2}$ and $x_\alpha \leq \wedge_{i\in I} \eta_i$. Then $\vee_{i\in I} \eta_i$ is a C_3 – connected (C_4 – connected) fuzzy set in X.

Hint: Since $x_\alpha \leq \wedge_{i\in I} \eta_i$, it follows that η_i and η_j are overlapping fuzzy sets, for each $i, j \in I$.

Corollary 4.4: Let $\{\eta_n\}$ be a sequence of C_3 – connected (C_4 – connected) fuzzy sets in X, such that η_n and η_{n+1} are overlapping for each n. Then $\vee_{n=1}^{\infty} \eta_n$ is a C_3 – connected (C_4 – connected) fuzzy set in X.

Theorem 4.8: If η is C_3 – connected (C_4 – connected) fuzzy sets in X and $\eta \le \delta \le \overline{\eta}$, then δ is also a C_3 – connected (C_4 – connected) fuzzy sets in X.

Proof: Let u and v be fuzzy open sets in X such that

$$\delta \le u \vee v,\; u \wedge v \le 1-\delta$$

Then, as η is C_3 – connected, we have

$$\eta \le 1-u \text{ or } \eta \le 1-v.$$

But, if $\eta \le 1-u$ then $\overline{\eta} \le \overline{(1-u)} = 1-u^0 = 1-u$ and on the other hand, if $\eta \le 1-v$ then $\overline{\eta} \le \overline{(1-v)} = 1-v^0 = 1-v$. Therefore,

$$\delta \le \overline{\eta} \le 1-u \text{ or } \delta \le \overline{\eta} \le 1-v.$$

However, the above theorem fails in case of C_1 – connectedness and C_2 – connectedness which is a departure from general topology. The following example will illustrate that the closure of a C_1 – connected fuzzy set need not to be a C_1 – connected fuzzy set.

Example 4.4: Let $X = [0,1]$ and define fuzzy sets u and v as follows:

$$u(x) = \begin{cases} 1 \text{ if } \frac{2}{3} < x \le 1 \\ 0 \text{ if } 0 \le x \le \frac{2}{3} \end{cases}$$

$$v(x) = \begin{cases} 0 \text{ if } \frac{2}{3} < x \le 1 \\ \frac{2}{3} \text{ if } 0 \le x \le \frac{2}{3} \end{cases}$$

Then $t = \{0, u, v, u \vee v, 1_X\}$ is a fuzzy topology in X. Here, u is a C_1 – connected fuzzy set but its closure $\overline{u}$ is not C_1 – connected.

To show theorem 4.8 fails for C_2 – connectedness consider in example 4.2 the fuzzy set η to be the fuzzy point $x_{1/3}$, where $x \in \left(\frac{2}{3}, 1\right]$. Then the closure $\overline{x}_{1/3}$ is not C_2 – connected.

CHAPTER – 4

(C3) – Connectedness in Fuzzy topological spaces

1. Introduction

In this chapter we study a new type of fuzzy connectedness and establish that it has several desirable features. We introduce here (C3) – connectedness in fuzzy topological spaces and also establish its appropriateness. In particular, we note that it is a "good extension" of the concept of connectedness in topology, is preserved under fuzzy continuity, second – additive and productive. However, we also show that it cannot be viewed as a "Preuss - Connectedness" concepts (c.f. e.g. Lowen [19] and Srivastrava [33]).

2. (C3) – connectedness

Let E be a set and I the unit interval. Put $T(E)$ the set of all topologies on E and $W(E)$ the set of all fuzzy topologies on E.The topological space one obtains giving I as the induced topology we denote I_r. We then define the mapping

$$\omega : T(E) \to W(E) : T \to \omega(T)$$

where $\omega(T) = \ell(E, I_r)$, the continuous functions from (E,T) to I_r. It is trivial to cheek that $\omega(T)$ is indeed a fuzzy topology since it is the set of all lower semi continuous functions from (E,T) to the unit interval equipped with the usual topology.

Definition: We call a fuzzy topological space (X,t) (C3) – connected iff for all $\varepsilon \in I_0$ and for all t – clopen fuzzy sets $\mu \geq \varepsilon$ there do not exist $u, v \in t$ with the following properties

(i) $u \neq \mu \neq v$

(ii) $u \vee v = \delta$

(iii) $u \wedge v = 0$

A pair (u,v) satisfying (i), (ii), (iii) shall be referred to as a (C3) – separation of μ.

Remarks: The concept of (C3) – connectedness is closely related to the concept of (C1) – connectedness as introduced in Lowen [15]. Obviously, (C1) – connectedness implies (C3) – connectedness and examples can easily be constructed to show that the converse is false.

We first show that this is a "good extension" (in the sense of Lowen [17]) of the concept of connectedness in topology.

Proposition 1: A topological space (X,T) is connected iff $(X,\omega(T))$ is (C3) – connected.

Proof: One way implication is obvious as each non – trivial open separation $\{Y_1,Y_2\}$ of X leads to a (C3) – separation viz. $\{1_{Y_1},1_{Y_2}\}$ of 1. To prove the other implication, let $(X,\omega(T))$ be not (C3) – connected. Then there exist $\varepsilon \in I_0$, an $\omega(T)$ - clopen $\mu \geq \varepsilon$ and $\xi_1,\xi_2 \in \omega(T)$ such that $\xi_1 \neq \mu \neq \xi_2$, $\xi_1 \vee \xi_2 = \mu$ and $\xi_1 \wedge \xi_2 = 0$.

Fix any $\varepsilon' \in (0,\varepsilon)$. Then $\xi_1^{-1}(\varepsilon',1],\xi_2^{-1}(\varepsilon',1] \in T$ and it is easily verified that these make a non – trivial T - open separation of X.

Proposition 2: If $(A_i,t_i)_{i\in J}$ is a family of (C3) – connected subspaces of an fts (X,t) with $\cap_{i\in J} A_i \neq \Phi$ then (A_i,t_A) is (C3) – connected, where $A = \cup_{i\in J} A_i$.

Proof: Let A be not (C3) – connected then there exist $\varepsilon \in I_0$ and a (C3) – connection (ξ_1,ξ_2) of some t_A - clopen $\mu \geq \varepsilon$. i.e. $\xi_1 \neq \mu \neq \xi_2$ with $\xi_1 \vee \xi_2 = \mu$ and $\xi_1 \wedge \xi_2 = 0$. Pick some $a_0 \in \cap_{i\in J} A_i$. For any $i_0 \in J$, $\xi_1 / A_{i_0} \wedge \xi_2 / A_{i_o} = 0$ and $\xi_1 / A_{i_0} \vee \xi_2 / A_{i_o} = \mu / A_{i_o}$. So, since $\xi_1 / A_{i_0}, \xi_2 / A_{i_o}$ are t_{i_0} - open, $\mu / A_{i_0} \geq \varepsilon$ is t_{i_0} - clopen and since (A_{i_0},t_{i_0}) is (C3) – connected, either $\xi_1 / A_{i_0} = \mu / A_{i_0}$ or $\xi_2 / A_{i_0} = \mu / A_{i_0}$. Suppose $\xi_1 / A_{i_0} = \mu / A_{i_0}$. Then, since, $a_0 \in A_{i_0}$, we have $\xi_1(a_0) = \mu(a_0) \geq \varepsilon$. Thus, $\xi_2(a_0) = 0$.

Now, take any other $j \in J$, then $\xi_2 / A_j \neq \mu / A_j$ since $a_0 \in A_j$. Thus $\xi_1 / A_j = \mu / A_j$ showing thereby that, for all $i \in J$, $\xi_1 / A_i = \mu / A_i$. As $A = \cup_{i \in J} A_i$, $\xi_1 = \mu$, which is a contradiction.

Remarks: Given an fts (X,t) and $D \subseteq X$, we recall, from Lowen [18], that D is called fuzzy dense in (X,t) iff for each $\mu \in t$,

$$\sup_{x \in X} \mu(x) = \sup_{x \in D} \mu(x).$$

Further, Lowen [18] has shown that if $\overline{Y} = cl_{L(t)} Y$, then (Y, t_Y) is the largest subspace of (X,t) in which Y is a fuzzy dense (for the definition of L(t), see Lowen [18])

Proposition 3: Let (X,t) be an fts and (Y,t_Y) be (C3) – connected subspace of an fts (X,t) with $Y \subseteq Z \subseteq \overline{Y}$. Then (Z,t_Z) is (C3) – connected.

Proof: Let (Z,t_Z) not be (C3) – connected. Then there exist $\varepsilon \in I_0$, t_Z- clopen $\mu \geq \varepsilon$, and $\alpha \neq \mu \neq \beta$ with $\alpha \vee \beta = \mu$ and $\alpha \wedge \beta = 0$. Clearly, μ / Y is t_Y- clopen and $\mu / Y \geq \varepsilon$. Also, $\alpha / Y, \beta / Y \in t_Y$.
Now, keeping in view that Z is fuzzy dense in Y, we can verify, as in Lowen [18], that $\{\alpha / Y, \beta / Y\}$ forms a (C3) – separation of μ / Y. Thus Y is not (C3) – connected which is a contradiction.

Proposition 4: Let $f:(X,s) \to (Y,t)$ be an onto fuzzy continuous function between fuzzy topological spaces and let (X,s) be (C3) – connected. Then (Y,t) is also (C3) – connected.

Proof: If possible, let (Y,t) be not (C3) – connected. Then there exist $\varepsilon \in I_0$ an t– clopen $\mu \geq \varepsilon$, and $\xi_1, \xi_2 \in t$ such that $\xi_1 \neq \mu \neq \xi_2$ with $\xi_1 \vee \xi_2 = \mu$ and $\xi_1 \wedge \xi_2 = 0$. As

f is fuzzy continuous, $f^{-1}(\mu)$ is $s-$ clopen. Also, $\mu \geq \varepsilon \Rightarrow f^{-1}(\mu) = \mu \circ f \geq \varepsilon$. Put $\eta_1 = f^{-1}(\xi_1)$ and $\eta_2 = f^{-1}(\xi_2)$. Then, $\eta_1, \eta_2 \in s$. For each $x \in X$,

$$\begin{aligned}(\eta_1 \wedge \eta_2)(x) &= \left(f^{-1}(\xi_1) \wedge f^{-1}(\xi_2)\right)(x) \\ &= (\xi_1 \wedge \xi_2)(f(x)) \\ &= 0.\end{aligned}$$

Thus, $\eta_1 \wedge \eta_2 = 0$. For each $x \in X$, we have

$$\begin{aligned}(\eta_1 \vee \eta_2)(x) &= \left(f^{-1}(\xi_1) \vee f^{-1}(\xi_2)\right)(x) \\ &= (\xi_1 \vee \xi_2)(f(x)) \\ &= \mu f(x) \\ &= f^{-1}\mu(x).\end{aligned}$$

Hence, $\eta_1 \vee \eta_2 = f^{-1}\mu$. Finally, it is obvious that $\eta_1 \neq f^{-1}\mu \neq \eta_2$. So, (X,s) is not (C3) – connected which is a contradiction.

Proposition 5: Let $\{(x_j, t_j) : j \in J\}$ be a family of fuzzy topological spaces. Then $\prod_{j \in J} X_j$ is (C3) – connected iff each X_j is (C3) – connected.

Proof: In view of proposition 4, we only need to prove the ' if ' part. Let (X_j, t_j) be (C3) – connected and put $(X,t) = \prod_{j \in J}(X_j, t_j)$. Fix some $x_0 = (x_j)_{j \in J} \in X$. We show, by induction, that all those points of X which differ from x_0 at most finitely many co – ordinates, are in a (C3) – connected subspaces of (X,t). Put $Z = \{x \in X : x_i = x_{i_0} \mathit{for\ all}\ i \neq j\}$. As (Z, t_Z) is fuzzy homeomorphic to (X_j, t_j), (Z, t_Z) must be (C3) – connected.

Next, assume that all points of X which differ from x_0 at most (n-1) co – ordinates are in a (C3) – connected subspaces of (X,t).

Now suppose x_0 and $x \in X$ differ by 'n' co – ordinates. Choose $y \in X$ such that x_0 and y differ by (n-1) co – ordinates and x and y only by 1. Then induction hypothesis x_0 and x are in some (C3) – subspaces, say (S,t_S) of (X,t) and y and x are in some (C3) – subspaces, say (T,t_T) of (X,t). As $y \in S \cap T$, $(S \cup T, t_{S \cup T})$ must be (C3) – connected in view of proposition 2. Thus if we substitute Y = Union of all (C3) – subspaces of (X,t) containing x_0 and

$D = \{x : x_0 \text{ and } x \text{ differ atmost finite co-ordinates}\}$, then $Y \supseteq D$ and Y is a (C3) – connected from proposition 2. We now show that D is fuzzy dense in X. Let $\mu \in t$ be basic open (in the standard basis). Then, there exists $j_1, j_2, j_3, j_4, \ldots\ldots.. j_n$ such that $\mu = \wedge_{i=1}^{n} P_{j_i}^{-1}(\mu_{j_i})$, where $\mu_{j_i} = t_{j_i}$, for each $i = 1,2,3,\ldots\ldots..n$ (P_{j_i} are the projection maps). Thus, for each $x \in X$

$$\mu(x) = \wedge_{i=1}^{n} \mu_{j_i}(x_{j_i}).$$

Given $x = (x_j)_{j \in J} \in X$, define $\tilde{x} = (\tilde{x}_j)_{j \in J} \in X$, where

$$\tilde{x}_j = \begin{cases} x_{0j}, \text{ if } j \neq j_1, j_2, j_3, \ldots\ldots\ldots, j_n \\ x_j, \text{ otherwise} \end{cases}$$

Clearly, $\tilde{x} \in D$ and $\mu(x) = \mu(\tilde{x})$. Also $\sup_{x \in D} \mu(x) = \sup_{x \in X} \mu(x)$ shows that D is fuzzy dense in (X,t).

In view of proposition 3 and the fact that $\overline{Y} = X$, it follows that (X,t) is (C3) – connected.

Let $|FTS|$ denote the class of all fuzzy topological spaces. For $A \subseteq |FTS|$, we put $\ell A = \{X \in |FTS| : each\ fuzzy\ continuous\ f : X \to E \in A\ is\ consant\}$. Following G. Preuss [24, 25, 26], members of A may be called A - connected spaces.

Proposition 6: There does not exist a class $A \subseteq |FTS|$ such that $\ell A = \{X \in |FTS| : Xis(C3) - connected\}$.

Proof: Let there exist a class $A \subseteq |FTS|$ such that

$$\ell A = \{X \in |FTS| : X\ is\ (C3) - connected\}$$

Then the inclusion map $i : B \to E$, being continuous, is constant. Thus B must be a singleton. Hence, for any $E \in A, B \subseteq E$ is (C3) – connected iff B is a singleton. Fix some $\varepsilon \in I_0$ and consider the fts $((0,2], \Omega)$ where Ω is generated by

$$\{\mu, 1-\mu, \mu \wedge 1_{(0,1]}, \mu \wedge 1_{(1,2]}\}$$

Where μ is given by

$$\mu(x) = \begin{cases} (1-\varepsilon)x + \varepsilon, where\ x \in (0,1] \\ (1-\varepsilon)x + 2\varepsilon - 1, when\ x \in (1,2] \end{cases}$$

Then $\mu \wedge 1_{(0,1]}$ and $\mu \wedge 1_{(1,2]}$ constitute a (C3) – connected of Ω - clopen $\mu \geq \varepsilon$ whereby $((0,2], \Omega)$ is not (C3) – connected. So, there exist $(X,t) \in A$ and a non – constant fuzzy continuous function. Evidently, (0,1] and (1,2] are (C3) – connected subspaces of $((0,2], \Omega)$. So, $f((0,1])$ and $f((1,2])$ must be (C3) – connected. Hence there exist distinct $x_1, x_2 \in X$ with $f((0,1]) = \{x_1\}$ and $f((1,2]) = \{x_2\}$. Further, since X is not 'totally

(C3) – connected', it follows that $(\{x_1,x_2\},t/\{x_1,x_2\})$ is not indiscrete, so there exists $\xi \in t$ such that

$$\xi(x_1) \neq \xi(x_2)$$

However $f^{-1}(\xi) \in \Omega$ is two – valued which, by simple inspection of Ω, is impossible.

Proposition 7: For an fts (X,t), the following conditions are equivalent:

(i) (X,t) is (C3) – connected.

(ii) There is no fuzzy continuous map $f:(X,t) \to ((0,2],\Omega)$ such that

$$f(X) \cap (0,1] \neq \Phi \text{ and } f(X) \cap (1,2] \neq \Phi.$$

Proof: $(ii) \Rightarrow (i)$ Let (X,t) not be (C3) – connected. Then there exist $\varepsilon \in I_0$, t - clopen $v \geq \varepsilon$ and $\lambda_1, \lambda_2 \in t - \{v\}$ such that

$$\lambda_1 \vee \lambda_2 = v \text{ and } \lambda_1 \wedge \lambda_2 = 0.$$

Put $X_1 = \lambda_1^{-1}((0,1])$ and $X_2 = \lambda_2^{-1}((0,1])$, and define $f: X \to (0,2]$ by

$$f(x) = \begin{cases} \dfrac{\lambda_1(x) - \varepsilon}{1-\varepsilon}, & if\ x \in X_1 \\ \dfrac{\lambda_2(x) + 1 - 2\varepsilon}{1-\varepsilon}, & if\ x \in X_2 \end{cases}$$

Then $f:(X,t) \to ((0,2],\Omega)$ is fuzzy continuous. For $x \in X$, it is easily checked that $f^{-1}(\mu) = v$ where

$$f^{-1}(x) = \begin{cases} \lambda_1(x), if\ x \in X_1, f(x) \leq 1 \\ \lambda_2, if\ x \in X_2, f(x) > 1 \end{cases}$$

Consequently, $f^{-1}(1-\mu) = 1 - v$.

Similarly, it can be checked that

$$f^{-1}(\mu \wedge 1_{(0,1]}) = \lambda_1 \text{ and } f^{-1}(\mu \wedge 1_{(1,2]}) = \lambda_2 .$$

Hence, f is fuzzy continuous. Since, $X_1 \neq \Phi$, $X_2 \neq \Phi$, $f(X \cap (0,1]) \neq \Phi$ and $f(X \cap (1,2]) \neq \Phi$, which is a contradiction.

$(i) \Rightarrow (ii)$. If there exists a continuous function $f : (X,t) \to ((0,2],\Omega)$ with $f(X \cap (0,1]) = \Phi$ and $f(X \cap (1,2]) \neq \Phi$, then by putting $v = f^{-1}(\mu)$, we see that v is t – clopen with $v \geq \varepsilon$. Moreover, $f^{-1}(\mu \wedge 1_{(0,1]})$ and $f^{-1}(\mu \wedge 1_{(1,2]})$ form a (C3) – connection on v. Hence, (X,t) cannot be (C3) – connected which is a contradiction.

CHAPTER – 5

Stronger Forms of Connectedness in FTS

1. Introduction

In this chapter we study some stronger forms of connectedness in fuzzy topological spaces. We introduce here the notion of fuzzy super connected spaces (fuzzy D – spaces) and fuzzy strongly connected spaces following closely the definition of D – space Levine [15] and strongly connected space Levine [16], due to Levine in general topological spaces. We also give some characterization of super connected spaces and fuzzy strongly connected spaces.

2. Fuzzy Super – Connectedness

Levine [15] introduced the notion of a D – space to be a topological space in which every non – empty open set is dense. In note Fatteh and Bassan [7] we defined super – connected spaces to be a topological space which has no proper regular open subset and have shown that a space is super – connected iff it is a D – space. So, we define a fuzzy D – space to be a fuzzy topological space in which there is no proper fuzzy regular open set and we shall also call such a space to be a fuzzy super – connected space. Since, a fuzzy clopen set is a fuzzy regular open set, fuzzy super – connectedness implies fuzzy connectedness but the following example shows that the converse is not true.

Example 2.1: Let $X=[0,1]$. For each $x \in X$ we define $\lambda(x)=\frac{1}{6}$ and $\mu(x)=\frac{2}{3}$. Let $t=\{0,1,\lambda,\mu\}$. Then clearly fts X is fuzzy connected but it is not fuzzy super – connected, since it has a proper fuzzy regular open set μ.

3. Characterizations of Fuzzy Super Connectedness

By giving some theorem we will discuss some characterization of fuzzy super – connectedness.

Theorem 3.1: If X is an fts then the following statements are equivalent:

(i) X is fuzzy super connected.

(ii) Closer of every non – zero fuzzy open set in X is 1.

(iii) Interior of every fuzzy closed set (in X), different from 1, is zero.

(iv) X does not have non – zero fuzzy open sets λ and μ satisfying $\lambda+\mu\le 1$.

(v) X does not have non – zero fuzzy open sets λ and μ satisfying $\lambda+\mu=\lambda+\overline{\mu}=1$.

(vi) X does not have non – zero fuzzy closed sets f and k satisfying $f^{i}+k=f+k^{i}=1$.

Proof: $(i)\Rightarrow(ii)$. If X has a non – zero fuzzy open set λ such that $\overline{\lambda}\neq 1$, then $\overline{\lambda}^{\prime}$ is a proper fuzzy open set.

$(ii)\Rightarrow(iii)$. Let f be a fuzzy closed set in X different from 1. Now $f^{i}=1-\overline{1-f}=0$, as $1-f$ is a non – zero open set. Since $1-f^{i}=\overline{1-f}$.

$(iii)\Rightarrow(iv)$. If X has non – zero fuzzy open sets λ and μ such that $\lambda+\mu\le 1$, then $\overline{\lambda}+\mu\le 1$. So $\mu\neq 0$ implies $\overline{\lambda}\neq 1$. Since $\lambda\neq 0$, $\lambda^{i}\neq 0$, which contradicts (iii).

$(iv)\Rightarrow(i)$. If X has a proper fuzzy regular open set λ, then λ and $\mu=1-\lambda$ are non – zero fuzzy open sets satisfying $\lambda+\mu\le 1$.

$(i)\Leftrightarrow(v)$. If X is not fuzzy super – connected, then it has a proper fuzzy regular open set say λ. If we put $\mu=1-\overline{\lambda}$, then $\mu\neq 0$ and $\overline{\lambda}+\mu=1$. Also according to since $1-\overline{\lambda}=(1-\lambda)^{i}$ we have $\overline{\mu}=\overline{1-\overline{\lambda}}=\overline{(1-\lambda)^{i}}=1-\lambda$. Since, $1-\lambda$ is fuzzy regular closed set. Therefore $\lambda+\overline{\mu}=1$. So (v) violated.

Conversely, if X has non – zero fuzzy open set λ and μ such that $\overline{\lambda}+\mu=\lambda+\overline{\mu}=1$, then

$$\overline{\lambda}^{i}=(1-\mu)^{i}=1-\overline{\mu}=\lambda$$

Since $\mu\neq 0$ and $\overline{\lambda}+\mu=1$, $\lambda\neq 1$. Also $\lambda\neq 0$ is given. Therefore λ is a proper fuzzy regular open set. Therefore X cannot be fuzzy super – connected.

$(v)\Leftrightarrow(vi)$. $(v)\Rightarrow(vi)$ Follows if we take $f=1-\lambda$ and $k=1-\mu$. Reverse implication can be proved similarly.

Theorem 3.2: An fts X is fuzzy super – connected iff it has no proper fuzzy open set which is also fuzzy semi – closed or equivalent iff it has no proper fuzzy closed set which is also fuzzy semi – open.

Proof: This follows immediately from the definition of fuzzy regular open sets, fuzzy semi – open sets and fuzzy semi – closed sets.

Theorem 3.3: If X and Y are fuzzy topological spaces and a function F from X onto Y is fuzzy continuous then X is fuzzy super – connected implies Y is fuzzy super – connected.

Proof: Deny. Then there exists a fuzzy open set $\lambda\neq 0$ in Y such that $\lambda\neq 1$. F is fuzzy continuous implies

$$\overline{F^{-1}(\lambda)}\leq F^{-1}(\lambda) \qquad \text{----------------(i)}$$

(see Theorem 4.2 of Warren [34]). Since $\lambda\neq 0$ and $\lambda\neq 1$, then there exist $y_1,y_2\in Y$ such that $\lambda(y_1)\neq 0$ and $\lambda(y_2)\neq 1$. Now F is onto. Therefore there exist $x_1,x_2\in X$ such that

$$F(x_1)=y_1 \text{ and } F(x_2)=y_2.$$

So $F^{-1}(\lambda)(x_1) = \lambda(F(x_1)) = \lambda(y_1) \neq 0$. Similarly $F^{-1}(\lambda)(x_2) \neq 0$. So by (i), $F^{-1}(\lambda)$ is a nonzero fuzzy open set in X such that $F^{-1}(\lambda) \neq 1$. This is a contradiction as X is fuzzy supper – connected.

Theorem 3.4: A finite product of fuzzy super – connected spaces is fuzzy super – connected.

Proof: Let (X,s) and (Y,t) be fuzzy super – connected topological spaces. Suppose that $(X \times Y, s \times t)$ is not fuzzy supper – connected. Then there exist $\lambda, \mu \in s$ and $\xi, \eta \in t$ such that $\lambda \times \xi \neq 0$, $\mu \times \eta \neq 0$ and

$$\lambda \times \xi(x,y) + \mu \times \eta(x,y) \leq 1$$

for every $(x,y) \in X \times Y$ where $(\lambda \times \xi, \mu \times \eta) \in s \times t$, $\lambda \times \xi = P_x^{-1}(\lambda) \cap P_y^{-1}(\xi)$, P_x is a projection map of $(X \times Y)$ onto X, etc. So $\min\{\lambda(x), \xi(y)\} + \min\{\mu(x), \eta(y)\} \leq 1$ for every $(x,y) \in X \times Y$ which implies that for any $(x,y) \in X \times Y$ either

(i) $\lambda(x) + \mu(x) \leq 1$

or

(ii) $\lambda(x) + \eta(y) \leq 1$

or

(iii) $\xi(y) + \mu(x) \leq 1$

or

(iv) $\xi(y) + \eta(y) \leq 1$.

Now $\lambda \wedge \mu \in s$ and $\xi \wedge \eta \in t$.

As X and Y are fuzzy super – connected topological spaces, if $\lambda \wedge \mu \neq 0$, $\xi \wedge \eta \neq 0$, then there exist $x_1 \in X$ and $y_1 \in Y$ such that $(\lambda \wedge \mu)(x_1) > \frac{1}{2}$ and

$(\xi \wedge \eta)(y_1) > \frac{1}{2}$. So $\lambda(x_1) > \frac{1}{2}$, $\mu(x_1) > \frac{1}{2}$, $\xi(y_1) > \frac{1}{2}$ and $\eta(y_1) > \frac{1}{2}$. Therefore if $x = x_1$ and $y = y_1$, then there none of the above four possibilities will be true.

If $\lambda \wedge \mu = 0$, then for each $x \in X$ either $\lambda(x) = 0$ or $\mu(x) = 0$. So for every $x \in X$, $\lambda(x) + \mu(x) \leq 1$. Note that $\lambda, \mu \neq 0$ as $\lambda \times \xi \neq 0$, $\mu \times \eta \neq 0$ which implies that (X, s) is not fuzzy super – connected. Similarly $\xi \wedge \eta = 0$ will imply that (Y, t) is not fuzzy super – connected.

Theorem 3.5: Any fuzzy product of fuzzy super – connected spaces is fuzzy super – connected.

Proof: Let $(X_i, t_i)_{i \in I}$ be a family of fuzzy super – connected spaces. Now it will be enough to show that at some point of X, the addition of any two non – zero basic fuzzy sets of (X, t), where $X = \prod_{i \in I} X_i$, exceeds 1.

Suppose not, Then there exist $i_1, i_2, i_3, \ldots\ldots, i_m, j_1, j_2, j_3, \ldots\ldots, j_m \in I$ and $\lambda_{i_k} \in t_{i_k}$, fo $k = 1,2,3,\ldots\ldots, m$ and $\mu_{k_{k'}} \in t_{j_{k'}}$, for $k' = 1,2,3,\ldots\ldots, n$ such that

$$\min\{\lambda_{i_1}(x_{i_1}), \ldots\ldots, \lambda_{i_m}(x_{i_m})\} + \min\{\mu_{j_1}(x_{j_1}), \ldots\ldots, \mu_{j_n}(x_{j_n})\} \leq 1 \quad \text{---------} \ (i)$$

for every $x_{i_k} \in X_{i_k}$, $k = 1,2,3,\ldots\ldots, m$ and for every $x_{j_{k'}} \in X_{j_{k'}}$, $k' = 1,2,3,\ldots\ldots, n$.

Case – I: If $\{i_1, i_2, i_3, \ldots\ldots, i_m\} \cap \{j_1, j_2, j_3, \ldots\ldots, j_m\} = \Phi$ then since each $X_i (i \in I)$ is fuzzy super – connected, then there exist $x^0{}_{i_k} \in X_{i_k}$, $k = 1,2,3,\ldots\ldots, m$ and $x^0{}_{j_{k'}} \in X_{j_{k'}}$, $k' = 1,2,3,\ldots\ldots, n$, such that

$$\lambda_{i_k}(x^0{}_{i_k}) > \frac{1}{2} \text{ and } \mu_{j_{k'}}(x^0{}_{j_{k'}}) > \frac{1}{2}$$

for $k = 1,2,3,\ldots\ldots, m$ and $x^0{}_{j_{k'}} \in X_{j_{k'}}$, $k' = 1,2,3,\ldots\ldots, n$. So (i) is not true for some $x_{i_k} \in X_{i_k}$ and $x_{j_{k'}} \in X_{j_{k'}}$, $k = 1,2,3,\ldots\ldots, m$ and $k' = 1,2,3,\ldots\ldots, n$.

Case – II: Let $p \in \{i_1, i_2, i_3, ...i_m\} \cap \{j_1, j_2, j_3, ...j_n\}$ .Now if $\min\{\lambda_{i_1}(x_{i_1}), ..., \lambda_{i_m}(x_{i_m})$ and $\min\{\mu_{j_1}(x_{j_1}), ..., \mu_{j_n}(x_{j_n})\}$ have different subscripts then by argument of Case – I we can prove that (i) does not hold but if the two minimum terms have the same subscript p then (i) becomes

$$\lambda_p(x_p) + \mu_p(x_p) \le 1 \text{ for all } x_p \in X_p.$$

Then $\lambda_p \wedge \mu_p \in t_p$ (where t_p is fuzzy topology on X_p) and as argued in Theorem 3.4.

$$\lambda_p(x_p) + \mu_p(x_p) \nleq 1 \text{ for all } x_p \in X_p.$$

4. Fuzzy Super – Connected Subspace:

Definition 4.1: A subset of a fuzzy topological space X is said to be a fuzzy super – connected subset of X if it is a fuzzy super – connected topological space as a fuzzy sub – space of X.

Theorem 4.1: If $A \subset Y \subset X$, then A is a fuzzy super – connected subset of X iff it is a fuzzy super – connected subset of the fuzzy subspace Y of X.

Proof: Easy.

Theorem 4.2: Let A be a fuzzy super – connected subset of an fts X. If there exist fuzzy closed sets f and k in X such that $f^i + k = f + k^i = 1$, then $f / A = 1$ and $k / A = 1$.

Proof: If $f(x_0) \neq 1$ and $k(y_0) \neq 1$ for some $x_0, y_0 \in A$, then

$$f^i(y_0) + k(y_0) = 1 \text{ and } f(x_0) + k^i(x_0) = 1.$$

This imply that $f^i(y_0) \neq 0$ and $k^i(x_0) \neq 0$. Thus f^i / A and k^i / A are not non – zero fuzzy open sets in A such that $f^i / A + k^i / A \le 1$, which contradicts the fact that A is a fuzzy super – connected subset of X.

Theorem 4.3: Let X be an fts and $A \subset X$ be a fuzzy super – connected subset of X such that μ_A is a fuzzy open set in X. If λ is a fuzzy regular open set in X, either $\mu_A \leq \lambda$ or $\mu_A \leq 1-\lambda$.

Proof: If $\lambda = 0$ or 1 then the result holds. Suppose that $\lambda \neq 0$ and $\lambda \neq 1$. Let $f = \overline{\lambda}$ an $k = 1-\lambda$. Then f and k are such that $f^i + k = f + k^i = 1$. By the previous theorem $\mu_A \leq f$ or $\mu_A \leq k$. So, $\mu_A \leq f^i$ or $\mu_A \leq k^i$, as μ_A is fuzzy open. Therefore $\mu_A \leq \overline{\lambda}^i = \lambda$ or $\mu_A \leq (1-\lambda)^i \leq \overline{(1-\lambda)^i} = 1-\lambda$.

Theorem 4.4: Let $\{O_\alpha\}_{\alpha \in A}$ be a family of subsets of an fts X such that each μ_{O_α} is fuzzy open set. If $\bigcap_{\alpha \in A} O_\alpha \neq \Phi$ and each O_α is a fuzzy super - connected subset of X, then $\bigcup_{\alpha \in A} O_\alpha$ is also a fuzzy super - connected subset of X.

Proof: Let $Y = \bigcup_{\alpha \in A} O_\alpha$ and suppose that Y is not a fuzzy super - connected subset of X. Then there exists a proper fuzzy regular open set λ_Y in the fuzzy subspace Y of X.

Each μ_0, is fuzzy open in X. So, each μ_{0_α}/Y is fuzzy open in Y. Also each O_α is a fuzzy super - connected subset of the subspace Y as it is so in X. Therefore by previous result for each $\alpha \in A$ either $\mu_{0_\alpha}/Y \leq \lambda_Y$ or $\mu_{0_\alpha}/Y \leq 1-\lambda_Y$. Suppose $x_0 \in \bigcap_{\alpha \in A} O_\alpha$. Then either $\lambda_Y(x_0) = 1$ or $\lambda_Y(x_0) = 0$. If $\lambda_Y(x_0) = 1$, then $\mu_{0_\alpha}/Y \leq \lambda_Y$ for every $\alpha \in A$. Hence $\mu_Y/Y = \vee_{\alpha \in A}(\mu_{0_\alpha}/Y) \leq \lambda_Y$. But $\lambda_Y \leq \mu_Y/Y$. So, $\lambda_Y = 1$, which is prohibited since $\lambda_Y \neq 1$. By a similar argument, if $\lambda_Y(x_0) = 0$ then we shall get $\lambda_Y = 0$, which is also a contradiction.

Theorem 4.5: If A and B are fuzzy super - connected subsets of an fts X and μ'_B / A or $\mu'_A / A \neq 0$, then $A \cup B$ is a fuzzy super - connected subset of X.

Proof: Suppose that $Y = A \cup B$ is not a fuzzy super - connected subset of X. Then there exist fuzzy open sets λ and δ such that $\lambda / Y \neq 0, \delta / Y \neq 0$, and $\lambda / Y + \delta / Y \leq 1$. Since A is a fuzzy super - connected subset of X either $\lambda / A = 0$ or $\delta / A = 0$. Without loss of generality assume that $\delta / A = 0$. In that case since B is also fuzzy super - connected, we have (i) $\lambda / A \neq 0$, (ii) $\delta / B \neq 0$, (iii) $\delta / A = 0$ and (iv) $\lambda / b = 0$. Therefore

$$\lambda / A + \mu'_B / A \leq 1 \qquad (\text{Because } \lambda / B = 0)$$

If $\mu'_B / A \neq 0$ then (i) and (v) imply that A is not a fuzzy super - connected subset of X. Similarly if $\mu'_A / B \neq 0$, then (ii) and $\delta / B + \mu'_A / B \leq 1$ imply that B is not a fuzy super - connected subset of X. We thus get a contradiction.

Theorem 4.6: If $\{A_\alpha\}_{\alpha \in A}$ is a family of fuzzy super - connected subsets of an fts X such that $\left[\wedge_{\alpha \in A} \mu_{A_\alpha}\right] \neq 0$. Then $\bigcup_{\alpha \in A} A_\alpha$ is a fuzzy super - connected subset of X.

Proof: Suppose $Y = \{A_\alpha\}_{\alpha \in A}$ is not a fuzzy super - connected subset of X. Then there exist fuzzy open sets λ and δ in X such that

$$\lambda / Y \neq 0 \text{ ---------------------------------- (i)}$$

$$\delta / Y \neq 0 \text{ ---------------------------------- (ii)}$$

and

$$\lambda / Y + \delta / Y \leq 1 \text{ ------------------------- (iii)}$$

Equations (i) and (ii) imply that there exist β and γ in A such that $\lambda / A_\beta \neq 0$ and $\delta / A_\gamma \neq 0$.

Case - I: If $\beta = \gamma$, then A_β will not be a fuzzy super - connected subset of X, which is prohibited..

Case - II: If $\beta \neq \gamma$, then

$$0 \neq \left[\wedge_{\alpha \in A} \mu_{A_\alpha}\right]^i \leq \wedge_{\alpha \in A} \mu^i{}_{A_\alpha}$$

implies that $\mu^i{}_{A_\beta} \wedge \mu^i{}_{A_\gamma} \neq 0$. So $\mu^i{}_{A_\beta} / A_\gamma \neq 0$. Hence by the previous theorem $A_\beta \cup A_\gamma$ is a fuzzy super - connected subset of X. On the contrary it can be seen that $\lambda / A_\beta \cup A_\gamma \neq 0$, $\delta / A_\beta \cup A_\gamma \neq 0$ and $\lambda / A_\beta \cup A_\gamma + \delta / A_\beta \cup A_\gamma \leq 1$. So $A_\beta \cup A_\gamma$ is not a fuzzy super - connected subset of X.

Theorem 4.6: Suppose an fts X is fuzzy super - connected and C is a fuzzy super - connected subset of X. Further suppose that X - C contains a set V such that $\mu_V / X - C$ is a fuzzy open set in the fuzzy subspace X - C of X. Then $C \cup V$ is a fuzzy super - connected subset of X.

Proof: Suppose $Y = C \cup V$ is not a fuzzy super - connected subset of X. Then there exist fuzzy open sets λ and δ in A such that $\lambda / Y \neq 0$, $\delta / Y \neq 0$ and $\lambda / Y + \delta / Y \leq 1$. As C is a fuzzy super - connected subset of X, either $\lambda / C = 0$ or $\delta / C = 0$.

Without loss of generality assume that $\lambda / C = 0$. Therefore

$$\lambda / V \neq 0 \quad \text{-------------------------------} \quad \text{(i)}$$

If we define a fuzzy set λ_V in X as $\lambda_V(c) = \lambda(x)$ if $x \in V$, $\lambda_V(x) = 0$ if $x \in X - V$, then λ_V is open in X as $\lambda_V = \lambda \wedge \mu_V$. So $\overline{\lambda}_V$ is a fuzzy regular closed set in X. Now we show that $\overline{\lambda}_V$ is a proper fuzzy set in X. $\lambda / Y + \delta / Y \leq 1$ implies $\lambda_V + \delta \leq 1$. So $\overline{\lambda}_V + \delta \leq 1$. Therefore $\lambda_V \neq 1$ as $\delta \neq 0$. Also if $\overline{\lambda}_V = 0$, then $\lambda_V = 0$, so $\lambda / V = 0$, but by (i), $\lambda / V \neq 0$. Thus X is not a fuzzy super connected space, which is a contradiction.

Theorem 4.7: If A and B are subsets of an fts X such that $\mu_A \leq \mu_B \leq \overline{\mu}_A$ and A is a fuzzy super - connected subset of X then so is B.

Proof: Easy.

5. Fuzzy Strong Connectedness:

Definition: An fts X is said to be fuzzy strongly connected if it has no non - zero fuzzy closed sets f and k such that $f+k \le 1$. If X is not fuzzy strongly connected then it will be called fuzzy weakly disconnected.

Theorem 5.1: An fts X is fuzzy strongly connected iff it has no (non - zero) fuzzy open sets λ and δ such that $\lambda \ne 1, \delta \ne 1$ and $\lambda + \delta \ge 1$.

Proof: An fts X is fuzzy weakly disconnected

$\Leftrightarrow$ if it has no non - zero fuzzy closed sets f and k such that $f+k \le 1$.

$\Leftrightarrow$ if it has no non - zero fuzzy open sets $\lambda = f'$ and $\delta = k'$ such that $\lambda \ne 1, \delta \ne 1$ and $\lambda + \delta \ge 1$.

Remarks 5.1: Fuzzy strong connectedness implies fuzzy connectedness, however the converse is not true. Also the following example shows that fuzzy strong connectedness and fuzzy super - connectedness are unrelated.

Example 5.1: If $X=[0,1]$ and for $0 \le x \le 1$, $\lambda(x)=\frac{2}{3}$, $t_1=\{0,1,\lambda\}$ and $t_2=\{0,1,\lambda'\}$ then (X,t_1) is fuzzy connected, fuzzy super - connected, but not fuzzy strongly connected and (X,t_2) is fuzzy strongly connected but not fuzzy super - connected.

Theorem 5.2: If $A \subset X$ and X is an fts A is a fuzzy strongly connected subset of X iff for any fuzzy open sets λ and δ in X, $\mu_A \le \lambda + \delta$ implies either $\lambda \le \mu_A$ or $\mu_A \le \delta$.

Proof: If A is not a fuzzy strongly connected subset of X, then there exist fuzzy closed sets f and k in X such that

(i) $f/A \ne 0$

(ii) $k/A \ne 0$ and

(iii) $f/A + k/A \le 1$.

If we put $\lambda = 1 - f$ and $\delta = 1 - k$, then $\lambda / A = 1 - f / A$, $\delta / A = 1 - k / A$. So (i), (ii) and (iii) imply that $\mu_A \leq \lambda + \delta$ but $\mu_A \nleq \lambda$ and $\mu_A \nleq \delta$.

Conversely if there exist fuzzy open sets λ and δ in X, $\mu_A \leq \lambda + \delta$, but $\mu_A \nleq \lambda$ and $\mu_A \nleq \delta$, then $\lambda / A \neq 1, \delta / A \neq 1$ and $\lambda / A + \delta / A \geq 1$. So A is not fuzzy strongly connected.

Theorem 5.3: If F is a subset of an fts X such that μ_F is fuzzy closed in X, then X is fuzzy strongly connected implies that F is a fuzzy strongly connected subset of X.

Proof: Suppose that F is not so. Then there exist fuzzy closed sets f and k in X such that

(i) $f / A \neq 0$

(ii) $k / A \neq 0$

(iii) $f / A + k / A \leq 1$.

(iii) implies that $(f \wedge \mu_F) + (k \wedge \mu_F) \leq 1$, where by (i) and (ii) $(f \wedge \mu_F) \neq 0$, $(k \wedge \mu_F) \neq 0$. So X is not fuzzy strongly connected, which is a contradiction.

Remarks 5.2: One can prove a similar result for a subset G of a fuzzy super - connected space X if μ_G is fuzzy open in X.

Theorem 5.4: If X and Y are fuzzy topological spaces and a function F from X onto Y is fuzzy continuous, then X is fuzzy strongly connected implies Y is fuzzy strongly connected.

Hint: Inverse image of a fuzzy closed set in Y under F is a fuzzy closed set in X.

Theorem 5.5: A finite product of fuzzy strongly connected spaces is fuzzy strongly connected.

Proof: Let $(X,t(X))$ and $(Y,t(Y))$ be fuzzy strongly connected topological spaces. We show that $(X\times Y,t(X\times Y))$ is fuzzy strongly connected. suppose not. Since the members of $t(X\times Y)$ are precisely of the type $\lambda\times\delta$, where $\lambda(x)\in t(X)$, $\delta(x)\in t(Y)$, there exist non - zero fuzzy sets $\lambda,\mu\in t(X)$ and $\delta,\eta\in t(Y)$ such that $\lambda\times\delta\neq 1$, $\mu\times\eta\neq 1$ and for every $x\in X$, $y\in Y$

$$\min\{\lambda(x),\delta(y)\}+\min\{\mu(x),\ \eta(y)\}\geq 1 \text{ --------------- (i)}$$

$\lambda\vee\mu\in t(X)$ and $\delta\vee\eta\in t(Y)$. X and Y are fuzzy super connected, so if $\lambda\vee\mu\neq 1$ and $\delta\vee\eta\neq 1$ then there is $x_1\in X$ and $y_1\in Y$ such that $(\lambda\vee\mu)(x_1)<\frac{1}{2}$ and $(\delta\vee\eta)(y_1)<\frac{1}{2}$, which implies each $\lambda(x_1),\mu(x_1),\delta(y_1)$ and $\eta(y_1)<\frac{1}{2}$. So for $x=x_1$ and $y=y_1$, (i) does not hold. If $\lambda\vee\mu=1$ then for each $x\in X$,

$$\lambda(x)=1 \text{ and } \mu(x)=1 \text{ ---------------------------------- (ii)}$$

Now we show that $\lambda\neq 1$. Suppose $\lambda=1$. So $\lambda\times\delta\neq 1$ and Y is fuzzy strongly connected implies that there exists $y_0\in Y$ such that $\delta(y_0)<\frac{1}{2}$. Now $\mu\times\eta\neq 1$. So either $\mu\neq 1$ or $\eta\neq 1$.

Case - I: If $\mu\neq 1$, then as X is fuzzy strongly connected, there is $x_0\in X$ such that $\mu(x_0)<\frac{1}{2}$. So for $x=x_0$, $y=y_0$, (i) is not true.

Case - II: If $\eta\neq 1$, then since $\delta\neq 1$ and Y is fuzzy strongly connected there is $y_1\in Y$ such that $\delta(y_1)+\eta(y_1)<1$. So for any $x\in X$ and $y=y_1$

$$\min\{\lambda(x), \delta(y_1)\} + \{\mu(x), \eta(y_1)\} \leq 1$$

$$\delta(y_1) + \eta(y_1) < 1$$

This is a contradiction because of (i). Thus $\lambda = 1$ is not possible. Similarly, we can prove that $\mu \neq 1$. By (ii) $\lambda + \mu \geq 1$. So $(X, t(X))$ is not strongly connected, which is a contradiction. Therefore $\lambda \vee \mu = 1$ is not possible. Similarly we can show that $\delta \vee \eta = 1$ is not possible.

Remarks 5.3: An infinite fuzzy product of fuzzy strongly connected spaces may not be fuzzy strongly connected.

Example 5.2: Let $X_n = [0,1]$, $n = 1, 2, 3, -----$ such that

$$\lambda_n(x) = \frac{n}{2(n+1)}, \qquad \text{if } x \in X,$$

$$t(X_n) = \{0, 1, \lambda_n\}, \qquad n = 1, 2, 3, -----$$

Then each $(X, t(X_n))$ is fuzzy strongly connected. But the fuzzy product space $X = \prod_{n=1}^{\infty} X_n$ is not so as $t(X)$ contains a member $\vee^{\infty}{}_{n=1} P^{-1}{}_n(\lambda_n) \neq 1$ such that $(\vee^{\infty}{}_{n=1} P^{-1}{}_n(\lambda_n))(x) = \frac{1}{2}$ where $x \in X$.

Remarks 5.4: In the general topology any product of strongly connected spaces is strongly connected. thus in the fuzzy setting we have a divergence.

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