2011

On Supra Fuzzy Topological Spaces

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On Supra Fuzzy Topological Spaces

A Thesis Submitted to the Department of Mathematics, University of Rajshahi, Rajshahi – 6205, Bangladesh, for the Degree of Master of Philosophy in Mathematics

By

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June, 2011

Dedicated

To

My Beloved

Parents

STATEMENT OF ORIGINALITY

I declare that the contents in my M. Phill. Thesis entitled **"On Supra Fuzzy Topological Spaces"** is original and accurate to the best of my knowledge. I also declare that the materials contained in my research work have not been previously published or written any person for degree or diploma.

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CERTIFICATE

I have the pleasure in certifying that the M. Phill. Thesis entitled **"On Supra Fuzzy Topological Spaces"** submitted by Md. Fazlul Hoque in fulfillment of the requirement for the degree of M. Phill. in Mathematics, University of Rajshahi, Rajshahi-6205, Bangladesh has been completed under my supervision. I believe that the research work is an original one and it has not been submitted elsewhere for any degree.

I wish him a bright future and every success in life.

Supervisor

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ACKNOWLEDGEMENT

Firstly, I give my limitless thanks to the Almighty Allah for giving me strength, endurance and ability to complete this course of study.

I would like to convey an immense grateful to a number of individuals for their kind advices, directions, suggestions and co-operation, which have enabled me to complete my thesis successfully.

I am bound to express my deepest respect and thanks to my respected supervisor Professor Dr. Dewan Muslim Ali, Department of Mathematics, University of Rajshahi, Bangladesh for his day–to-day monitoring, supervision and maximum assistance for the completion of the present study successfully.

I wish to thank the Chairman, Deparetment of Mathematics, Rajshahi University, for providing necessary laboratory and other facilities during the course of thesis work.

I am thankful to Md. Sahadat Hossain, Assistant Professor, Department of Mathematics, Rajshahi University, for providing me the source of information and materials about my study. I also wish to express my sincere admiration, appreciation and gratitude to all the teachers of the Department of Mathematics, University of Rajshahi for providing me with invaluable suggestions during the period of my research work.

A special thank will go to the shoulders of my colleagues and friends at Pabna Science and Technology University, Pabna, for their friendly support to overcome the hurdles during the M. Phill. program.

Finally, I am ever grateful to my parents for their moral and constant blessing, good advice and encouragement. I would like to thank all of my family members and well wishers for their inspiration.

I fine, I am alone responsible for the errors and shortcomings in this study if there be any, I am sorry for that.

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CONTENTS

Introduction

The fundamental concept of a fuzzy set was introduced by L. A. Zadeh[111] in 1965 to provide a foundation for the development of many areas of knowledge. Consequently, this provides a natural frame work for generalizing many algebraic and topological concepts in various directions such as fuzzy groups, fuzzy rings, fuzzy vector spaces, fuzzy supra topology, fuzzy infra topology, fuzzy bitopology etc. many other branches of mathematics have been developed all over the world during the last five decades. In 1968, Chang [19] introduced the concepts of a fuzzy topological space by using the fuzzy set. Wong [105], Lowen[60], Hutton[48], Katsaras[52], Ali[3], Pu and Liu[72], etc., discussed various aspects of fuzzy topological spaces. Ying [74] introduced fuzzifying topology and developed this in a new direction with the semantic methods of continuous valued logic. In the frame work of fuzzifying topology, Sinha[93] introduced and studied T_0-, T_1-, T_2(Hausdorff)-, T_3(regular)-, T_4(normal)-, separation axioms. A.S. Mashhour et al.[64] introduced and studied the concepts of the family of fuzzifying semi-open sets, fuzzifying neighbourhood structure of a point and fuzzifying semi-closure of a fuzzy set. A.S. Mashhour et al.[64] also introduced and studied the R_0 and R_1 separation axioms and studied their relations with the T_1 and T_2- separation axioms respectively. Also in fuzzifying topology they introduced and studied semi-T_0-, semi-R_0-, semi-T_1-, semi-R_1-, semi-T_2(semi Hausdorff)-, semi-T_3(semi regular)-, semi-T_4(semi normal)-, separation axioms. In 1983, A.S. Mashhour et al.[64] introduced supra topological spaces and studies s-continuous functions and s^* - continuous functions. In 1987, M.E. Abd EL-Monsef et al. [1] introduced the fuzzy supra topological spaces and studied fuzzy supra continuous functions and characterized a number of basic concepts.

The purpose of this thesis is to introduce some new definitions of separation axioms in supra fuzzy topological spaces using the ideas of Ali [8]. Some of their equivalent formulations along with various new characterizations and results concerning the existing ones are presented here. Our criterion for definitions has been preserving as much as possible the relation between the corresponding separation properties for supra fuzzy topological spaces. Moreover, it will be seen that the definitions of these axioms are 'good extensions' in the sense of Lowen [60].

We aim to develop theories of supra fuzzy T_0, supra fuzzy T_1, supra fuzzy T_2(Hausdorff), supra fuzzy SFR(supra fuzzy regular), supra fuzzy SFN(supra fuzzy normal), supra fuzzy R_0- and supra fuzzy R_1- separation axioms analogous to its counter part in ordinary topology. The materials of this thesis have been divided into seven chapters, a brief scenario of which we present as follows.

Chapter one incorporates some of the basic definitions and results of fuzzy set, fuzzy topology, mapping, supra fuzzy topology and its mapping. These results are ready references for the work in the subsequent chapter. Results are stated without proof and can be seen in the papers referred to.

Our work starts from second chapter. In second chapter, we have introduced and studied T_0 properties in supra fuzzy topological spaces. Here we add four more definitions to this list and we have established relations among them. All these four definitions are 'good extensions' of the corresponding concept T_0 in a topological space. We prove that all the definitions are hereditary, productive and projective. Also we have studied some other properties of these concepts.

In third chapter, we have introduced and studied T_1 properties in supra fuzzy topological spaces. Here we add four more definitions to this list and we have established relations among them. All these four definitions are 'good extensions' of the corresponding concept T_1 in a topological space. We prove that all these concepts are hereditary, productive and projective. Also we have studied some other properties of these concepts.

We have introduced and studied T_2(Hausdorff) properties for supra fuzzy topological spaces, in chapter four. We have introduced here four more definitions and we established relations among them. All these properties are 'good extensions' of the corresponding concept T_2(Hausdorff) in a topological space. We have observed that all the definitions are hereditary, productive and projective. Also we have studied some other properties of these concepts.

In fifth chapter, we have discussed SFR (supra fuzzy regular) properties in supra fuzzy topological spaces. Here we have obtained four more definitions and we established relations among them. We see that all these properties are 'good extensions' of the corresponding concept T_3 (regularity) in topological spaces. We have discussed that all the properties are hereditary, productive and projective. We have also studied several other properties of these concepts.

In chapter six, we have introduced and studied SFN (supra fuzzy normal) properties in supra fuzzy topological spaces. We have given here three more definitions and we have established relations among them. All these are 'good extensions' of the corresponding concept T_4 (normality) in a topological space. Some other pleasant properties of these concepts have been studied here.

In seventh chapter, we aim to introduce and study R_0 and R_1 properties in supra fuzzy topological spaces. Four more definitions of each concept are given and obtained implications among them. All these are found to be 'good extensions' of the corresponding concepts R_0 and R_1 in a topological space. We have found that all the concepts are hereditary, productive and projective. Some other properties of these concepts are also found.

CHAPTER-1

Preliminaries

1.1 Introduction:

In this chapter, we collect several basic definitions and results of the Fuzzy sets, Grade of membership, complement of fuzzy sets, some laws of fuzzy sets, different mappings on fuzzy sets, Fuzzy topological spaces, Supra fuzzy topological spaces, distinction between Fuzzy sets and Supra fuzzy sets, T_0, T_1, T_2 –spaces, Fuzzy product topological spaces and Supra fuzzy product topological spaces which are to be used as ready references for understanding the subsequent chapters. Most of the results are quoted from various research papers. We make use the following general notations in this thesis.

Λ	:	Index set.
$I = [0, 1]$	:	Closed unit interval.
$I_1 = [0, 1)$	:	Right open unit interval.
$I_0 = (0, 1]$	:	Left open unit interval.
u , v , w,..	:	Supra Fuzzy sets.
(X, t)	:	Fuzzy topological space.
(X, t^*)	:	Supra Fuzzy topological space.
(X, T^*)	:	General supra topological space.
(X, T)	:	General topological space.
$\Pi_{i \in \Lambda} X_i$	:	Usual product of X_i .

$(X, t^*_1 \times t^*_2)$: Product of supra fuzzy topologies t^*_1 and t^*_2 on the set X.

$I_\alpha(t^*) = \{ u^{-1}(\alpha, 1] : u \in t^* \}, \alpha \in I_1$: General supra topology on X.

Definition 1.2[111]: For a set X, a function $u: X \to [0,1]$ is called a fuzzy set in X. For every $x \in X$, u(x) represents the grade of membership of x in the fuzzy set u. Some authors say that u is a fuzzy subset of X. Thus a usual subset of X, is a special type of a fuzzy set in which the range of the function is {0, 1}.

Definition 1.3[111]: Let X be a nonempty set and A be a subset of X. The function $1_A : X \to [0, 1]$ defined by $1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$ is called the characteristic function of A. We also write 1_x for the characteristic function of {x}.

Definition 1.4[111]: The characteristic functions of subsets of a set X are referred to as the crisp sets in X.

Example 1.5: Suppose X is real number R and the fuzzy set of real numbers much greater than 5 in X that could be defined by the continuous function $U: X \to [0,1]$ such that

$$u(x) = \begin{cases} 0 \ \text{if} \ x \le 5 \\ \dfrac{x-5}{50} \ \text{if} \ 5 < x < 55. \\ 1 \ \text{if} \ x \ge 55 \end{cases}$$

Definition 1.6[75]: A fuzzy subset is empty if and only if grade of membership is identically zero in X. It is denoted by 0.

Definition 1.7[75]: A fuzzy subset is whole if and only if its grade of membership is identically one in X. It is denoted by 1.

Definition 1.8[111]: Let u and v be two fuzzy subsets of a set X. Then u is said to be subset of v, i.e., $u \subseteq v$ if and only if $u(x) \leq v(x)$ for every $x \in X$.

Definition 1.9[111]: Let u and v be two fuzzy subsets of a set X. Then u is said to be equal to v, i.e., $u = v$ if and only if $u(x) = v(x)$ for every $x \in X$.

Definition 1.10[111]: Let u and v be two fuzzy subsets of a set X. Then u is said to be the complement of v, i.e., $u = v^c$ if and only if $u(x) = 1 - v(x)$, for every $x \in X$. Obviously, $(v^c)^c = v$.

Definition 1.11[19]: Let u and v be two fuzzy subsets of a set X. Then the union w of u and v, i.e., $w = u \cup v$ if and only if $w(x) = (u \cup v)(x) = \max\{ u(x), v(x) \}$, for every $x \in X$. The union w is a fuzzy subset of X.

In general, if Λ be an index set and $A = \{u_i : i \in \Lambda\}$ be a family of fuzzy sets of X then the union $\cup u_i$ is defined by $(\cup u_i)(x) = \sup\{u_i(x) : i \in \Lambda\}$, $x \in X$.

Definition 1.12[19]: Let u and v be two fuzzy subsets of a set X. Then the intersection m of u and v, i.e., $m = u \cap v$ is a fuzzy subset of X if and only if $m(x) = (u \cap v)(x) = \min\{u(x), v(x)\}$, $\forall x \in X$, and $(\cap u_i)(x) = \inf\{u_i(x) : i \in \Lambda\}$, $x \in X$, where $\{u_i, i \in \Lambda\}$.

Definition 1.13[3]: Let u and v be two fuzzy subsets of a set X. Then the difference of u and v is defined by $u - v = u \cap v^c$.

Definition 1.14[3]: If $\alpha \in I$ and $u \in I^X$ define by $u(x) = \alpha$, for all $x \in X$, we refer to u as a constant fuzzy set and denote it by α itself. In particular, we have the constant fuzzy sets 0 and 1.

Example 1.15: Let $X = \{x, y, z\}$ and $u, v \in I^X$ are defined by $u(x) = .6$, $u(y) = .7$, $u(z) = .5$ and $v(x) = .7$, $v(y) = .5$, $v(z) = .4$. Then $(u \cup v)(x) = \max\{u(x), v(x)\} = .7$, $(u \cup v)(y)$

= max {u(y), v(y)} =.7, $(u \cup v)$ (y) =max {u(y), v(y)} =.5, $(u \cap v)$ (x) =min {u(x), v(x)} =.6, $(u \cap v)$ (y) =min {u(y), v(y)} =.5, $(u \cap v)$ (z) =min {u(z), v(z)} =.4., u^c (x) = 1-u(x)=.4, u^c (y)= 1-u(y)=.3, u^c (z)= 1-u(z)=.5.

Laws of the algebra of fuzzy sets 1.16:

As in ordinary set theory, idempotent laws, associative law, commutative law, distributive laws, identity law, demorgan's laws hold in the case of fuzzy sets also. But the complement laws are not necessarily true. For example, if X= {a, b, c} and u is a fuzzy subset of X where is defined by

$u=\{(a,.2),(b,.7),(c,1)\}$,

then $u^c=\{(a,.8),(b,.3),(c,0)\}$

so $u \cup u^c=\{(a,.8),(b,.7),(c,1)\} \neq 1$,

$u \cap u^c=\{(a,.2),(b,.3),(c,0)\} \neq 0$.

Also in ordinary set theory $U \cap V=\phi$ if and only if $U \subset V^c$. But in fuzzy subsets reverse is not necessary true. For example if

$v=\{(a,.6),(b,.2),(c,0)\}$ then $u \subset v^c$,

$u \cap v=\{(a,.2),(b,.2),(c,0)\} \neq 0$.

Definition 1.17[19]: Let f be a mapping from a set X into a set Y and u is a fuzzy subset of X. Then f and u induced a fuzzy subset v=f (u) of Y whose membership function is defined by

$$v(y)=(f(u))(y)=\sup\{u(x)\} \text{ if } f^{-1}[\{y\}] \neq \phi,\ x \in X$$

$$=0, \text{ otherwise.}$$

Definition 1.18[19]: Let f be a mapping from a set X into Y and v be a fuzzy subset of Y. Then the inverse of v written as $u=f^{-1}(v)$ is a fuzzy subset of X and is defined by

$u(x) = (f^{-1}(v))(x) = v(f(x))$, for $x \in X$.

Example 1.19: Suppose that X= {x, y, z, w} and Y= {a, b, c}. Define $f : X \longrightarrow Y$ by $f(x)=b$, $f(y)=c$, $f(z)=a$, $f(w)=a$. Let $u \in I^X$ be given by $u(x) =.2$, $u(y) =.3$, $u(z) =.5$ and $u(w) =.4$. Then $(f(u))(a) = \sup\{u(z), u(w)\} =.5$. Similarly, $(f(u))(b) = \sup u(x) =.2$, and $(f(u))(c) = \sup u(y) =.3$.

On the other hand, if v is a fuzzy set in Y given by $v(a) =.6$, $v(b) =.8$, $v(c) =.7$. Then $(f^{-1}(v))(x) = v(f(x))=v(b)=.8$, $(f^{-1}(v))(y) = v(f(y))=v(c)=.7$, $(f^{-1}(v))(z) = v(f(z))=v(a)=.6$, $(f^{-1}(v))(w) = v(f(w))=v(a)=.6$.

We now mention some properties of fuzzy subsets induced by mappings.

Let f be a mapping from X into Y, u be a fuzzy subset of X and v be a fuzzy subset of Y. Then the following properties are true [19].

(a) $f^{-1}(v^c) = (f^{-1}(v))^c$ for any fuzzy subset v of Y.

(b) $f(u^c) = (f(u))^c$ for any fuzzy subset u of X.

(c) $v_1 \subset v_2 \Rightarrow f^{-1}(v_1) \subset f^{-1}(v_2)$, where v_1 and v_2 are two fuzzy subsets of Y.

(d) $u_1 \subset u_2 \Rightarrow f(u_1) \subset f(u_2)$, where u_1 and u_2 are two fuzzy subsets of X.

(e) $v \supset f(f^{-1}(v))$, for any fuzzy subset v of Y.

(f) $u \subset f^{-1}(f(u))$, for any fuzzy subset u of X.

(g) Let f be a function from X into Y and g be a function from Y into Z. Then $(g \circ f)^{-1}(w) = f^{-1}(g^{-1}(w))$, for any fuzzy subset w in Z, where $(g \circ f)$ is the composition of g and f.

Definition 1.20[72] : A fuzzy point in X is a special type of fuzzy set in X with membership function $p(x) = r$, $p(y) = 0$, $\forall\, y \neq x$, where $0 < r < 1$. This fuzzy point is said to have support x and value r and this point is denoted by x_r or ${}_r1_x$.

Definition 1.21[72] : A fuzzy point p is said to belong a fuzzy set u in X ($p \in u$) if and only if $p(x) < u(x)$ and $p(y) \leq u(y)$ $\forall$ $y \neq x$. Evidently, every fuzzy set u can be expressed as the union of all the fuzzy points which belong to u.

Definition 1.22[72] : Two fuzzy sets u and v in X are said to be intersected if and only if there exist a point $x \in X$ such that $(u \cap v)(x) \neq 0$. In this case we say that u and v intersect at x.

Definition 1.23[3]: Let X be a non empty set and u be a fuzzy set in X. A α- cut of u is defined by $\alpha_u = \{x : u(x) \geq \alpha, \alpha \in I\}$.

Definition 1.24[3]: Let X be a non empty set and u be a fuzzy set in X. A strong α- cut of u is defined by $\alpha_{+u} = \{x : u(x) > \alpha, \alpha \in I\}$. We see that α-cut and strong α-cut are crisp subsets of X.

Definition 1.25[3]: Let X be a non empty set and u be a fuzzy set in X. The support of u in X is the crisp subset of X that contains all the elements of X that have non-zero membership grads in u, i.e., $\operatorname{supp} u = \{x : u(x) > 0\}$. The 1-cut is called the core of u.

Definition 1.26[3]: The height h(u) of a fuzzy set u is the largest membership grade obtained by any element in that set, i.e., $h(u) = \frac{\sup}{x \in X} u(x)$.

Definition 1.27[3]: For a finite fuzzy set, the cardinality $|\alpha|$ defined as $|\alpha| = \sum_{x \in X} \alpha(x)$.

$\|\alpha\| = \frac{|\alpha|}{|x|}$ is called the relative cardinality of α.

Definition 1.28[3]: A fuzzy set u is called normal when h(u) = 1; it is called subnormal when h(u) < 1. The height of u may also be viewed as the supremum of α for which $\alpha_u \neq \phi$.

Definition 1.29[19]: Let X be a non empty set and t be the collection of fuzzy sets in X. Then t is called a fuzzy topology on X if it satisfies the following conditions:

(i) $1, 0 \in t$,

(ii) If $u_i \in t$ for each $i \in \Lambda$, then $\cup_{i \in \Lambda} u_i \in t$.

(iii) If $u_1, u_2 \in t$ then $u_1 \cap u_2 \in t$.

If t is a fuzzy topology on X, then the pair (X , t) is called a fuzzy topological space (fts, in short) and members of t are called t- open (or simply open) fuzzy sets . If u is open fuzzy set, then the fuzzy sets of the form 1-u are called t- closed (or simply closed) fuzzy sets.

Definition 1.30[60]: Let X be a nonempty set and t be a collection of fuzzy sets in X such that

(i) $1, 0 \in t$,

(ii) If $u_i \in t$ for each $i \in \Lambda$, then $\cup_{i \in \Lambda} u_i \in t$.

(iii) If $u_1, u_2 \in t$ then $u_1 \cap u_2 \in t$.

(iv) all constant fuzzy sets in X belong to t.

Then t is called a fuzzy topology on X.

Example 1.31: Let $X = \{a, b, c, d\}$, $t = \{0, 1, u, v\}$,

where $1 = \{(a, 1), (b, 1), (c, 1), (d, 1)\}$

$0 = \{(a, 0), (b, 0), (c, 0), (d, 0)\}$

$u = \{(a, .2), (b, .5), (c, .7), (d, .9)\}$

$v = \{(a, .3), (b, .5), (c, .8), (d, .95)\}$

Then (X, t) is a fuzzy topological space.

Definition 1.32[64]: Let X be a nonempty set. A subfamily t^* of X is said to be a supra topology on X if and only if

(i) $1, 0 \in t^*$,

(ii) If $u_i \in t^*$ for each $i \in \Lambda$, then $\cup_{i \in \Lambda} u_i \in t^*$.

Then the pair (X, t^*) is called a supra fuzzy topological spaces. The elements of t^* are called supra open sets in (X, t^*) and complement of supra open set is called supra closed set.

Example 1.33: Let X={x, y} and $u, v \in I^X$ are defined by u(x) =.8, u(y) =.6 and v(x) =.6, v(y) =.8. Then we have $w(x) = (u \cup v)(x) = .8$, $w(y) = (u \cup v)(y) = .8$ and $k(x) = (u \cap v)(x) = .6$, $k(y) = (u \cap v)(y) = .6$. If we consider t^* on X generated by {0, u, v, w, 1}, then t^* is supra fuzzy topology on X but t^* is not fuzzy topology. Thus we see that every fuzzy topology is supra fuzzy topology but the converse is not always true.

Definition 1.34[72]: Let u be a fuzzy set in (X, t). The interior of u is defined as the union of all t – open sets contained in u. It is denoted by u^{o}. Evidently u^{o} is the largest open fuzzy set contained in u and $(u^{o})^{o} = u^{o}$.

Definition 1.35[72]: The intersection of all the t – closed set containing u is called the closure of u denoted by $\bar{u}$. Obviously $\bar{u}$ is the smallest closed set containing u and $\bar{\bar{u}} = \bar{u}$.

Definition 1.36[101] : A fuzzy set n in a fuzzy topological space (X, t) is called a neighborhood of a point $x \in X$, if and only if there exit $u \in t$ such that $u \subseteq n$ and u (x) = n (x) > 0.

Example 1.37: Let us consider the example 1.15, and n = { (a , .5) , (b , .6) , (c , .8), (d , .9) }. Now n is a neighborhood of $d \in X$. Since $u \in t$ such that $u \subseteq n$ and u (d) = n (d) > 0, n is a neighborhood of d. Similarly, n_1= { (a , .7) , (b , .6) , (c , .75) , (d , .9)} is a neighborhood of d . We denoted the family of all neighborhoods of x by N_x.

Definition 1.38[72]: A fuzzy set u in a fuzzy topological space (X, t) is called a neighborhood of a fuzzy point x_r if and only if there exist a fuzzy set $u_1 \in t$ such that $x_r \in u_1 \subseteq u$. A neighborhood u is called an open neighborhood if u is open. The family consisting of all the neighborhoods of x_r is called the system of x_r.

Definition 1.39[73]: Let (X, t) and (Y, s) be two topological spaces. Let t^* and s^* are associated supra topologies with t and s respectively. The function $f : (X, t^*) \longrightarrow (Y, s^*)$ is called supra fuzzy continuous if and only if for every $v \in s$, $f^{-1}(v) \in t^*$, the function f is called supra fuzzy homeomerphic if and only if f is bijective and both f and f^{-1} are supra fuzzy continuous.

Definition 1.40[64] : The function $f:(X,t^*) \longrightarrow (Y,s^*)$ is called supra fuzzy open if and only if for each open supra fuzzy set u in (X,t^*), f (u) is open supra fuzzy set in (Y,s^*).

Definition 1.41[64]: The function $f:(X,t^*) \longrightarrow (Y,s^*)$ is called supra fuzzy closed if and only if for each closed supra fuzzy set u in (X,t^*), f(u) is closed supra fuzzy set in (Y,s^*).

Proposition 1.42 ([73] Theorem 1.1): Let $f:(X,t^*) \longrightarrow (Y,s^*)$ be supra fuzzy continuous function, then the following properties hold :

(i) For every s^* – closed v, $f^{-1}(v)$ is t^* – closed.

(ii) For each supra fuzzy point p in X and each neighborhood u of f (u), then there exist a neighborhood v of p such that f (v) = u.

(iii) For any supra fuzzy set u in X, $f(u) \subset (f(u))$.

(iv) For any supra fuzzy set v in Y, $(f^{-1}(v)) \subset f^{-1}(v)$.

Proposition 1.43([62] Theorem 3.1): Let $f:(X,t^*) \longrightarrow (Y,s^*)$ be supra fuzzy open function , then the following properties hold:

(i) $f(u^o) \subseteq (f(u))^o$, for each supra fuzzy set u in X .

(ii) $(f^{-1}(v))^o \subseteq f^{-1}(v^o)$, for each supra fuzzy set v in Y.

Proposition 1.44([63] Theorem 1.5) : Let $f:(X,t^*) \longrightarrow (Y,s^*)$ be a function. Then f is closed if and only if $f(u) \subseteq f(u)$ for each supra fuzzy set u in X.

Definition 1.45[72]: Let (X,t^*) be a supra fuzzy topological space and A be an ordinary subset of X. The class $t^*_A = \{u / A : u \in t^*\}$ determines a supra fuzzy topology on A . This topology is called the subspace supra fuzzy topology on A.

Definition 1.46[106]: Let (X, t^*) be a supra fuzzy topological space. A subfamily B of t^* is a base for t^* if and only if each member of t^* can be express as the union of some members of B.

Definition 1.47[106]: Let (X, t^*) be a supra fuzzy topological space. A subfamily S of t^* is a sub- base for t^* if and only if the family of finite intersection of members of S forms a base for t^*.

Definition 1.48[111] : If u_1 and u_2 are two supra fuzzy subsets of X and Y respectively, then the Cartesian product $u_1 \times u_2$ of two supra fuzzy subsets u_1 and u_2 is a supra fuzzy subsets of $X \times Y$ defined by $(u_1 \times u_2)(x, y) = \min(u_1(x), u_2(y))$, for each pair $(x, y) \in X \times Y$.

Definition 1.49[105] : Let $\{X_i, i \in \Lambda\}$, be any class of sets and let X denoted the Cartesian product of these sets, i.e., $X = \Pi_{i\in\Lambda} X_i$. Note that X consists of all points $p = < a_i, i \in \Lambda >$, where $a_i \in X_i$. Recall that , for each $j_o \in \Lambda$, we define the projection π_{jo} from the product set X to the coordinate space X_{jo}. ie $\pi_{jo} : X \longrightarrow X_{jo}$

by $\pi_{jo}(< a_i : i \in \Lambda >) = a_{jo}$,

These projections are used to define the product supra topology.

Definition 1.50[106]: If (X_1, t^*_1) and (X_2, t^*_2) be two supra fuzzy topological space and $X = X_1 \times X_2$ be the usual product and t^* be the coarsest supra fuzzy topology on X, then each projection $\pi_i : X \longrightarrow X_i$, $i = 1, 2$ is supra fuzzy continuous. The pair (X, t^*) is called the product space of the supra fuzzy topological spaces (X_1, t^*_1) and (X_2, t^*_2).

Proposition 1.51([13] Theorem 3.6) : If u is a supra fuzzy subset of a supra fuzzy topological space (X, t^*_1) and v is a supra fuzzy subsets of a supra fuzzy topological space (Y, t^*_2), then $u \times v \subseteq u \times v$.

Definition 1.52[105]: Let $\{ X_\alpha \}_{\alpha \in \Lambda}$ be a family of nonempty sets. Let $X = \Pi_{\alpha \in \Lambda} X_\alpha$ be the usual product of X_α 's and let π_α be the projection from X into X_α. Further assume that each X_α is an supra fuzzy topological spaces with supra fuzzy topology t^*_α. Now the supra fuzzy topology generated by $\{ \pi_\alpha^{-1}(b_\alpha) : b_\alpha \in t^*_\alpha, \alpha \in \Lambda \}$ as a sub basis, is called the product supra fuzzy topology on X. Clearly if w is a basis element in the product, then there exist $\alpha_1, \alpha_2, \ldots \alpha_n \in \Lambda$ such that $w(x) = \min \{ b_\alpha (x_\alpha) : \alpha = 1, 2, 3, \ldots n \}$, where $x = (x_\alpha)_{\alpha \in \Lambda} \in X$.

Definition 1.53[111]: Let (X, T) be a topological spaces and T^* be associated supra topology with T. Then the function $f: X \to R$ is semi continuous if and only if $\{x \in X : f(x) > \alpha\}$ is open for all $\alpha \in R$.

Definition 1.54[60]: Let (X, T) be a topological spaces and T^* be associated supra topology with T. Then the lower semi continuous topology on X is $\omega (T^*) = \{ u \in I^X : u^{-1} (\alpha, 1] \in T^* \}$ for each $\alpha \in I_1$. It can be shown that $\omega (T^*)$ is a supra fuzzy topology on X.

Let P be the property of a topological space (X, T^*) and FP be its supra fuzzy topological analogue. Then FP is called a 'good extension' of P " iff the statement (X, T^*) has P iff $(X, \omega(T^*))$ has FP " holds good for every topological space (X, T^*).

Definition 1.55[3]: A fuzzy topological space (X, t) is said to be fuzzy T_0 if and only if

(a) for all distinct elements x, y $\in$X, there exists u$\in$ t such that u(x) =1, u(y) = 0 or u(x) = 0, u(y) =1;

(b) for all distinct elements x, y $\in$X, there exists u$\in$ t such that u(x) < u(y) or u(y) < u(x);

(c) for all distinct elements x, y $\in$X, $\overline{x_1} \cap \overline{y_1} < 1$.

Definition 1.56[3]: A fuzzy topological space (X, t) is said to be fuzzy T_1 if and only if

(a) for all distinct elements x, y $\in$X, there exist u, v$\in$ t such that u(x) =1, u(y) =0 and v(x) =0, u(y) =1;

(b) for all distinct elements x, y $\in$X, there exist u, v$\in$ t such that u(x) >0, u(y) =0 and v(x) =0, v(y) >0;

(c) for all distinct elements x, y $\in$X, there exist u, v$\in$ t such that u(x) > u(y) and v(y) > v(x).

Definition 1.57[3]: A fuzzy topological space (X, t) is said to be fuzzy Hausdorff or T_2 if and only if

(a) for all distinct elements x, y $\in$X, there exist u, v$\in$ t such that u(x) =1=u(y) and $u \cap v = 0$;

(b) for all pair of distinct fuzzy points x_r, y_s $\in$X, there exist u, v$\in$ t such that $x_r \in$u, $y_s \in$v and $u \cap v = 0$; (c) for all distinct elements x, y $\in$X, there exist u, v$\in$ t such that u(x) > 0, v(y) >0 and $u \cap v = 0$.

Definition 1.58[3]: A fuzzy topological space (X, t) is said to be fuzzy regular if and only if for each x $\in$ X and closed fuzzy set u with u(x) =0, there exists open fuzzy sets v, w $\in$t such that v(x)=1, $u \subseteq w$ and $v \subseteq 1-w$.

Definition 1.59[3]: A fuzzy topological space (X, t) is said to be fuzzy normal if and only if for each close fuzzy set m and open fuzzy set u with $m \subseteq u$, there exists a fuzzy set v such that $m \subseteq v^{\circ} \subseteq \bar{v} \subseteq u$.

CHAPTER-2

T_0 Spaces in Supra Fuzzy Topology

2.1 Introduction

Four concepts of T_0 supra fuzzy topological spaces are introduced and studied in this chapter. We also establish some relationships among them and study some other properties of these spaces.

Definition 2.2: Let (X, t) be a fuzzy topological space and t^* be associated supra topology with t and $\alpha \in I_1$. Then

(a) (X, t^*) is an α - T_0(i) space if and only if for all distinct elements $x, y \in X$, there exists $u \in t^*$ such that $u(x) = 1$, $u(y) \leq \alpha$ or there exists $v \in t^*$ such that $v(x) \leq \alpha$, $v(y) = 1$.

(b) (X, t^*) is an α - T_0(ii) space if and only if for all distinct elements $x, y \in X$, there exists $u \in t^*$ such that $u(x) = 0$, $u(y) > \alpha$ or there exists $v \in t^*$ such that $v(x) > \alpha$, $v(y) = 0$.

(c) (X, t^*) is an α - T_0(iii) space if and only if for all distinct elements $x, y \in X$, there exists $u \in t^*$ such that $0 \leq u(x) \leq \alpha < u(y) \leq 1$ or there exists $v \in t^*$ such that $0 \leq v(y) \leq \alpha < v(x) \leq 1$.

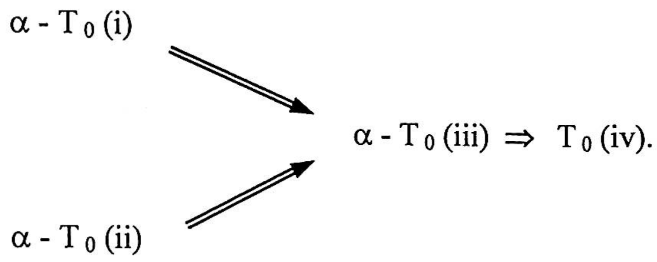
(d) (X, t^*) is a T_0(iv) space if and only if for all distinct elements $x, y \in X$, there exists $u \in t^*$ such that $u(x) \neq u(y)$.

Lemma 2.3: Suppose (X, t) is a topological space and t^* is associated supra topology with t and $\alpha \in I_1$. Then the following implications are true:

(a) (X, t^*) is α - T_0 (i) implies (X, t^*) is α - T_0 (iii) implies (X, t^*) is T_0 (iv).

(b) (X, t^*) is α - T_0 (ii) implies (X, t^*) is α - T_0 (iii) implies (X, t^*) is T_0 (iv).

Also, these can be shown in a diagram as follows:



Proof: Suppose (X, t^*) is α - T_0 (i). We have to prove that (X, t^*) is α - T_0 (iii). Let x and y be any two distinct elements in X. Since (X, t^*) is α - T_0 (i), for $\alpha \in I_1$, then by definition, there exists $u \in t^*$ such that $u(x) = 1$, $u(y) \le \alpha$ which shows that $0 \le u(y) \le \alpha < u(x) \le 1$. Hence by definition (c), (X, t^*) is α - T_0 (iii).

Suppose (X, t^*) is α - T_0 (iii). Then, for $x, y \in X$ with $x \neq y$, there exist $u \in t^*$ such that $0 \le u(x) \le \alpha < u(y) \le 1$, i.e., $u(x) \neq u(y)$, hence by definition, (X, t^*) is α - T_0(iv).

Let (X, t^*) is α - T_0 (ii). Then, for $x, y \in X$ with $x \neq y$, there exists $u \in t^*$ such that $u(x) = 0$ and $u(y) > \alpha$, which implies $0 \le u(x) \le \alpha < u(y) \le 1$. Hence, by definition, (X, t^*) is α - T_0 (iii) and hence (X, t^*) is α - T_0(iv).

Therefore, the proof is complete.

The non-implications among α - T_0 (i), α - T_0 (ii), α - T_0 (iii) and T_0 (iv) are shown in the following examples, i.e., The following examples show that

(a) T_0 (iv) does not imply α - T_0 (iii), so, not imply α - T_0 (i) and α - T_0 (ii).

(b) α - T_0 (iii) does not imply α - T_0 (i) and α - T_0 (ii).

(c) α - T_0 (i) does not imply α - T_0 (ii).

(d) α - T_0 (ii) does not imply α - T_0 (i).

Example 2.4: Let X = {x, y} and $u \in I^X$ is defined by u(x) = 0.4, u(y) = 0.7. Let the supra fuzzy topology t^* on X generated by { 0 , u , 1, Constants }. Then for $\alpha = 0.8$, we can easily show that (X, t^*) is T_0(iv) but (X, t^*) is not α - T_0(iii), so, not α - T_0 (i) and α - T_0 (ii).

Example 2.5: Let X = {x, y} and $u \in I^X$ be defined by u(x) = 0.5, u(y) = 0.9. Let the supra fuzzy topology t^* on X generated by {0, u, 1, Constants}. For $\alpha = 0.7$, we have $0 \le u(x) \le 0.7 < u(y) \le 1$. Thus according to the definition, (X, t^*) is α - T_0 (iii) but (X, t^*) is not α - T_0 (i). Also we can easily show that (X, t^*) is not α -T_0(ii).

Example 2.6: Let X = {x, y} and $u \in I^X$ be defined by u(x) = 1, u(y) = 0.5. Consider the supra fuzzy topology t^* on X generated by {0, u, 1, Constants}. For $\alpha = 0.7$, we have $u(x) = 1$ and $u(y) \le \alpha$. Thus according to the definition (X, t^*) is α -T_0(i) but (X, t^*) is not α -T_0 (ii).

Example 2.7: Let X = {x, y} and $u \in I^X$ be defined by u(x) = 0, u(y) = 0.8. Let the supra fuzzy supra topology t^* on X generated by {0, u, 1, Constants}. For $\alpha = 0.4$, it can easily show that (X , t^*) is α - T_0(ii) but (X , t*) is not α -T_0(i) .

This completes the proof.

Lemma 2.8: Let (X, t^*) be a supra fuzzy topological space and $\alpha, \beta \in t^*$ with $0 \le \alpha \le \beta < 1$, then

(a) (X, t^*) is α - T_0 (i) implies (X, t^*) is β -T_0 (i).

(b) (X, t^*) is β- T_0(ii) implies (X, t^*) is α -T_0 (ii).

(c) (X, t^*) is $0 - T_0$ (ii) if and only if (X, t^*) is $0 - T_0$ (iii).

Proof: Suppose that (X, t^*) is a supra fuzzy topological space and (X, t^*) is α - T_0 (i). We have to show that (X, t^*) is β- T_0(i). Let any two distinct elements $x, y \in X$. Since (X, t^*) is α - T_0(i) , for $\alpha \in I_1$, there is $u \in t^*$ such that $u(x) = 1$, and $u(y) \leq \alpha$. This implies that $u(x) = 1$, and $u(y) \leq \beta$, since $0 \leq \alpha \leq \beta < 1$. Hence by definition, (X, t^*) is β - T_0 (i).

Suppose that (X, t^*) is β - T_0 (ii). Then, for $x, y \in X$ with $x \neq y$, there exist $u \in t^*$ such that $u(x) = 0$ and $u(y) > \beta$, which implies $u(x) = 0$ and $u(y) > \alpha$, since $0 \leq \alpha \leq \beta < 1$. Hence we have (X, t^*) is α -T_0 (ii).

Example 2.9: Let $X = \{x, y\}$ and $u \in I^X$ be defined by $u(x) = 1$, $u(y) = 0.6$. Let the supra fuzzy topology t^* on X generated by {0, u, 1, Constants}. Then by definition, for $\alpha = 0.5$ and $\beta = 0.8$; (X, t^*) is β -T_0 (i) but (X , t^*) is not α -T_0 (i).

Example 2.10: Let $X = \{x, y\}$ and $u \in I^X$ be defined by $u(x) = 0$, $u(y) = 0.65$. Let the supra fuzzy topology t^* on X generated by {0, u, 1, Constants}. Then by definition, for $\alpha = 0.45$ and $\beta = 0.75$; (X , t^*) is α -T_0 (ii) but (X , t^*) is not β -T_0 (ii).

In the same way, we can prove that (X, t^*) is $0 - T_0$(ii) if and only if (X, t^*) is $0 - T_0$(iii).

Theorem 2.11: Let (X, T) be a topological space and T^* be associated supra topology with T and $\alpha \in I_1$. Suppose that the following statements:

(1) (X, T^*) be a T_0 – space.

(2) $(X, \omega(T^*))$ be an α- T_0(i) space.

(3) (X, ω (T^*)) be an α- T_0(ii) space.

(4) (X, ω (T^*)) be an α- T_0 (iii) space.

(5) (X, ω (T^*)) be a T_0 (iv) space.

Then the following implications are true:

(a) (1) ⇒(2) ⇒(4) ⇒(5) ⇒(1).

(b) (1) ⇒(3) ⇒(4) ⇒(5) ⇒(1).

Proof: Suppose (X, T^*) is a T_0 – topological space. We have to prove that (X, ω (T^*)) is α - T_0 (i) space. Suppose x and y are two distinct elements in X. Since (X, T^*) is T_0, there is U ∈ T^* such that x∈ U, y ∉ U. By the definition of lsc , we have $1_U \in \omega(T^*)$ and $1_U(x) = 1$, $1_U(y) = 0$. Hence we have (X, ω (T^*)) is α -T_0(i) space. Also we have (X, ω (T^*)) is α - T_0 (ii) space. Further, it is easy to show that (2) ⇒ (4), (3) ⇒ (4) and (4) ⇒ (5). We therefore prove that (4) ⇒ (1).

Suppose (X, ω (T^*)) be a T_0(iv) space. We have to prove that (X, T^*) is T_0 – space. Let x, y ∈ X with x ≠ y. Since (X , ω(T^*)) is T_0 (iv), there is u∈ ω(T^*) such that u(x) < u(y) or u(x) > u(y) . Suppose u(x) < u(y). Then for r ∈ I_1, such that u(x) < r < u(y). We observe that x ∉ u^{-1}(r , 1], y∈ u^{-1}(r , 1], and by definition of lsc, u^{-1}(r , 1] ∈ T^*. Hence (X, T^*) is T_0 – space.

Thus it is seen that α – T_0(p) is a good extension of its topological counter part (p = i , ii, iii, iv) .

Theorem 2.12: Let (X, t^*) be a supra fuzzy topological space, α ∈ I_1 and $I_\alpha(t^*) = \{u^{-1}(\alpha, 1] : u \in t^*\}$, then

(a) (X, t^*) is an α -T_0(i) implies $(X, I_\alpha(t^*))$ is T_0.

(b) (X, t^*) is an α -T_0(ii) implies $(X, I_\alpha(t^*))$ is T_0.

(c) (X, t^*) is an α -T_0(iii) if and only if $(X, I_\alpha(t^*))$ is T_0.

Proof: (a) Let (X, t^*) be a supra fuzzy topological space and (X, t^*) be α - T_0(i). Suppose x and y be any two distinct elements in X. Since (X, t^*) is α -T_0(i), for $\alpha \in I_1$, there exists $u \in t^*$ such that $u(x) = 1$, $u(y) \le \alpha$. Since $u^{-1}(\alpha, 1] \in I_\alpha(t^*)$, $y \notin u^{-1}(\alpha, 1]$ and $x \in u^{-1}(\alpha, 1]$, We have that $(X, I_\alpha(t^*))$ is T_0 – space .

(b) Let (X, t^*) be a supra fuzzy topological space and (X, t^*) be α - T_0(i). Suppose x and y be any two distinct elements in X. Since (X, t^*) is α -T_0(ii), for $\alpha \in I_1$, there exists $u \in t^*$ such that $u(x) = 0$, $u(y) > \alpha$. Since $u^{-1}(\alpha, 1] \in I_\alpha(t^*)$, $y \in u^{-1}(\alpha, 1]$ and $x \notin u^{-1}(\alpha, 1]$ thus, we have that $(X, I_\alpha(t^*))$ is T_0 – space .

(c) Finally, suppose that (X, t^*) is α - T_0(iii). Let $x, y \in X$ with $x \neq y$. Since (X, t^*) is α - T_0(iii), for $\alpha \in I_1$, there exists $u \in t^*$ such that $0 \le u(x) \le \alpha < u(y) \le 1$. Since $u^{-1}(\alpha, 1] \in I_\alpha(t^*)$, $x \notin u^{-1}(\alpha, 1]$ and $y \in u^{-1}(\alpha, 1]$, so these implies $(X, I_\alpha(t^*))$ is T_0 – space.

Conversely, suppose $(X, I_\alpha(t^*))$ be T_0 – space. We shall prove that (X, t^*) is a α - T_0(iii) space. Let $x, y \in X$ with $x \neq y$. Since $(X, I_\alpha(t^*))$ is T_0 – space, there exists $u^{-1}(\alpha, 1] \in I_\alpha(t^*)$ such that $x \in u^{-1}(\alpha, 1]$ and $y \notin u^{-1}(\alpha, 1]$, where $u \in t^*$. Then we have $u(x) > \alpha$, $u(y) \le \alpha$, i.e., $0 \le u(y) \le \alpha < u(x) \le 1$, and hence by definition, (X, t^*) is α - T_0(iii) space.

Example 2.13: Let $X = \{x, y\}$ and $u \in I^X$ be defined by $u(x) = 0.7$, $u(y) = 0$. Suppose the supra fuzzy topology t^* on X generated by {0, u, 1, Constants}. Then by definition, for

$\alpha = 0.5$, (X, t^*) is not α-T_0(i) and (X, t^*) is not α - T_0(ii). Now $I_\alpha(t^*) = \{X, \phi, \{x\}\}$. Then we see that $I_\alpha(t^*)$ is a supra topology on X and $(X, I_\alpha(t^*))$ is a T_0 -space .This completes the proof.

Theorem 2.14: Let (X, t^*) be a supra fuzzy topological space, $A \subseteq X$, and $t^*_A = \{u/A : u \in t^*\}$, then

(a) (X, t^*) is α - T_0 (i) implies (A, t^*_A) is α - T_0 (i).

(b) (X, t^*) is α - T_0 (ii) implies (A, t^*_A) is α - T_0 (ii).

(c) (X, t^*) is α - T_0 (iii) implies (A, t^*_A) is α - T_0 (iii).

(d) (X, t^*) is T_0 (iv) implies (A, t^*_A) is T_0 (iv).

Proof: (a) Suppose that (X, t^*) be a supra fuzzy topological space and (X, t^*) is α - T_0(i). Let $x, y \in A$ with $x \neq y$, so that $x, y \in X$, as $A \subseteq X$. Since (X, t^*) is α - T_0 (i), for $\alpha \in I_1$, there exists $u \in t^*$ such that $u(x) = 1$, $u(y) \leq \alpha$. For $A \subseteq X$, we have $u/A \in t^*_A$ and $(u/A)(x) = 1$ and $(u/A)(y) \leq \alpha$ as $x, y \in A$. Hence by definition, (A, t^*_A) is α -T_0(i).

(b) Suppose that (X, t^*) be a supra fuzzy topological space and (X, t^*) is α - T_0(ii). Let x, $y \in A$ with $x \neq y$, so that $x, y \in X$, as $A \subseteq X$. Since (X, t^*) is α - T_0 (ii), for $\alpha \in I_1$, there exists $u \in t^*$ such that $u(x) = 0$, $u(y) > \alpha$. For $A \subseteq X$, we have $u/A \in t^*_A$ and $(u/A)(x) = 0$ and $(u/A)(y) > \alpha$ as $x, y \in A$. Hence by definition, (A, t^*_A) is α -T_0(ii).

(c) Suppose that (X, t^*) is α - T_0(iii) . Then for $x, y \in A$ with $x \neq y$, so that $x, y \in X$, as $A \subseteq X$. Since (X, t^*) is α - T_0 (iii), for $\alpha \in I_1$, there exists $u \in t^*$ such that $0 \leq u(x) \leq \alpha < u(y) \leq 1$. For $A \subseteq X$, we have $u/A \in t^*_A$ and $0 \leq (u/A)(x) \leq \alpha < (u/A)(y) \leq 1$ as x, $y \in A$. Hence, by definition, (A, t^*_A) is α -T_0(iii).

Similarly, we can prove (d).

Theorem 2.15: Suppose that (X_i, t^*_i) , $i \in \Lambda$ be supra fuzzy topological spaces and $X = \Pi_{i \in \Lambda} X_i$ and t^* be the product supra fuzzy topology on X, then

(a) $\forall i \in \Lambda$, (X_i, t^*_i) is α - T_0(i) if and only if (X, t^*) is α - T_0 (i).

(b) $\forall i \in \Lambda$, (X_i, t^*_i) is α - T_0(ii) if and only if (X, t^*) is α - T_0 (ii).

(c) $\forall i \in \Lambda$, (X_i, t^*_i) is α - T_0(iii) if and only if (X, t^*) is α - T_0 (iii).

(d) $\forall i \in \Lambda$, (X_i, t^*_i) is T_0(iv) if and only if (X, t^*) is T_0 (iv).

Proof: (a) Suppose that $\forall i \in \Lambda$, (X_i, t^*_i) is α - T_0(i). Let $x, y \in X$ with $x \neq y$, then $x_i \neq y_i$, for some $i \in \Lambda$. Since (X_i, t^*_i) is α - T_0 (i), for $\alpha \in I_1$, there exists $u_i \in t^*_i$, $i \in \Lambda$ such that $u_i(x_i) = 1$ and $u_i(y_i) \leq \alpha$. But we have $\pi_i(x) = x_i$, and $\pi_i(y) = y_i$. Then $u_i(\pi_i(x)) = 1$ and $u_i(\pi_i(y)) \leq \alpha$, i.e., $(u_i \circ \pi_i)(x) = 1$ and $(u_i \circ \pi_i)(y) \leq \alpha$. It follows that there exists $(u_i \circ \pi_i) \in t^*$ such that $(u_i \circ \pi_i)(x) = 1$, $(u_i \circ \pi_i)(y) \leq \alpha$. Hence by definition, (X, t^*) is α - T_0(i).

Conversely, suppose that (X, t^*) is α - T_0(i) space. We have to show that (X_i, t^*_i), $i \in \Lambda$ is α - T_0(i). Let a_i be a fixed element in X_i and $A_i = \{ x \in X = \Pi_{i \in \Lambda} X_i : x_j = a_j$ for some $i \neq j \}$. Thus A_i is a subset of X and hence $(A_i, t^*_{A_i})$ is also a subspace of (X, t^*). Since (X, t^*) is α - T_0(i), $(A_i, t^*_{A_i})$ is also α- T_0 (i). Now we have A_i is homeomorphic image of X_i. Thus, we have (X_i, t^*_i), $i \in \Lambda$ is α - T_0 (i).

(b) Suppose that $\forall i \in \Lambda$, (X_i, t^*_i) is α - T_0(ii). Let $x, y \in X$ with $x \neq y$, then $x_i \neq y_i$, for some $i \in \Lambda$. Since (X_i, t^*_i) is α - T_0 (ii), for $\alpha \in I_1$, there exists $u_i \in t^*_i$, $i \in \Lambda$ such that $u_i(x_i) = 0$ and $u_i(y_i) > \alpha$. But we have $\pi_i(x) = x_i$, and $\pi_i(y) = y_i$. Then

$u_i(\pi_i(x)) = 0$ and $u_i(\pi_i(y)) > \alpha$, i.e., $(u_i \circ \pi_i)(x) = 0$ and $(u_i \circ \pi_i)(y) > \alpha$. It follows that, there exists $(u_i \circ \pi_i) \in t^*$ such that $(u_i \circ \pi_i)(x) = 0$, $(u_i \circ \pi_i)(y) > \alpha$. Hence, by definition, (X, t^*) is $\alpha - T_0$ (ii).

Conversely, suppose that (X, t^*) is $\alpha - T_0$(ii) space. We have to show that (X_i, t^*_i), $i \in \Lambda$ is $\alpha - T_0$(ii). Let a_i be a fixed element in X_i and $A_i = \{ x \in X = \Pi_{i\in\Lambda} X_i : x_j = a_j$ for some $i \neq j \}$. Thus A_i is a subset of X and hence (A_i, t^*_{Ai}) is also a subspace of (X, t^*). Since (X, t^*) is $\alpha - T_0$(ii), (A_i, t^*_{Ai}) is also $\alpha - T_0$ (ii). Then, we have A_i is homeomorphic image of X_i. Thus, we have (X_i, t^*_i), $i\in\Lambda$ is $\alpha - T_0$ (ii)

(c) Suppose that $\forall\, i \in \Lambda$, (X_i, t^*_i) is $\alpha - T_0$(iii). Let $x, y \in X$ with $x \neq y$, then $x_i \neq y_i$, for some $i \in\Lambda$. Since (X_i, t^*_i) is $\alpha - T_0$ (iii), for $\alpha \in I_1$, there exists $u_i \in t^*_i$, $i \in \Lambda$. such that $0 \leq (u_i)(x_i) \leq \alpha < (u_i)(y_i) \leq 1$. But we have $\pi_i(x) = x_i$, and $\pi_i(y) = y_i$. Then, we have $0 \leq u_i(\pi_i(x) \leq \alpha < u_i(\pi_i(y) \leq 1$, i.e., $0 \leq (u_i \circ \pi_i)(x) \leq \alpha < (u_i \circ \pi_i)(y) \leq 1$. It follows that, there exists $(u_i \circ \pi_i) \in t^*$ such that $0 \leq (u_i \circ \pi_i)(x) \leq \alpha < (u_i \circ \pi_i)(y) \leq 1$. Hence, by definition, (X, t^*) is $\alpha - T_0$ (iii).

Conversely, suppose that (X, t^*) is $\alpha - T_0$(iii) space. We have to show that (X_i, t^*_i), $i \in \Lambda$ is $\alpha - T_0$(iii). Let a_i be a fixed element in X_i and $A_i = \{ x \in X = \Pi_{i\in\Lambda} X_i : x_j = a_j$ for some $i \neq j \}$. Thus A_i is a subset of X and hence (A_i, t^*_{Ai}) is also a subspace of (X, t^*). Since (X, t^*) is $\alpha - T_0$(iii), so (A_i, t^*_{Ai}) is also $\alpha - T_0$ (iii). Then, we have A_i is homeomorphic image of X_i. Thus, we have (X_i, t^*_i), $i\in\Lambda$ is $\alpha - T_0$ (iii).

Similarly, (d) can be proved.

Theorem 2.16: Let (X, t^*) and (Y, s^*) be two supra fuzzy topological spaces and $f : X \longrightarrow Y$ be a one-one, onto and open map, then

(a) (X, t^*) is α-T_0 (i) implies (Y, s^*) is α-T_0 (i).

(b) (X, t^*) is α-T_0 (ii) implies (Y, s^*) is α-T_0 (ii).

(c) (X, t^*) is α-T_0 (iii) implies (Y, s^*) is α-T_0 (iii).

(d) (X, t^*) is T_0 (iv) implies (Y, s^*) is T_0 (iv).

Proof: (a) Suppose (X, t^*) be α-T_0 (i). We have to prove that (Y, s^*) is α-T_0 (i). Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since f is onto, there exist $x_1, x_2 \in X$ with $f(x_1) = y_1$, $f(x_2) = y_2$ and hence $x_1 \neq x_2$ as f is one-one. Again, since (X, t^*) is α-T_0(i), for $\alpha \in I_1$, there exists $u \in t^*$ such that $u(x) = 1$, $u(y) \leq \alpha$.

Now, $f(u)(y_1) = \{\text{Sup } u(x_1) : f(x_1) = y_1\}$

$= 1.$

$f(u)(y_2) = \{\text{Sup } u(x_2) : f(x_2) = y_2\}$

$\leq \alpha.$

Since f is open, $f(u) \in s^*$ as $u \in t^*$. We observe that there exists $f(u) \in s^*$ such that $f(u)(y_1) = 1$, $f(u)(y_2) \leq \alpha$. Hence by definition, (Y, s^*) is α-T_0 (i).

(b) Suppose (X, t^*) be α-T_0 (ii). Then for $y_1, y_2 \in Y$ with $y_1 \neq y_2$, there exist $x_1, x_2 \in X$ with $f(x_1) = y_1$, $f(x_2) = y_2$, since f is onto, and hence $x_1 \neq x_2$ as f is one-one. Again, since (X, t^*) is α-T_0 (ii), for $\alpha \in I_1$, there exists $u \in t^*$ such that $u(x) = 0$, $u(y) > \alpha$.

Now, $f(u)(y_1) = \{ \text{Sup } u(x_1) : f(x_1) = y_1 \}$

$= 0.$

$f(u)(y_2) = \{ \text{Sup } u(x_2) : f(x_2) = y_2 \}$

$> \alpha.$

Since f is open, $f(u) \in s^*$ as $u \in t^*$. We observe that there exists $f(u) \in s^*$ such that $f(u)(y_1) = 0$, $f(u)(y_2) > \alpha$. Hence by definition, (Y, s^*) is α - T_0 (ii).

Similarly, (c) and (d) can be proved.

Theorem 2.17: Let (X, t^*) and (Y, s^*) be two supra fuzzy topological spaces and $f: X \longrightarrow Y$ be continuous and one-one map, then

(a) (Y, s^*) is α- T_0 (i) implies (X, t^*) is α - T_0 (i).

(b) (Y, s^*) is α- T_0 (ii) implies (X, t^*) is α - T_0 (ii).

(c) (Y, s^*) is α- T_0 (iii) implies (X, t^*) is α - T_0(iii).

(d) (Y, s^*) is T_0 (iv) implies (X, t^*) is T_0(iv).

Proof: (a) Suppose (Y, s^*) be α - T_0 (i). We have to prove that (X, t^*) is α -T_0 (i). Let $x_1, x_2 \in X$ with $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$ in Y, since f is one-one. Also since (Y, s^*) is α-T_0 (i), for $\alpha \in I_1$, there exists $u \in s^*$ such that $u(f(x_1)) = 1$ and $u(f(x_2)) \leq \alpha$. This implies that $f^{-1}(u)(x_1) = 1$, $f^{-1}(u)(x_2) \leq \alpha$, since $u \in s^*$ and f is continuous, then $f^{-1}(u) \in t^*$. Thus there is a $f^{-1}(u) \in t^*$ such that $f^{-1}(u)(x_1) = 1$, $f^{-1}(u)(x_2) \leq \alpha$. Hence (X, t^*) is α -T_0 (i).

(b) Suppose (Y, s^*) be α - T_0 (ii). Then, for $x_1, x_2 \in X$ with $x_1 \neq x_2$, we have, $f(x_1) \neq f(x_2)$ in Y, since f is one-one. Also since (Y, s^*) is α-T_0 (ii), for $\alpha \in I_1$, there exists $u \in s^*$ such that $u(f(x_1)) = 0$ and $u(f(x_2)) > \alpha$. This implies that $f^{-1}(u)(x_1) = 0$, $f^{-1}(u)(x_2) > \alpha$, since $u \in s^*$ and f is continuous, then $f^{-1}(u) \in t^*$. Thus there is a $f^{-1}(u) \in t^*$ such that $f^{-1}(u)(x_1) = 0$, $f^{-1}(u)(x_2) > \alpha$. Hence (X, t^*) is α-T_0(ii).

(c) Suppose (Y, s^*) be α - T_0(iii). Then, for $x_1, x_2 \in X$ with $x_1 \neq x_2$, we have, $f(x_1) \neq f(x_2)$ in Y, since f is one-one. Also, since (Y, s^*) is α-T_0(iii), for $\alpha \in I_1$, there exists $u \in s^*$ such that $0 \leq u(f(x_1)) \leq \alpha < u(f(x_2)) \leq 1$. This implies that $0 \leq f^{-1}(u)(x_1) \leq \alpha < f^{-1}(u)(x_2) \leq 1$, since $u \in s^*$ and f is continuous, then $f^{-1}(u) \in t^*$. Thus, there is a $f^{-1}(u) \in t^*$ such that $0 \leq f^{-1}(u)(x_1) \leq \alpha < f^{-1}(u)(x_2) \leq 1$. Hence (X, t^*) is α -T_0(iii).

Similarly, (d) can be proved.

CHAPTER 3

T_1 Spaces in Supra Fuzzy Topology

3.1 Introduction

In this chapter, we introduce and study four concepts of T_1 supra fuzzy topological spaces. We also establish some relationships among them and study some other properties of these spaces.

Definition 3.2: Let (X, t) be a fuzzy topological space and t^* be associated supra topology with t and $\alpha \in I_1$. Then

(a) (X, t^*) is an α - T_1(i) space if and only if for all distinct elements $x, y \in X$, there exists $u \in t^*$ such that $u(x) = 1$, $u(y) \leq \alpha$ and there exists $v \in t^*$ such that $v(x) \leq \alpha$, $v(y) = 1$.

(b) (X, t^*) is an α - T_1(ii) space if and only if for all distinct elements $x, y \in X$, there exists $u \in t^*$ such that $u(x) = 0$, $u(y) > \alpha$ and there exists $v \in t^*$ such that $v(x) > \alpha$, $v(y) = 0$.

(c) (X, t^*) is an α - T_1(iii) space if and only if for all distinct elements $x, y \in X$, there exists $u \in t^*$ such that $0 \leq u(y) \leq \alpha < u(x) \leq 1$ and there exists $v \in t^*$ such that $0 \leq v(x) \leq \alpha < v(y) \leq 1$.

(d) (X, t^*) is a T_1(iv) space if and only if for all distinct elements $x, y \in X$, there exists $u, v \in t^*$ such that $u(x) < u(y)$ and $v(x) > v(y)$.

Lemma 3.3: Suppose (X, t) is a topological space and t^* is associated supra topology with t and $\alpha \in I_1$. Then the following implications are hold:

(a) (X, t^*) is α - T_1 (i) implies (X, t^*) is α - T_1 (iii) implies (X, t^*) is T_1 (iv).

(b) (X, t^*) is α - T_1 (ii) implies (X, t^*) is α - T_1 (iii) implies (X, t^*) is T_1 (iv).

One can show these in a diagram given below:

α -T_1(i)

$\searrow$

α - T_1(iii) $\Rightarrow$ T_1(iv) .

$\nearrow$

α -T_1(ii)

Proof: Suppose (X, t^*) is α - T_1 (i). We have to prove that (X, t^*) is α - T_1 (iii). Let x and y be any two distinct elements in X. Since (X, t^*) is α - T_1 (i), for $\alpha \in I_1$, then by definition, there exist $u, v \in t^*$ such that $u(x) = 1$, $u(y) \leq \alpha$ and $v(y) = 1$, $v(x) \leq \alpha$ which shows that $0 \leq u(y) \leq \alpha < u(x) \leq 1$ and $0 \leq v(x) \leq \alpha < v(y) \leq 1$. Hence by definition (c), (X, t^*) is α - T_1(iii) .

Let (X, t^*) is α - T_1 (iii). Then for any x and y be two distinct elements in X, there exist $u, v \in t^*$ such that $0 \leq u(y) \leq \alpha < u(x) \leq 1$, i.e., $u(y) < u(x)$. Hence, by definition, (X, t^*) is T_1(iv).

Let (X, t^*) is α - T_1 (ii). Then for any x and y be two distinct elements in X, there exist $u, v \in t^*$ such that $u(x) = 0$, $u(y) > \alpha$ and $v(x) = 0$, $v(x) > \alpha$, i.e., $0 \leq u(x) \leq \alpha < u(y) \leq 1$, and $0 \leq v(y) \leq \alpha < v(x) \leq 1$. Hence, by definition, (X, t^*) is T_1 (iii) and (X, t^*) is T_1 (iv).

The non-implications among α - T_1 (i) , α - T_1 (ii), α - T_1 (iii) and T_1 (iv) are shown in the following examples: .

Example 3.4: Let $X = \{x, y\}$ and $u \in I^X$ be defined by $u(x) = 0.52$, $u(y) = 0.95$ and $v(x) = 0.95$, $v(y) = 0.52$. Let the supra fuzzy topology t^* on X generated by {0, u, v, 1, Constants}. For $\alpha = 0.81$, we have $0 \leq u(x) \leq 0.81 < u(y) \leq 1$ and $0 \leq v(y) \leq 0.81 < v(x) \leq 1$. Thus according to the definition, (X, t^*) is α - T_1(iii) but (X, t^*) is not α - T_1(i). Also, we can easily show that (X, t^*) is not α -T_1(ii).

Example 3.5: Let X = {x, y} and u ∈ I^X be defined by u(x) = 1, u(y) = 0.53 and v(x) =0.53, v(y) = 1. Consider the supra fuzzy topology t^* on X generated by { 0 , u , v, 1 , Constants } . For α = 0.76, we have $u(x)=1,\ u(y)\le\alpha$ and $v(y)=1,\ v(x)\le\alpha$. Thus according to the definition (X, t^*) is α -T_1(i) but (X, t^*) is not α -T_1(ii).

Example 3.6: Let X = { x , y } and u ∈ I^X be defined by u(x) = 0 , u(y) = 0.83 and v(x) = 0.83 , v(y) = 0. Let the supra fuzzy supra topology t^* on X generated by {0, u, v, 1, Constants}. For α = 0.45 , it can easily show that (X , t^*) is α - T_1(ii) but (X , t*) is not α -T_1(i) .

Example 3.7: Let X = {x, y} and u ∈ I^X is defined by u(x) = 0.35, u(y) = 0.72 and v(x) = 0.72, v(y) = 0.35. Let the supra fuzzy topology t^* on X generated by {0, u, v, 1, Constants}. Then for α = 0.82, we can easily show that (X, t^*) is T_1(iv) but (X, t^*) is not α - T_1(iii) and hence not α -T_1(i) and α -T_1(ii).

Thus the proof is complete.

Theorem 3.8: Suppose that (X, t^*) is a supra fuzzy topological space and $\alpha,\beta\in t^*$ with $0\le\alpha\le\beta<1$. Then

(a) (X, t^*) is α - T_1 (i) implies (X, t^*) is β -T_1 (i).

(b) (X, t^*) is β - T_1 (ii) implies (X, t^*) is α -T_1(ii).

(c) (X, t^*) is 0 – T_1(iii) if and only if (X, t^*) is 0 – T_1(iii).

Proof: (a) Suppose that (X, t^*) is a supra fuzzy topological space and (X, t^*) is α - T_1(i). We have to show that (X, t^*) is β- T_1(i). Let any two distinct elements x, y ∈ X. Since (X, t^*) is α- T_1(i) , for α ∈ I_1 , there is u, v ∈ t^* such that u(x) = 1, u(y) ≤ α and v(y) = 1, v(x) ≤ α. This implies that u(x) = 1 , u(y) ≤ β, and v(x) ≤ β, v(y) = 1 , since $0\le\alpha\le\beta<1$. Hence by definition, (X, t^*) is β - T_1 (i).

(b) Suppose (X, t^*) is β - T_1(ii). We have to show that (X, t^*) is α- T_1(ii). Then for any two distinct elements $x, y \in X$, there are $u, v \in t^*$ such that $u(x) = 0$, $u(y) > \beta$ and $v(y) = 0$, $v(x) > \beta$, for $\beta \in I_1$. This implies that $u(x) = 0$, $u(y) > \alpha$ and $v(y) = 0$, $v(x) > \alpha$, since $0 \leq \alpha \leq \beta < 1$. Hence, by definition, (X, t^*) is α - T_1 (ii).

The proof of (c) is trivial.

Example 3.9: Let $X = \{x, y\}$ and $u, v \in I^X$ be defined by $u(x) = 1$, $u(y) = 0.62$ and $v(x) = 0.62$, $v(y) = 1$. Let the supra fuzzy topology t^* on X generated by {0, u, v, 1, Constants}. Then by definition, for $\alpha = 0.51$ and $\beta = 0.84$; Thus (X, t^*) is β -T_1 (i) but (X, t^*) is not α -T_1 (i).

Example 3.10: Let $X = \{x, y\}$ and $u, v \in I^X$ be defined by $u(x) = 0$, $u(y) = 0.65$ and $v(x) = 0.66$, $v(y) = 0$. Let the supra fuzzy topology t^* on X generated by {0, u, v, 1, Constants}. Then by definition, for $\alpha = 0.41$ and $\beta = 0.84$; (X, t^*) is α-T_1 (ii) but (X, t^*) is not β -T_1 (ii).

Theorem 3.11: Let (X, T) be a topological space and T^* be associated supra topology with T and $\alpha \in I_1$. Suppose that the following statements:

(1) (X, T^*) be a T_1 – space.

(2) $(X, \omega(T^*))$ be an α- T_1(i) space.

(3) $(X, \omega(T^*))$ be an α- T_1(ii) space.

(4) $(X, \omega(T^*))$ be an α- T_1(iii) space .

(5) $(X, \omega(T^*))$ be a T_1(iv) space .

Then the followings are true:

(a) $(1) \Rightarrow (2) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$.

(b) $(1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$.

Proof: Suppose (X, T^*) is a T_1 – topological space. We have to prove that $(X, \omega(T^*))$ is α - T_1(i) space. Suppose x and y are two distinct elements in X. Since (X, T^*) is T_1 -spaces, there are $U, V \in T^*$ such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$. By the definition of lsc , we have $1_U, 1_V \in \omega(T^*)$ and $1_U(x) = 1$, $1_U(y) = 0$ and $1_V(x) = 0$, $1_V(y) = 1$. Hence we have $(X, \omega(T^*))$ is α -T_1(i) -space. Also we have $(X, \omega(T^*))$ is α - T_1 (ii) space. Further, it is easy to show that (2) $\Rightarrow$ (4), (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (5). We therefore prove that (5) $\Rightarrow$ (1).

Suppose $(X, \omega(T^*))$ be a T_1(iv) - space. We have to prove that (X, T^*) is T_1 – space. Let $x, y \in X$ with $x \neq y$. Since $(X, \omega(T^*))$ is T_1 (iv), there are $u, v \in \omega(T^*)$ such that $u(x) < u(y)$ and $v(x) > v(y)$. Suppose that $r, s \in I_1$ such that $u(x) < r < u(y)$ and $v(x) > s > v(y)$. We observe that $x \notin u^{-1}(r, 1]$, $y \in u^{-1}(r, 1]$ and $x \in v^{-1}(s, 1]$, $y \notin v^{-1}(s, 1]$, and by definition of lsc , $u^{-1}(r, 1]$ and $v^{-1}(s, 1] \in T^*$. Therefore (X, T^*) is T_1 – space.

Thus it is seen that α – T_1 (p) is a good extension of its topological counter part (p = i , ii, iii, iv).

Theorem 3.12: Let (X, t^*) be a supra fuzzy topological space, $\alpha \in I_1$ and $I_\alpha(t^*) = \{u^{-1}(\alpha, 1] : u \in t^*\}$, then

(a) (X, t^*) is an α -T_1(i)-space implies $(X, I_\alpha(t^*))$ is T_1 -space.

(b) (X, t^*) is an α -T_1(ii)-space implies $(X, I_\alpha(t^*))$ is T_1 -space.

(c) (X, t^*) is an α -T_1(iii)-space if and only if $(X, I_\alpha(t^*))$ is T_1 -space.

Proof: (a) Let (X, t^*) be a supra fuzzy topological space and (X, t^*) be α - T_1(i). Suppose x and y be any two distinct elements in X. Since (X, t^*) is α -T_1(i), for $\alpha \in I_1$, there exist $u, v \in t^*$ such that $u(x) = 1$, $u(y) \leq \alpha$ and $v(x) \leq \alpha$, $v(y) = 1$, Since $u^{-1}(\alpha, 1]$,

$v^{-1}(\alpha, 1] \in I_\alpha(t^*)$, $y \notin u^{-1}(\alpha, 1]$, $x \in u^{-1}(\alpha, 1]$ and $x \notin v^{-1}(\alpha, 1]$, $y \in v^{-1}(\alpha, 1]$. We have that $(X, I_\alpha(t^*))$ is T_1 – space .

(b) Suppose that (X, t^*) be α - T_1(i). Then for any x and y be two distinct elements in X, there exist $u, v \in t^*$ such that $u(x) = 0$, $u(y) > \alpha$ and $v(x) > \alpha$, $v(y) = 0$, for $\alpha \in I_1$. Since $u^{-1}(\alpha, 1]$, $v^{-1}(\alpha, 1] \in I_\alpha(t^*)$, $y \in u^{-1}(\alpha, 1]$, $x \notin u^{-1}(\alpha, 1]$ and $x \in v^{-1}(\alpha, 1]$, $y \notin v^{-1}(\alpha, 1]$. We have that $(X, I_\alpha(t^*))$ is T_1 – space .

(c) Suppose that (X, t^*) is α - T_1(iii)-space. Let $x, y \in X$ with $x \neq y$. Since (X, t^*) is α - T_1(iii), for $\alpha \in I_1$, there exist $u, v \in t^*$ such that $0 \leq u(x) \leq \alpha < u(y) \leq 1$ and $0 \leq v(y) \leq \alpha < v(x) \leq 1$. Since $u^{-1}(\alpha, 1]$, $v^{-1}(\alpha, 1] \in I_\alpha(t^*)$, $x \notin u^{-1}(\alpha, 1]$, $y \in u^{-1}(\alpha, 1]$ and $x \in v^{-1}(\alpha, 1]$, $y \notin v^{-1}(\alpha, 1]$, so these implies $(X, I_\alpha(t^*))$ is T_1 – space .

Conversely, suppose $(X, I_\alpha(t^*))$ be T_1 – space. We shall prove that (X, t^*) is a α - T_1(iii) space. Let $x, y \in X$ with $x \neq y$. Since $(X, I_\alpha(t^*))$ is T_1 – space , there exist $u^{-1}(\alpha, 1]$, $v^{-1}(\alpha, 1] \in I_\alpha(t^*)$ such that $x \in u^{-1}(\alpha, 1]$ and $y \notin u^{-1}(\alpha, 1]$; $x \notin v^{-1}(\alpha, 1]$ and $y \in v^{-1}(\alpha, 1]$, where $u, v \in t^*$. Then we have $u(x) > \alpha$, $u(y) \leq \alpha$ and $v(x) \leq \alpha$, $v(y) > \alpha$, i.e., $0 \leq u(y) \leq \alpha < u(x) \leq 1$and $0 \leq v(x) \leq \alpha < v(y) \leq 1$, and hence by definition, (X, t^*) is α - T_1(iii) -space .

Example 3.13: Let $X = \{x, y\}$and $u, v, w \in I^X$ be defined by , $u(x) = 1$, $u(y) = 0.63$, $v(x) = 0.43$, $v(y) = 0.83$, $w(x) = 0.23$, $w(y) = 0.29$. Suppose the supra fuzzy topology t^* on X generated by $\{0, u, v, w, 1, \text{Constants}\}$.Then by definition, for $\alpha = 0.51$, (X, t^*) is not α - T_1(i) and (X, t^*) is not α - T_1(ii) . Now $I_\alpha(t^*) = \{X, \phi, \{x\}, \{y\}\}$. Then we see that $I_\alpha(t^*)$ is a supra topology on X and $(X, I_\alpha(t^*))$ is a T_1 -space. This completes the proof.

Theorem 3.14: Let (X, t^*) be a supra fuzzy topological space, $A \subseteq X$, and $t^*_A = \{u/A : u \in t^*\}$, then

(a) (X, t^*) is α - T_1 (i) implies (A, t^*_A) is α - T_1 (i).

(b) (X, t^*) is α - T_1 (ii) implies (A, t^*_A) is α - T_1 (ii).

(c) (X, t^*) is α - T_1 (iii) implies (A, t^*_A) is α - T_1 (iii).

(d) (X, t^*) is T_1 (iv) implies (A, t^*_A) is T_1 (iv).

Proof: (a) Suppose that (X, t^*) be a supra fuzzy topological space and (X, t^*) is α - T_1(i) space. Let $x, y \in A$ with $x \neq y$, so that $x, y \in X$, as $A \subseteq X$. Since (X, t^*) is α - T_1 (i), for $\alpha \in I_1$, there exist $u, v \in t^*$ such that $u(x) = 1$, $u(y) \leq \alpha$ and $v(x) \leq \alpha$, $v(y) = 1$. For $A \subseteq X$, we have $u / A, v / A \in t^*_A$ and $(u / A)(x) = 1$ and $(u /A)(y) \leq \alpha$ and $(v /A)(x) \leq \alpha$, $(v / A)(y) = 1$, as $x, y \in A$. Hence, by definition, (A, t^*_A) is α -T_1(i).

(b) Suppose that (X, t^*) is α - T_1(ii)-space. Then for $x, y \in A$ with $x \neq y$, so that $x, y \in X$, as $A \subseteq X$ and there exist $u, v \in t^*$ such that $u(x) = 0$, $u(y) > \alpha$ and $v(x) > \alpha$, $v(y) = 0$, for $\alpha \in I_1$. For $A \subseteq X$, we have $u / A, v / A \in t^*_A$ and $(u / A)(x) = 0$ and $(u /A)(y) > \alpha$ and $(v /A)(x) > \alpha$, $(v / A)(y) = 0$, as $x, y \in A$. Hence, by definition, (A, t^*_A) is α -T_1(ii).

(c) Suppose that (X, t^*) is α - T_1(iii)-space. Then for $x, y \in A$ with $x \neq y$, so that $x, y \in X$, as $A \subseteq X$ and there exist $u, v \in t^*$ such that $0 \leq u(y) \leq \alpha < u(x) \leq 1$ and $0 \leq v(x) \leq \alpha < v(y) \leq 1$, for $\alpha \in I_1$. For $A \subseteq X$, we have $u / A, v / A \in t^*_A$ and $0 \leq (u/A)(y) \leq \alpha < (u/A)(x) \leq 1$ and $0 \leq (v/A)(x) \leq \alpha < (v/A)(y) \leq 1$, as $x, y \in A$. Hence, by definition, (A, t^*_A) is α -T_1(iii).

Similarly, (d) can be proved.

Theorem 3.15: Suppose that (X_i, t^*_i), $i \in \Lambda$ be supra fuzzy topological spaces and $X = \Pi_{i \in \Lambda} X_i$ and t^* be the product supra fuzzy topology on X, then

(a) $\forall\, i \in \Lambda$, (X_i, t^*_i) is α - T_1(i) if and only if (X, t^*) is α - T_1 (i).

(b) $\forall\, i \in \Lambda$, (X_i, t^*_i) is α - T_1(ii) if and only if (X, t^*) is α - T_1 (ii).

(c) $\forall\, i \in \Lambda$, (X_i, t^*_i) is α - T_1(iii) if and only if (X, t^*) is α - T_1 (iii).

(d) $\forall\, i \in \Lambda$, (X_i, t^*_i) is T_1(iv) if and only if (X, t^*) is T_1 (iv).

Proof: (a) Suppose that $\forall\, i \in \Lambda$, (X_i, t^*_i) is α - T_1(i). Let x, y$\in$ X with $x \neq y$, then $x_i \neq y_i$, for some $i \in \Lambda$. Since (X_i, t^*_i) is α - T_1 (i), for $\alpha \in I_1$, there exists $u_i \in t^*_i$, $i \in \Lambda$ such that $u_i(x_i) = 1$, $u_i(y_i) \leq \alpha$ and $v_i(y_i) = 1$, $v_i(x_i) \leq \alpha$. But we have $\pi_i(x) = x_i$, and $\pi_i(y) = y_i$. Then $u_i(\pi_i(x)) = 1$, $u_i(\pi_i(y)) \leq \alpha$ and $v_i(\pi_i(y)) = 1$, $v_i(\pi_i(x)) \leq \alpha$, i.e., $(u_i \circ \pi_i)(x) = 1$, $(u_i \circ \pi_i)(y) \leq \alpha$ and $(v_i \circ \pi_i)(y) = 1$, $(v_i \circ \pi_i)(x) \leq \alpha$. It follows that there exist $(u_i \circ \pi_i)$, $(v_i \circ \pi_i) \in t^*$ such that $(u_i \circ \pi_i)(x) = 1$, $(u_i \circ \pi_i)(y) \leq \alpha$ and $(v_i \circ \pi_i)(y) = 1$, $(v_i \circ \pi_i)(x) \leq \alpha$. Hence by definition, (X, t^*) is α - T_1(i).

Conversely, suppose that (X, t^*) is α - T_1(i) space. We have to show that (X_i, t^*_i), $i \in \Lambda$ is α - T_1(i). Let a_i be a fixed element in X_i and $A_i = \{ x \in X = \Pi_{i \in \Lambda} X_i : x_j = a_j$ for some $i \neq j \}$. Thus A_i is a subset of X and hence $(A_i, t^*_{A_i})$ is also a subspace of (X, t^*). Since (X, t^*) is α - T_1(i), $(A_i, t^*_{A_i})$ is also α- T_1 (i). Now we have A_i is homeomorphic image of X_i. Thus, we have (X_i, t^*_i), $i \in \Lambda$ is α - T_1 (i).

(b) Suppose that $\forall\, i \in \Lambda$, (X_i, t^*_i) is α - T_1(ii). Then, for x, y$\in$ X with $x \neq y$, we have $x_i \neq y_i$, for some $i \in \Lambda$, there exists $u_i \in t^*_i$, $i \in \Lambda$ such that $u_i(x_i) = 0$, $u_i(y_i) > \alpha$ and $v_i(y_i) = 0$, $v_i(x_i) > \alpha$, for $\alpha \in I_1$. But, we have $\pi_i(x) = x_i$, and $\pi_i(y) = y_i$. Then

$u_i(\pi_i(x)) = 0$, $u_i(\pi_i(y)) > \alpha$ and $v_i(\pi_i(y)) = 0$, $v_i(\pi_i(x)) > \alpha$, i.e., $(u_i \circ \pi_i)(x) = 0$, $(u_i \circ \pi_i)(y) > \alpha$ and $(v_i \circ \pi_i)(y) = 0$, $(v_i \circ \pi_i)(x) > \alpha$. It follows that there exist $(u_i \circ \pi_i)$, $(v_i \circ \pi_i) \in t^*$ such that $(u_i \circ \pi_i)(x) = 0$, $(u_i \circ \pi_i)(y) > \alpha$ and $(v_i \circ \pi_i)(y) = 0$, $(v_i \circ \pi_i)(x) > \alpha$. Hence, by definition, (X, t^*) is α - T_1 (ii).

Conversely, suppose that (X, t^*) is α - T_1(ii) space. Let a_i be a fixed element in X_i and $A_i = \{ x \in X = \Pi_{i\in\Lambda} X_i : x_j = a_j \text{ for some } i \neq j \}$. Thus A_i is a subset of X and hence (A_i, t^*_{Ai}) is also a subspace of (X, t^*). Since (X, t^*) is α - T_1(i), (A_i, t^*_{Ai}) is also α- T_1 (i) . Thus we have A_i is homeomorphic image of X_i. Hence (X_i, t^*_i), $i\in\Lambda$ is α - T_1 (i) .

Similarly, (c) and (d) can be proved.

Theorem 3.16: Let (X, t^*) and (Y, s^*) be two supra fuzzy topological spaces and $f : X \longrightarrow Y$ be a one-one, onto and open map, then

(a) (X, t^*) is α -T_1 (i) implies (Y, s^*) is α- T_1 (i).

(b) (X, t^*) is α -T_1 (ii) implies (Y, s^*) is α- T_1 (ii).

(c) (X, t^*) is α -T_1 (iii) implies (Y, s^*) is α- T_1 (iii).

(d) (X, t^*) is T_1 (iv) implies (Y, s^*) is T_1 (iv).

Proof: (a) Let (X, t^*) be α -T_1 (i) . We have to prove that (Y, s^*) is α -T_1 (i). Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since f is onto, there exist $x_1, x_2 \in X$ with $f(x_1) = y_1$, $f(x_2) = y_2$ and also $x_1 \neq x_2$ as f is one-one. Again since (X, t^*) is α-T_1(i) , for $\alpha \in I_1$, there exist $u, v \in t^*$ such that $u(x_1) = 1$, $u(x_2) \leq \alpha$ and $v(x_1) \leq \alpha$, $v(x_2) = 1$.

Now, $f(u)(y_1) = \{\text{Sup } u(x_1) : f(x_1) = y_1\}$

$= 1.$

$f(u)(y_2) = \{\text{Sup } u(x_2) \ : f(x_2) = y_2\}$

$\leq \alpha,$

and $f(v)(y_1) = \{\text{Sup } v(x_1) \ : \ f(x_1) = y_1\}$

$\leq \alpha.$

$f(v)(y_2) = \{\text{Sup } v(x_2) \ : f(x_2) = y_2\}$

$= 1.$

Since f is open then $f(u) \in s^*$ as $u \in t^*$. We observe that there exist $f(u), f(v) \in s^*$ such that $f(u)(y_1) = 1$, $f(u)(y_2) \leq \alpha$ and $f(v)(y_1) \leq \alpha$, $f(v)(y_2) = 1$. Hence by definition, (Y, s^*) is $\alpha - T_1(i)$

(b) Let (X, t^*) be α -T_1(ii). Then for $y_1, y_2 \in Y$ with $y_1 \neq y_2$, there exist $x_1, x_2 \in X$ with $f(x_1) = y_1$, $f(x_2) = y_2$, since f is onto, and also $x_1 \neq x_2$ as f is one-one. Again, since (X, t^*) is α-T_1(i), for $\alpha \in I_1$, there exist $u, v \in t^*$ such that $u(x_1) = 0$, $u(x_2) > \alpha$ and $v(x_1) > \alpha$, $v(x_2) = 0$.

Now, $f(u)(y_1) = \{\text{Sup } u(x_1) \ : \ f(x_1) = y_1\}$

$= 0.$

$f(u)(y_2) = \{\text{Sup } u(x_2) \ : f(x_2) = y_2\}$

$> \alpha,$

and $f(v)(y_1) = \{\text{Sup } v(x_1) \ : \ f(x_1) = y_1\}$

$> \alpha.$

$f(v)(y_2) = \{\text{Sup } v(x_2) \ : f(x_2) = y_2\}$

$= 0.$

Since f is open, then $f(u) \in s^*$ as $u \in t^*$. We observe that there exist $f(u), f(v) \in s^*$ such that $f(u)(y_1) = 0$, $f(u)(y_2) > \alpha$ and $f(v)(y_1) > \alpha$, $f(v)(y_2) = 0$. Hence by definition, (Y, s^*) is α - T_1(ii).

Similarly, (c) and (d) can be proved.

Theorem 3.17: Let (X, t^*) and (Y, s^*) be two supra fuzzy topological spaces and $f: X \longrightarrow Y$ be continuous and one-one map, then

(a) (Y, s^*) is α- T_1 (i) implies (X, t^*) is α - T_1 (i).

(b) (Y, s^*) is α- T_1 (ii) implies (X, t^*) is α - T_1 (ii).

(c) (Y, s^*) is α- T_1 (iii) implies (X, t^*) is α - T_1(iii).

(d) (Y, s^*) is T_1(iv) implies (X, t^*) is T_1(iv).

Proof: (a) Let (Y, s^*) be α - T_1 (i). We have to prove that (X, t^*) is α -T_1 (i). Let $x_1, x_2 \in X$ with $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$ in Y, since f is one-one. Also since (Y, s^*) is α-T_1(i), for $\alpha \in I_1$, there exist $u, v \in s^*$ such that $u(f(x_1)) = 1$, $u(f(x_2)) \leq \alpha$ and $v(f(x_1)) \leq \alpha$, $v(f(x_2)) = 1$. This implies that $f^{-1}(u)(x_1) = 1$, $f^{-1}(u)(x_2) \leq \alpha$ and $f^{-1}(v)(x_1) \leq \alpha$, $f^{-1}(v)(x_2) = 1$, since $u, v \in s^*$ and f is continuous, then $f^{-1}(u)$, $f^{-1}(v) \in t^*$ .Thus there are $f^{-1}(u)$, $f^{-1}(v) \in t^*$ such that $f^{-1}(u)(x_1) = 1$, $f^{-1}(u)(x_2) \leq \alpha$ and $f^{-1}(v)(x_1) \leq \alpha$, $f^{-1}(v)(x_2) = 1$. Hence (X, t^*) is α -T_1 (i).

(b) Let (Y, s^*) be α - T_1 (ii). Then for $x_1, x_2 \in X$ with $x_1 \neq x_2$, we have $f(x_1) \neq f(x_2)$ in Y, since f is one-one. Also we have, for $\alpha \in I_1$, there exist $u, v \in s^*$ such that $u(f(x_1)) = 0$, $u(f(x_2)) > \alpha$ and $v(f(x_1)) > \alpha$, $v(f(x_2)) = 0$. This implies that $f^{-1}(u)(x_1) = 0$, $f^{-1}(u)(x_2) > \alpha$ and $f^{-1}(v)(x_1) > \alpha$, $f^{-1}(v)(x_2) = 0$, since $u, v \in s^*$ and f is continuous, then $f^{-1}(u)$, $f^{-1}(v) \in t^*$. Thus there are $f^{-1}(u)$, $f^{-1}(v) \in t^*$ such that $f^{-1}(u)(x_1) = 0$, $f^{-1}(u)(x_2) > \alpha$ and $f^{-1}(v)(x_1) > \alpha$, $f^{-1}(v)(x_2) = 0$. Hence (X, t^*) is α -T_1(ii).

Similarly, (c) and (d) can be proved.

CHAPTER 4

T_2 Spaces in Supra Fuzzy Topology

4.1 Introduction

Four concepts of T_2 supra fuzzy topological spaces are introduced and studied in this chapter. We also establish some relationships among them and study some other properties of these spaces.

Definition 4.2: Let (X, t) be a fuzzy topological space and t^* be associated supra topology with t and $\alpha \in I_1$. Then

(a) (X, t^*) is an T_2(i) space if and only if for all distinct elements $x, y \in X$, there exist $u, v \in t^*$ such that $u(x) = 1 = v(y)$ and $u \cap v = 0$.

(b) (X, t^*) is an α - T_2(ii) space if and only if for all distinct elements $x, y \in X$, there exist $u, v \in t^*$ such that $u(x) = 1 = v(y)$ and $u \cap v \leq \alpha$.

(c) (X, t^*) is an α - T_2(iii) space if and only if for all distinct elements $x, y \in X$, there exists $u, v \in t^*$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v = 0$.

(d) (X, t^*) is an α - T_2(iv) space if and only if for all distinct elements $x, y \in X$, there exist $u, v \in t^*$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v \leq \alpha$.

Lemma 4.3: Suppose (X, t) is a topological space and t^* is associated supra topology with t and $\alpha \in I_1$. Then the following implications are hold:

(a) (X, t^*) is T_2 (i) implies (X, t^*) is α - T_2 (ii) implies (X, t^*) is α -T_2 (iv).

(b) (X, t^*) is α - T_2 (i) implies (X, t^*) is α - T_2 (iii) implies (X, t^*) is α -T_2 (iv).

Also, these can be shown in a diagram as follows:

T_2 (i) $\Rightarrow \alpha$ - T_2 (ii) ↘

α - T_2 (iv).

T_2 (i) $\Rightarrow \alpha$ - T_2 (iii) ↗

Proof: (a) Suppose (X, t^*) is T_2 (i). We have to prove that (X, t^*) is α - T_2 (ii). Let x and y be any two distinct elements in X. Since (X, t^*) is T_2 (i), for $\alpha \in I_1$, then by definition, there exists $u, v \in t^*$ such that $u(x) = 1 = v(y)$ and $u \cap v = 0$, which shows that $u(x) = 1 = v(y)$ and $u \cap v \leq \alpha$. Hence by definition (b), (X, t^*) is α - T_2(ii). Also, we see that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v \leq \alpha$. Hence (X, t^*) is α - T_2(iv).

(b) Suppose (X, t^*) is T_2 (i). Then for any x and y be two distinct elements in X, there exist $u, v \in t^*$ such that $u(x) = 1 = v(y)$ and $u \cap v = 0$, for $\alpha \in I_1$ which shows that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v = 0$. Hence by definition (c), (X, t^*) is α - T_2(iii) and hence (X, t^*) is α - T_2(iv).

The following examples show the non-implications among T_2 (i), α - T_2 (ii), α - T_2 (iii) spaces, i.e.,

(1) α - T_2 (iv) does not imply α - T_2 (ii), so not imply T_2 (i).

(2) α - T_2 (iv) does not imply α - T_2 (iii), (3) α - T_2 (ii) does not imply α - T_2 (iii),

(4) α - T_2 (iii) does not imply α - T_2 (ii), (5) α - T_2 (iii) does not imply T_2 (i),

(6) α - T_2 (ii) does not imply T_2 (i).

Example 4.4: Let $X = \{x, y\}$ and $u, v \in I^X$ be defined by $u(x) = 0.95$, $u(y) = 0.43$ and $v(x) = 0.31$, $v(y) = 0.95$. Let the supra fuzzy topology t^* on X generated by {0, u, v, 1, Constants}. For $\alpha = 0.75$, we have $u(x) = 0.95 > \alpha$, $v(y) = 0.95 > \alpha$ and $u \cap v \leq \alpha$. Thus according to the definition, (X, t^*) is α - T_2(iv) but (X, t^*) is not α - T_2(ii) and (X, t^*) is not α - T_2(iii). So, (X, t^*) is not T_2(i).

Example 4.5: Let X = {x, y} and u, v ∈ I^X be defined by u(x) = 1, u(y) = 0.43 and v(x) =0.43, v(y) = 1. Consider the supra fuzzy topology t^* on X generated by {0, u, v, 1, Constants}. For α = 0.53, we have, $u(x)=1$, $v(y)=1$, and u ∩ v ≤ α. Thus according to the definition, (X, t^*) is α -T_2(ii) but (X, t^*) is not α -T_2(iii) and (X, t^*) is not T_2(i).

Example 4.6: Let X = {x, y} and u, v ∈ I^X be defined by u(x) = 0.85, u(y) = 0 and v(x) = 0, v(y) = 0.85. Let the supra fuzzy supra topology t^* on X generated by {0, u, v, 1, Constants}. For α = 0.53, it can easily show that (X, t^*) is α - T_2(iii) but (X, t*) is not α -T_1(ii) and (X, t*) is not T_2(i).

Theorem 4.7: Suppose that (X, t^*) is a supra fuzzy topological space and $\alpha, \beta \in t^*$ with $0 \le \alpha \le \beta < 1$, then

(a) (X, t^*) is α - T_2 (ii) implies (X, t^*)) is β -T_2 (ii).

(b) (X, t^*) is β - T_2 (iii) implies (X, t^*) is α -T_2(iii).

(c) (X, t^*) is 0 – T_2(iii) if and only if (X, t^*) is 0 – T_2(iv).

Proof: (a) Suppose that (X, t^*) is a supra fuzzy topological space and (X, t^*) is α - T_2(ii). We have to show that (X, t^*) is β- T_2(ii). Let any two distinct elements x, y ∈ X. Since (X, t^*) is α - T_2(ii), for α ∈ I_1, there is u, v ∈ t^* such that u(x) = 1 = v(y) and u ∩ v ≤ α. This implies that u(x) = 1 = v(y) and u ∩ v ≤ β as 0 ≤ α ≤ β < 1. Hence by definition, (X, t^*) is β - T_2 (ii).

(b) Suppose that (X, t^*) is β - T_2(iii). We have to show that (X, t^*) is α- T_2(iii). Then, any two distinct elements x, y ∈ X, there is u, v ∈ t^* such that u(x) > β, v(y) > β and

$u \cap v = 0$, for $\alpha \in I_1$. This implies that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v = 0$ as $0 \le \alpha \le \beta < 1$. Hence by definition, (X, t^*) is $\alpha - T_2$ (iii).

(c) The proof is trivial.

Example 4.8: Let $X = \{x, y\}$ and $u, v \in I^X$ be defined by $u(x) = 1$, $u(y) = 0.62$ and $v(x) = 0.82$, $v(y) = 1$. Let the supra fuzzy topology t^* on X generated by $\{0, u, v, 1, \text{Constants}\}$. Then by definition, for $\alpha = 0.31$ and $\beta = 0.84$; (X, t^*) is $\beta - T_2$ (ii) but (X, t^*) is not $\alpha - T_2$ (ii).

Example 4.9: Let $X = \{x, y\}$ and $u, v \in I^X$ be defined by $u(x) = 0$, $u(y) = 0.72$ and $v(x) = 0.88$, $v(y) = 0$. Let the supra fuzzy topology t^* on X generated by $\{0, u, v, 1, \text{Constants}\}$. Then, by definition, for $\alpha = 0.33$ and $\beta = 0.85$; (X, t^*) is $\alpha - T_2$ (iii) but (X, t^*) is not $\beta - T_2$ (iii).

Theorem 4.10: Let (X, T) be a topological space and T^* be associated supra topology with T and $\alpha \in I_1$. Suppose that the following statements:

(1) (X, T^*) be a T_2 – space.

(2) $(X, \omega(T^*))$ be a T_2 (i) space.

(3) $(X, \omega(T^*))$ be an α- T_2 (ii) space.

(4) $(X, \omega(T^*))$ be an α- T_2 (iii) space.

(5) $(X, \omega(T^*))$ be an α- T_2 (iv) space.

Then the followings are true:

(a) $(1) \Rightarrow (2) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$.

(b) $(1) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (1)$.

Proof: Suppose (X, T^*) is a T_2 – topological space. We have to prove that $(X, \omega(T^*))$ is T_2(i) space. Suppose x and y are two distinct elements in X. Since (X, T^*) is T_2 -spaces, there are $U, V \in T^*$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$. By the definition of lsc, we have $1_U, 1_V \in \omega(T^*)$ and $1_U(x) = 1$, $1_V(y) = 1$ and $1_U \cap 1_V = 0$. If $1_U \cap 1_V \neq 0$, then there exists $z \in X$ such that $(1_U \cap 1_V)(z) \neq 0$ implies $1_U(z) \neq 0$, $1_V(z) \neq 0$ implies $z \in U$, $z \in V$ implies $z \in U \cap V \Rightarrow U \cap V \neq \phi$, a contradiction. So that $1_U \cap 1_V = 0$, and consequently $(X, \omega(T^*))$ is T_2 (i). Also, we see that $(X, \omega(T^*))$ is α - T_2 (ii).

Further, it is easy to show that (2) ⇒ (4), (3) ⇒ (4) and (4) ⇒ (5). We therefore prove that (5) ⇒ (1).

Now, suppose $(X, \omega(T^*))$ be a T_2(iv) - space. We have to prove that (X, T^*) is T_2 – space. Let $x, y \in X$ with $x \neq y$. Since $(X, \omega(T^*))$ is T_2 (iv), for $\alpha \in I_1$, there are $u, v \in \omega(T^*)$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v \leq \alpha$. We have that $u^{-1}(\alpha, 1]$ and $v^{-1}(\alpha, 1] \in T^*$, $\alpha \in I_1$ and $x \in u^{-1}(\alpha, 1]$, $y \in v^{-1}(\alpha, 1]$. Moreover $u^{-1}(\alpha, 1] \cap v^{-1}(\alpha, 1] = \phi$. For if $z \in u^{-1}(\alpha, 1] \cap v^{-1}(\alpha, 1]$, then $z \in u^{-1}(\alpha, 1]$, $z \in v^{-1}(\alpha, 1]$ implies $u(z) > \alpha$ and $v(z) > \alpha$ implies $(u \cap v)(z) > \alpha$, a contradiction as $(u \cap v)(z) \leq \alpha$. Therefore (X, T^*) is T_2 – space.

Thus it is seen that α – T_2 (p) is a good extension of its topological counter part (p = i, ii, iii, iv).

Theorem 4.11: Let (X, t^*) be a supra fuzzy topological space, $\alpha \in I_1$ and $I_\alpha(t^*) = \{u^{-1}(\alpha, 1] : u \in t^*\}$. Then

(a) (X, t^*) is an α -T_2(ii)-space implies $(X, I_\alpha(t^*))$ is T_2 -space.

(b) (X, t^*) is an α -T_2(iii)-space implies $(X, I_\alpha(t^*))$ is T_2 -space.

(c) (X, t^*) is an α -T_2(iv)-space if and only if $(X, I_\alpha(t^*))$ is T_2 -space

Proof: (a) Let (X, t^*) be a supra fuzzy topological space and (X, t^*) be α - T_2(ii). Suppose x and y be any two distinct elements in X. Since (X, t^*) is α -T_2(ii), for $\alpha \in I_1$, there exist u, v $\in t^*$ such that u(x) = 1, v(y) = 1 and $u \cap v \le \alpha$. But for every $\alpha \in I_1$, $u^{-1}(\alpha, 1]$, $v^{-1}(\alpha, 1] \in I_\alpha(t^*)$ and $y \in v^{-1}(\alpha, 1]$, $x \in u^{-1}(\alpha, 1]$ and $u^{-1}(\alpha, 1] \cap v^{-1}(\alpha, 1] = \phi$ as $u \cap v \le \alpha$. We have that $(X, I_\alpha(t^*))$ is T_2 – space.

(b) Let (X, t^*) be α - T_2(iii). Then for any x and y be two distinct elements in X, there exist u, v $\in t^*$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v = 0$, for $\alpha \in I_1$. But for every $\alpha \in I_1$, $u^{-1}(\alpha, 1]$, $v^{-1}(\alpha, 1] \in I_\alpha(t^*)$ and $y \in v^{-1}(\alpha, 1]$, $x \in u^{-1}(\alpha, 1]$ and $u^{-1}(\alpha, 1] \cap v^{-1}(\alpha, 1] = \phi$ as $u \cap v = 0$. We have that $(X, I_\alpha(t^*))$ is T_2 – space.

(c) Suppose (X, t^*) is α -T_2(iv)- space . We have to prove that $(X, I_\alpha(t^*))$ is T_2 -space. Suppose x and y are two distinct elements in X. Since (X, t^*) is α -T_2(iv)- space, for $\alpha \in I_1$, there are u, v $\in t^*$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v \le \alpha$. But for every $\alpha \in I_1$, $u^{-1}(\alpha, 1]$, $v^{-1}(\alpha, 1] \in I_\alpha(t^*)$. So we have $x \in u^{-1}(\alpha, 1]$, $y \in v^{-1}(\alpha, 1]$ and $u^{-1}(\alpha, 1] \cap v^{-1}(\alpha, 1] = \phi$, as $u \cap v \le \alpha$. Hence it is clear that $(X, I_\alpha(t^*))$ is T_2 – space.

Conversely, suppose that $(X, I_\alpha(t^*))$ is T_2 – space. Let x, y $\in$ X with $x \neq y$. Since $(X, I_\alpha(t^*))$ is T_2-space, there are $U, V \in I_\alpha(t^*)$, such that $x \in U$, $y \in V$ and $U \cap V = \phi$. Again since $U, V \in I_\alpha(t^*)$, so we get $u, v \in t^*$ such that $U = u^{-1}(\alpha, 1]$, $V = v^{-1}(\alpha, 1]$. This implies that $u(x) > \alpha$, $v(y) > \alpha$ and $u^{-1}(\alpha, 1] \cap v^{-1}(\alpha, 1] = \phi$ $\Rightarrow (u \cap v)^{-1}(\alpha, 1] = \phi$, i.e., $u \cap v \le \alpha$. So, we see that (X, t^*) is α -T_2(iv).

This completes the proof.

Example 4.12: Suppose that X={x, y} and u, v $\in t^*$ defined by u(x) = 0.82, u(y) = 0.21, v(x) = 0.31 and v(y) = 0.74. Suppose the supra fuzzy topology t^* on X generated by {0, u, v, 1, Constants}. Then by definition, for $\alpha = 0.53$, (X, t^*) is not α -T_2 (ii). Now

$I_\alpha(t^*) = \{X, \phi, \{x\}, \{y\}\}$. Then we can easily show that $I_\alpha(t^*)$ is a topology on X and $(X, I_\alpha(t^*))$ is T_2 space.

Similarly, we can easily prove that (X, t^*) is α -T_2(iii)-space implies $(X, I_\alpha(t^*))$ is T_2 -space.

Theorem 4.13: Let (X, t^*) be a supra fuzzy topological space, $A \subseteq X$, and $t^*_A = \{u/A : u \in t^*\}$, then

(a) (X, t^*) is T_2 (i) implies (A, t^*_A) is T_2 (i).

(b) (X, t^*) is α - T_2 (ii) implies (A, t^*_A) is α - T_2 (ii).

(c) (X, t^*) is α - T_2 (iii) implies (A, t^*_A) is α - T_2 (iii).

(d) (X, t^*) is α–T_2 (iv) implies (A, t^*_A) is α–T_2 (iv).

Proof: **(a)** Suppose that (X, t^*) be a supra fuzzy topological space and (X, t^*) is T_2(i) space. Let $x, y \in A$ with $x \neq y$, so that $x, y \in X$, as $A \subseteq X$. Since (X, t^*) is T_2 (i) , for $\alpha \in I_1$, there exist $u, v \in t^*$ such that $u(x) = 1 = v(y)$ and $u \cap v = 0$. For $A \subseteq X$, we have $u/A, v/A \in t^*_A$ and $(u/A)(x) = 1 = (v/A)(y)$ and $u/A \cap v/A = 0$. Hence by definition, (A, t^*_A) is T_2(i).

(b) Suppose that (X, t^*) be a supra fuzzy topological space and (X, t^*) is α - T_2(ii)-space. Let $x, y \in A$ with $x \neq y$, so that $x, y \in X$, as $A \subseteq X$. Since (X, t^*) is α - T_2 (ii) , for $\alpha \in I_1$, there exist $u, v \in t^*$ such that $u(x) = 1 = v(y)$ and $u \cap v \leq \alpha$. For $A \subseteq X$, we have u/A, $v/A \in t^*_A$ and $(u/A)(x) = 1 = (v/A)(y)$ and $u/A \cap v/A \leq \alpha$. Hence by definition, (A, t^*_A) is α -T_2(ii).

(c) Suppose that (X, t^*) is α - T_2(iii)-space. Then for $x, y \in A$ with $x \neq y$, so that $x, y \in X$, as $A \subseteq X$. and for $\alpha \in I_1$, there exist $u, v \in t^*$ such that $u(x) > \alpha$, $v(y) > \alpha$

and $u \cap v = 0$. For $A \subseteq X$, we have $u / A, v / A \in t^*_A$ and $(u / A)(x) > \alpha$, $(v /A)(y) > \alpha$ and $u/A \cap v/A = 0$. Hence by definition, (A, t^*_A) is α-T_2(iii).

(d) Suppose that (X, t^*) be a supra fuzzy topological space and (X, t^*) is α - T_2(iv)-space. Let $x, y \in A$ with $x \neq y$, so that $x, y \in X$, as $A \subseteq X$. Since (X, t^*) is α - T_2 (iii), for $\alpha \in I_1$, there exist $u, v \in t^*$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v \leq \alpha$. For $A \subseteq X$, we have $u / A, v / A \in t^*_A$ and $(u / A)(x) > \alpha$, $(v /A)(y) > \alpha$ and $u/A \cap v/A \leq \alpha$. Hence by definition, (A, t^*_A) is α-T_2(iv).

Theorem 4.14: Suppose that (X_i, t^*_i), $i \in \Lambda$ be supra fuzzy topological spaces and $X = \Pi_{i \in \Lambda} X_i$ and t^* be the product supra fuzzy topology on X, then

(a) $\forall\, i \in \Lambda$, (X_i, t^*_i) is T_2(i) if and only if (X, t^*) is T_2 (i).

(b) $\forall\, i \in \Lambda$, (X_i, t^*_i) is α - T_2(ii) if and only if (X, t^*) is α - T_2 (ii).

(c) $\forall\, i \in \Lambda$, (X_i, t^*_i) is α - T_2(iii) if and only if (X, t^*) is α - T_2 (iii).

(d) $\forall\, i \in \Lambda$, (X_i, t^*_i) is α - T_2(iv) if and only if (X, t^*) is α - T_2 (iv).

Proof: (b) Suppose $\forall\, i \in \Lambda$, (X_i, t^*_i) be an α-T_2(ii). Let x, y be two distinct points in $X = \Pi_{i \in \Lambda} X_i$, then there exist an $x_i \neq y_i$ in X_i. Thus, for $\alpha \in I_1$, there exist u_i, $v_i \in t^*_i$ such that $u_i(x_i) = 1 = v_i(y_i)$ and $u_i \cap v_i \leq \alpha$. But we have $\pi_i(x) = x_i$, $\pi_i(y) = y_i$, then $u_i(\pi_i(x)) = 1 = v_i(\pi_i(y))$ and $(u_i \cap v_i) \circ \pi_i \leq \alpha$. Hence $(u_i \circ \pi_i)(x) = 1 = (v_i \circ \pi_i)(y)$ and $(u_i \circ \pi_i) \cap (v_i \circ \pi_i) \leq \alpha$. Put $u = u_i \circ \pi_i$, $v = v_i \circ \pi_i$, then $u, v \in t^*$ with $u(x) = 1 = v(y)$ and $u \cap v \leq \alpha$. Hence by definition, (X, t^*) is α - T_2 (ii).

Conversely, suppose that (X, t^*) is α - T_2 (ii). For some $i \in \Lambda$, let a_i be a fixed element in X_i. Suppose that $Ai = \{x \in X = \Pi_{i \in \Lambda} X_i : x_j = a_j \text{ for some } i \neq j\}$. Then A_i is a subsets of X and therefore (A_i, t^*_{Ai}) is a subspace of (X, t^*). Since (X, t^*) is α - T_2 (ii) space.

Then, we have also (A_i, t^*_{Ai}) is α - T_2(ii) space. Furthermore, A_i is homeomorphic image of X_i. Hence, by definition, we have (Xi, t^*_i) is α -T_2(ii) space.

(c) Suppose $\forall\ i \in \Lambda, (X_i, t^*_i)$ be an α -T_2(iii) . Let x, y be two distinct points in $X = \Pi_{i\in\Lambda} X_i$, then there exist an $x_i \neq y_i$ in X_i. Since (X_i, t^*_i) is an α-T_2(iii), for $\alpha \in I_1$, there exist $u_i, v_i \in t^*_i$ such that $u_i(x_i) > \alpha$, $v_i(y_i) > \alpha$ and $u_i \cap v_i = 0$. But we have $\pi_i(x) = x_i$, $\pi_i(y) = y_i$, then $u_i(\pi_i(x)) > \alpha$, $v_i(\pi_i(y)) > \alpha$ and $(u_i \cap v_i) \circ \pi_I = 0$. Hence $(u_i \circ \pi_i)(x) > \alpha$, $(v_i \circ \pi_i)(y) > \alpha$ and $(u_i \circ \pi_i) \cap (v_i \circ \pi_i) = 0$. Put $u = u_i \circ \pi_i$, $v = v_i \circ \pi_i$, then $u, v \in t^*$ with $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v = 0$. Hence, we have (X, t^*) is α - T_2 (iii).

Conversely, suppose that (X, t^*) is α - T_2 (iii). For some $i \in \Lambda$, let a_i be a fixed element in Xi. Suppose that $Ai = \{x \in X = \Pi_{i\in\Lambda} X_i : x_j = a_j \text{ for some } i \neq j\}$. Then A_i is a subsets of X and therefore (A_i, t^*_{Ai}) is a subspace of (X, t^*). Since (X, t^*) is α - T_2 (iii) space. Then, we have also (A_i, t^*_{Ai}) is α - T_2(iii) space. Furthermore, A_i is homeomorphic image of X_i. Hence by definition, we have (Xi, t^*_i) is α -T_2 (iii) space.

(d) Suppose $\forall\ i \in \Lambda, (X_i, t^*_i)$ be an α -T_2(iv) . Let x , y be two distinct points in $X = \Pi_{i\in\Lambda} X_i$, then there exist an $x_i \neq y_i$ in X_i. Since (X_i, t^*_i) is an α-T_2(iv), for $\alpha \in I_1$, $\exists\ u_i, v_i \in t^*_i$ such that $u_i(x_i) > \alpha$, $v_i(y_i) > \alpha$ and $u_i \cap v_i \leq \alpha$. But we have $\pi_i(x) = x_i$, $\pi_i(y) = y_i$, then $u_i(\pi_i(x)) > \alpha$, $v_i(\pi_i(y)) > \alpha$ and $(u_i \cap v_i) \circ \pi_i \leq \alpha$. Hence $(u_i \circ \pi_i)(x) > \alpha$, $(v_i \circ \pi_i)(y) > \alpha$ and $(u_i \circ \pi_i) \cap (v_i \circ \pi_i) \leq \alpha$. Put $u = u_i \circ \pi_i$, $v = v_i \circ \pi_i$, then $u, v \in t^*$ with $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v \leq \alpha$. Hence we have (X, t^*) is α - T_2 (iv).

Conversely, suppose that (X, t^*) is α - T_2 (iv). For some $i \in \Lambda$, let a_i be a fixed element in X_i. Suppose that $Ai = \{x \in X = \Pi_{i \in \Lambda} X_i : x_j = a_j \text{ for some } i \neq j\}$. Then A_i is a subsets of X and therefore (A_i, t^*_{Ai}) is a subspace of (X, t^*). Since (X, t^*) is α - T_2 (iv) space. Then, we have also (A_i, t^*_{Ai}) is α - T_2(iv) space. Furthermore, A_i is homeomorphic image of X_i. Hence by definition, we have (X_i, t^*_i) is α -T_2(iv) space.

Similarly, (a) can be proved.

Theorem 4.15: Let (X, t^*) and (Y, s^*) be two supra fuzzy topological spaces and $f: X \longrightarrow Y$ be a one-one, onto and open map, then

(a) (X, t^*) is T_2 (i) implies (Y, s^*) is T_2 (i).

(b) (X, t^*) is α -T_2 (ii) implies (Y, s^*) is α- T_2 (ii).

(c) (X, t^*) is α -T_2 (iii) implies (Y, s^*) is α- T_2 (iii).

(d) (X, t^*) is α -T_2 (iv) implies (Y, s^*) is α -T_2 (iv).

Proof: (b) Let (X, t^*) be α -T_2 (ii). We have to prove that (Y, s^*) is α -T_2(ii). Let $y_1, y_2 \in Y$ with $y_1 \neq y_2$. Since f is onto, there exist $x_1, x_2 \in X$ with $f(x_1) = y_1$, $f(x_2) = y_2$ and also $x_1 \neq x_2$ as f is one-one. Again since (X, t^*) is α-T_2(i), for $\alpha \in I_1$, there exist $u, v \in t^*$ such that $u(x_1) = 1 = v(x_1)$ and $u \cap v \leq \alpha$.

$$\text{Now, } f(u)(y_1) = \{\text{Sup } u(x_1) \quad : f(x_1) = y_1\}$$

$$= 1$$

$$f(v)(y_2) = \{\text{Sup } v(x_2) \quad : f(x_2) = y_2\}$$

$$= 1$$

$$\text{and } f(u \cap v)(y_1) = \{\text{Sup } (u \cap v)(x_1) \quad : f(x_1) = y_1\}$$

$$f(u \cap v)(y_2) = \{\text{Sup } (u \cap v)(x_2) \quad : f(x_2) = y_2\}$$

Hence $f(u \cap v) \leq \alpha \Rightarrow f(u) \cap f(v) \leq \alpha$.

Since f is open then f(u), f(v) $\in s^*$ as u, v $\in t^*$. We observe that there exist f(u), f(v) $\in s^*$ such that $f(u)(y_1) = 1$, $f(v)(y_2) = 1$ and $f(u) \cap f(v) \leq \alpha$. Hence by definition, (Y, s^*) is α - T_2(ii).

(c) Let (X, t^*) be α -T_2(iii). Then for $y_1, y_2 \in Y$ with $y_1 \neq y_2$, there exist $x_1, x_2 \in X$ with $f(x_1) = y_1$, $f(x_2) = y_2$, since f is onto, and also $x_1 \neq x_2$ as f is one-one. Again, since (X, t^*) is α-T_2(iii), for $\alpha \in I_1$, there exist u, v $\in t^*$ such that $u(x_1) > \alpha$, $v(x_1) > \alpha$ and $u \cap v = 0$.

Now, $f(u)(y_1) = \{ \text{Sup } u(x_1) \ : f(x_1) = y_1 \}$

$> \alpha$

$f(v)(y_2) = \{ \text{Sup } v(x_2) \ : f(x_2) = y_2 \}$

$> \alpha$

and $f(u \cap v)(y_1) = \{ \text{Sup } (u \cap v)(x_1) \ : f(x_1) = y_1 \}$

$f(u \cap v)(y_2) = \{ \text{Sup } (u \cap v)(x_2) \ : f(x_2) = y_2 \}$

Hence $f(u \cap v) = 0 \Rightarrow f(u) \cap f(v) = 0$.

Since f is open then f(u), f(v) $\in s^*$ as u, v $\in t^*$. We observe that there exist f(u), f(v) $\in s^*$ such that $f(u)(y_1) > \alpha$, $f(v)(y_2) > \alpha$ and $f(u) \cap f(v) = 0$. Hence by definition, (Y, s^*) is α - T_2(iii).

Similarly, (a) and (d) can be proved.

Theorem 4.16: Let (X, t^*) and (Y, s^*) be two supra fuzzy topological spaces and $f : X \longrightarrow Y$ be continuous and one-one map, then

(a) (Y, s^*) is T_2 (i) implies (X, t^*) is T_2 (i).

(b) (Y, s^*) is α- T_2 (ii) implies (X, t^*) is α - T_2 (ii).

(c) (Y, s^*) is α- T_2(iii) implies (X, t^*) is α - T_2(iii).

(d) (Y, s^*) is α- T_2(iv) implies (X, t^*) is α- T_2(iv).

Proof: **(b)** Let (Y, s^*) be α - T_2(ii). Let $x_1, x_2 \in X$ with $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$ in Y, since f is one-one. Also since (Y, s^*) is α-T_2(ii), for $\alpha \in I_1$, there exist $u, v \in s^*$ such that $u(f(x_1)) = 1 = v(f(x_2))$ and $u \cap v \leq \alpha$. This implies that $f^{-1}(u)(x_1) = 1$, $f^{-1}(v)(x_2) = 1$ and $f^{-1}(u \cap v) \leq \alpha$, i.e., $f^{-1}(u) \cap f^{-1}(v) \leq \alpha$, since $u, v \in s$ and f is continuous then $f^{-1}(u)$, $f^{-1}(v) \in t^*$. Now it is clear that there exist $f^{-1}(u)$, $f^{-1}(v) \in t^*$ such that $f^{-1}(u)(x_1) = 1$, $f^{-1}(v)(x_2) = 1$ and $f^{-1}(u) \cap f^{-1}(v) \leq \alpha$. Hence (X, t^*) is α -T_2 (ii).

(c) Let (Y, s^*) be α - T_2 (iii). Then for $x_1, x_2 \in X$ with $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$ in Y, since f is one-one. Also since (Y, s^*) is α-T_2(iii), for $\alpha \in I_1$, there exist $u, v \in s^*$ such that $u(f(x_1)) > \alpha$, $v(f(x_2)) > \alpha$ and $u \cap v = 0$. This implies that $f^{-1}(u)(x_1) > \alpha$, $f^{-1}(v)(x_2) > \alpha$ and $f^{-1}(u \cap v) = 0$, i.e., $f^{-1}(u) \cap f^{-1}(v) = 0$, since $u, v \in s$ and f is continuous then $f^{-1}(u)$, $f^{-1}(v) \in t^*$. Thus there exist $f^{-1}(u)$, $f^{-1}(v) \in t^*$ such that $f^{-1}(u)(x_1) > \alpha$, $f^{-1}(v)(x_2) > \alpha$ and $f^{-1}(u) \cap f^{-1}(v) = 0$. Hence (X, t^*) is α -T_2 (iii).

Similarly, (a) and (d) can be proved.

CHAPTER 5

Regular Spaces in Supra Fuzzy Topology

5. 1 Introduction

We introduce and study supra fuzzy regular spaces and we establish some relationships among them in this chapter. We also study some other properties of these concepts and obtain their several features.

Definition 5.2: Let (X, t) be a fuzzy topological space and t^* be associated supra topology with t and $\alpha \in I_1$. Then

(a) (X, t^*) is an α - SFR (i) space if and only if for all $w \in (t^*)^c$ with $w(x) < 1$, $\forall\, x \in X$, there exist $u, v \in t^*$ such that $u(x) = 1$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$.

(b) (X, t^*) is an α -SFR(ii) space if and only if for all $w \in (t^*)^c$ with $w(x) < 1$, $\forall\, x \in X$, there exist $u, v \in t^*$ such that $u(x) > \alpha$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$.

(c) (X, t^*) is an α -SFR(iii) space if and only if for all $w \in (t^*)^c$ with $w(x) = 0$, $\forall\, x \in X$, there exist $u, v \in t^*$ such that $u(x) = 1$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$.

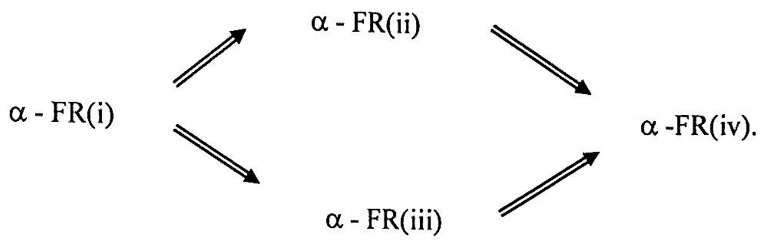
(d) (X, t^*) is an α -SFR(iv) space if and only if for all $w \in (t^*)^c$ with $w(x) = 0$, $\forall\, x \in X$, there exist $u, v \in t^*$ such that $u(x) > \alpha$, $v(y) = 1$, $y \in w^{-1}\{1\}$, $u \cap v \leq \alpha$.

Theorem 5.3: Let (X, t) be a fuzzy topological space and t^* be associated supra topology with t. Then the following implications are true:

(a) (X, t^*) is α - SFR(i) implies (X, t^*) is α - SFR(ii) implies (X, t^*) is α - SFR(iv).

(b) (X, t^*) is α - SFR(i) implies (X, t^*) is α - SFR(iii) implies (X, t^*) is α - SFR(iv).

Also, these can be shown in a diagram as follows:



Proof: First, suppose that (X, t^*) is α - SFR (i). We have to prove that (X, t^*) is α - SFR(ii). Let $w \in (t^*)^c$ with $w(x) < 1, x \in X$. Since (X, t^*) is α -SFR(i), for $\alpha \in I_1$, there exist $u, v \in t^*$ such that $u(x) = 1$, $v(y) = 1, y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$. Now, we see that $u(x) > \alpha$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$. Hence by definition (X, t^*) is α -SFR (ii).

Suppose that (X, t^*) is α - SFR (ii). Let $w \in (t^*)^c$ with $w(x) < 1, x \in X$, then for $\alpha \in I_1$, there exist $u, v \in t^*$ such that $u(x) > 1$, $v(y) = 1, y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$. Now, we see that $u(x) > \alpha$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$. Hence by definition (X, t^*) is α -SFR (iv).

In the same way, we can prove that

$$(X, t^*) \text{ is } \alpha\text{ - SFR(i)} \Rightarrow (X, t^*) \text{ is } \alpha\text{ - SFR(iii)}.$$

$$(X, t^*) \text{ is } \alpha\text{ - SFR(iii)} \Rightarrow (X, t^*) \text{ is } \alpha\text{ - SFR(iv)}.$$

Now, we give some examples to show the non implication among α- SFR (i), α - SFR (ii), α- SFR (iii) and α - SFR (iv).

Example 5.4: Let $X = \{x, y\}$ and $u, v \in I^X$ are defined by $u(x) = 0.9$, $u(y) = 0$, $v(x) = 0.5$ and $v(y) = 1$. Consider the supra fuzzy topology t^* on X generated by {0, u, v, 1, constants}. Then for $w = 1 - u$ and $\alpha = 0.7$, we see that (X, t^*) is α -SFR(ii) but (X, t^*) is not α-SFR(i).

Example 5.5: Let $X = \{x, y\}$ and $u, v \in I^X$ are defined by $u(x) = 0.2$, $u(y) = 0.3$, $v(x) = 0.3$, $v(y) = 0.2$. Consider the supra fuzzy topology t^* on X generated by {0, u, v, 1, Constants}. Then for $w = 1 - u$ and $\alpha = 0.5$, we see that (X, t^*) is α - SFR(iii) and (X, t^*) is α -SFR(iv), but (X, t^*) is not α - SFR(ii) as they do not exist any $u, v \in t^*$ such that $u(x) > \alpha$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$.

Example 5.6: Let $X = \{x, y\}$ and $u, v, w \in I^X$ are defined by $u(x) = 0.9$, $u(y) = 0$, $v(x) = 0.5$, $v(y) = 1$, $w(x) = 1$, $w(y) = 0$. Consider the supra fuzzy topology t^* on X generated by $\{0, u, v, w, 1, \text{Constants}\}$. Then for $\alpha = 0.6$ and $p = 1 - w$, it is seen that (X, t^*) is α - SFR(iv) but (X, t^*) is not α - SFR(iii).

This completes the proof.

Theorem 5.7: If $\alpha, \beta \in t^*$ with $0 \leq \alpha \leq \beta < 1$, then

(a) (X, t^*) is α -SFR(i) implies (X, t^*) is β - SFR(i) .

(b) (X, t^*) is α - SFR(iii) implies (X, t^*) is β - SFR(iii).

Proof: Suppose that (X, t^*) is α - SFR (i). We have to prove that (X, t^*) is β - SFR (i). Let $w \in (t^*)^c$ and $x \in X$ with $w(x) < 1$. Since (X, t^*) is α - SFR(i) , for $\alpha \in I_1$, there exist $u, v \in t^*$ such that $u(x) = 1$, $v(y) = 1$, $y \in w^{-1}\{1\}$ and $u \cap v \leq \alpha$. Since $\alpha \leq \beta$, then $u \cap v \leq \beta$. So it is observed that (X, t^*) is β - SFR (i).

Example 5.8: Let $X = \{x, y\}$ and $u, v \in I^X$ are defined by $u(x) = 1$, $u(y) = 0$, $v(x) = 0.7$, $v(y) = 1$. Consider the supra fuzzy topology t^* on X generated by {0, u, v,1, constants }. Then for $w = 1 - u$, $\alpha = 0.75$, $\beta = 0.6$. We see that (X, t^*) is β - SFR(i) but (X, t^*) is not α - SFR(i) .

In the same way, we can prove that

(X, t^*) is α - SFR(iii) implies (X, t^*) is β - SFR(iii).

Theorem 5.9: Let (X, t^*) be a supra fuzzy topological space and $I_\alpha(t^*) = \{ u^{-1}(\alpha, 1] : u \in t^* \}$, then (X, t^*) is 0 – SFR (i) implies $(X, I_0(t^*))$ is Regular.

Proof: Suppose (X, t^*) be a 0 –SFR (i). We have to prove that $(X, I_0(t^*))$ is Regular. Let V be a closed set in $I_0(t^*)$ and $x \in X$ such that $x \notin V$. Then $V^c \in I_0(t^*)$ and $x \in V^c$. So, by the definition of $I_0(t^*)$, there exists an $u \in t^*$ such that $V^c = u^{-1}(0, 1]$, i.e., $u(x) > 0$. Since $u \in t^*$, then u^c is closed supra fuzzy set in t^* and $u^c(x) < 1$. Since (X, t^*) is 0 –SFR(i), there exist $v, w \in t^*$ such that $v(x) = 1$, $w \geq 1_{(u^c)^{-1}\{1\}}$, $v \cap w = 0$.

(a) Since $v, w \in t^*$ then $v^{-1}(0, 1], w^{-1}(0, 1] \in I_0(t^*)$ and $x \in v^{-1}(0, 1]$

(b) Since $w \geq 1_{(u^c)^{-1}\{1\}}$ then $w^{-1}(0, 1] \supseteq \left(1_{(u^c)^{-1}\{1\}}\right)^{-1}(0, 1]$.

(c) And $v \cap w = 0$, mean $(v \cap w)^{-1}(0, 1] = v^{-1}(0, 1] \cap w^{-1}(0, 1] = \phi$.

Now, we have

$$\begin{aligned}
\left(1_{(u^c)^{-1}\{1\}}\right)^{-1}(0, 1] &= \{ x : 1_{(u^c)^{-1}\{1\}}(x) \in (0, 1] \} \\
&= \{x : 1_{(u^c)^{-1}\{1\}}(x) = 1 \} \\
&= \{ x : x \in (u^c)^{-1}\{1\} \} \\
&= \{ x : u^c(x) = 1 \} \\
&= \{ x : u(x) = 0 \} \\
&= \{ x : x \notin V^c \} \\
&= \{ x : x \in V \} \\
&= V.
\end{aligned}$$

Put $W = v^{-1}(0, 1]$ and $W^* = w^{-1}(0, 1]$, then $x \in W$, $W^* \supseteq V$ and $W \cap W^* = \phi$. Hence it is clear that $(X, I_0(t^*))$ is Regular.

Theorem 5.10: Let (X, t^*) be a supra fuzzy topological space $A \subseteq X$, and $t^*_A = \{ u/A : u \in t^* \}$, then $1_{((u/A)^c)^{-1}\{1\}}(x) = (1_{(u^c)^{-1}\{1\}}/A)(x)$.

Proof: Let w be a closed supra fuzzy set in t^*_A, i.e., $w \in {t^*_A}^c$, then $u/A = w^c$, where $u \in t^*$.

Now, we have

$$1_{((u/A)^c)^{-1}\{1\}}(x) = \begin{cases} 0 & if \ x \notin ((u/A)^c)^{-1}\{1\} \\ 1 & if \ x \in ((u/A)^c)^{-1}\{1\} \end{cases}$$

$$= \begin{cases} 0 & if \ x \notin \{y : (u/A)^c(y) = 1\} \\ 1 & if \ x \in \{y : (u/A)^c(y) = 1\} \end{cases}$$

$$= \begin{cases} 0 & if \ (u/A)^c(x) < 1 \\ 1 & if \ (u/A)^c(x) = 1 \end{cases}$$

$$= \begin{cases} 0 & if \ w(x) < 1 \\ 1 & if \ w(x) = 1 \end{cases}$$

Again, $$1_{(u^c)^{-1}\{1\}}(x) = \begin{cases} 0 & if \ x \notin (u^c)^{-1}\{1\} \\ 1 & if \ x \in (u^c)^{-1}\{1\} \end{cases}$$

$$= \begin{cases} 0 & if \ x \notin \{y : u^c(y) = 1\} \\ 1 & if \ x \in \{y : u^c(y) = 1\} \end{cases}$$

$$= \begin{cases} 0 & if \ u^c(x) < 1 \\ 1 & if \ u^c(x) = 1 \end{cases}$$

Now, $$(1_{(u^c)^{-1}\{1\}}/A)(x) = \begin{cases} 0 & if \ (u^c/A)(x) < 1 \\ 1 & if \ (u^c/A)(x) = 1 \end{cases}$$

$$= \begin{cases} 0 & if \ (u/A)^c(x) < 1 \\ 1 & if \ (u/A)^c(x) = 1 \end{cases}$$

$$= \begin{cases} 0 & if \quad w(x) < 1 \\ 1 & if \quad w(x) = 1 \end{cases}$$

Hence it is clear that $1_{((u/_A)^c)^{-1}\{1\}}(x) = (1_{(u^c)^{-1}\{1\}} / A)(x)$.

Theorem 5.11: Let (X, t^*) be a supra fuzzy topological space and $A \subseteq X$ and $t^*_A = \{ u/A : u \in t^* \}$, then

(a) (X, t^*) is α -SFR (i) implies (A, t^*_A) is α - SFR (i).

(b) (X, t^*) is α -SFR (ii) implies (A, t^*_A) is α - SFR (ii).

(c) (X, t^*) is α -SFR (iii) implies (A, t^*_A) is α - SFR (iii).

(d) (X, t^*) is α -SFR(iv) implies (A, t^*_A) is α - SFR(iv).

Proof: Let (X, t^*) be α - SFR (i). We have to prove that (A, t^*_A) is α -SFR (i). Let w be a closed fuzzy set in t^*_A, and $x^* \in A$ such that $w(x^*) < 1$. This implies that $w^c \in t^*_A$ and $w^c(x^*) > 0$. So there exists an $u \in t^*$ such that $u/A = w^c$ and clearly u^c is closed in t^* and $u^c(x^*) = (u/A)^c(x^*) = w(x^*) < 1$, i.e., $u^c(x^*) < 1$. Since (X, t^*) is α -SFR (i), for $\alpha \in I_1$, there exist $v, v^* \in t^*$ such that $v(x^*) = 1$, $v^* \geq 1_{(u^c)^{-1}\{1\}}$ and $v \cap v^* \leq \alpha$. Since $v, v^* \in t^*$, then v/A, $v^*/A \in t^*_A$ and $v/A(x^*) = 1$, $v^*/A \geq (1_{(u^c)^{-1}\{1\}}/A)$ and $v/A \cap v^*/A$ $= (v \cap v^*)/A \leq \alpha$.

But $1_{(u^c)^{-1}\{1\}}/A = 1_{((u/_A)^c)^{-1}\{1\}} = 1_{w^{-1}\{1\}}$, then $v^*/A \geq 1_{w^{-1}\{1\}}$. Hence it is clear that (A, t^*_A) is α - SFR (i).

The proofs of (b), (c) and (d) are similar.

Theorem 5.12: Let (X, T) be a topological space and T^* be associated supra topology with T. Consider the following statements:

1) (X, T^*) is a Regular space.

2) (X, ω (T^*)) is α -SFR (i).

3) (X, ω (T^*)) is α -SFR (ii).

4) (X, ω (T^*)) is α -SFR (iii).

5) (X, ω (T^*)) is α -SFR (iv).

Then the following are true:

(a) (1) implies (2) implies (3) implies (5) implies (1),

(b) (1) implies (2) implies (4) implies (5) implies (1).

Proof: First, suppose that (X, T^*) be regular space. We shall prove that (X, ω (T^*)) is α -SFR (i). Let w be a closed supra fuzzy set in ω (T^*) and $x \in X$ such that $w(x) < 1$, then $w^c \in \omega(T^*)$ and $w^c(x) > 0$. Now we have $(w^c)^{-1}(0,1] \in T^*$, $x \in (w^c)^{-1}(0,1]$. Also it is clear that $[(w^c)^{-1}(0,1]]^c = w^{-1}\{1\}$ be a closed in T^* and $x \notin w^{-1}\{1\}$. Since (X, T^*) is Regular , then there exist $V, V^* \in T$ such that $x \in V$, $V^* \supseteq w^{-1}\{1\}$ and $V \cap V^* = \phi$. But by the definition of lower semi continuous functions $1_V, 1_{V^*} \in \omega(T^*)$ and $1_V(x) = 1$, $1_{V^*} \supseteq 1_{w^{-1}\{1\}}$, $1_V \cap 1_{V^*} = 1_{V \cap V^*} = 0$. Put $u = 1_V$ and $v = 1_{V^*}$, then, we have $u(x) = 1$, $v \supseteq 1_{w^{-1}\{1\}}$ and $u \cap v \leq \alpha$. Hence (X, ω (T^*)) is α - SFR (i).

We can easily be shown that (2) implies (3), (3) implies (5), (2) implies (4), (4) implies (5).

We therefore prove that (5) implies (1).

Let (X, ω (T^*)) be α - SFR (iv). Let $x \in X$ and V be a closed set in T^*, such that $x \notin V$. This implies that $V^c \in T^*$ and $x \in V^c$. But from the definition of $\omega(T^*)$, $1_{V^c} \in \omega(T^*)$, and $(1_{V^c})^c = 1_V$ closed in $\omega(T^*)$ and $1_V(x) = 0$. Since (X , ω (T^*)) is α - SFR (iv) , for $\alpha \in I_1$, there exist $u, v \in \omega(T^*)$ such that $u(x) > \alpha$, $v \geq 1_{(1_V)^{-1}\{1\}} = 1_V$ and $u \cap v \leq \alpha$.

Since u , v $\in \omega(T^*)$, then $u^{-1}(\alpha, 1], v^{-1}(\alpha, 1] \in T^*$ and $x \in u^{-1}(\alpha, 1]$. Since $v \geq 1_V$, then $v^{-1}(\alpha, 1] \supseteq (1_V)^{-1}(\alpha, 1] = V$, and $u \cap v \leq \alpha$ implies $(u \cap v)^{-1}(\alpha, 1] = u^{-1}(\alpha, 1] \cap v^{-1}(\alpha, 1] = \phi$. Then by definition, (X, T^*) is Regular space.

Thus it is seen that α - SFR (p) is a good extension of its topological counter part (p = i , ii , iii, iv).

Theorem 5.13: Let (X, t^*) and (Y, s^*) be two supra fuzzy topological spaces and $f : X \longrightarrow Y$ be continuous , one-one , onto and open map, then

(a) (X, t^*) is α - SFR (i) implies (Y, s^*) is α - SFR (i).

(b) (X, t^*) is α - SFR (ii) implies (Y, s^*) is α - SFR (ii).

(c) (X, t^*) is α - SFR (iii) implies (Y, s^*) is α - SFR (iii).

(d) (X, t^*) is α - SFR (iv) implies (Y, s^*) is α - SFR (iv).

Proof: Suppose (X, t^*) be α - SFR (i). For $w \in (s^*)^c$ and $p \in Y$ such that $w(p) < 1$, $f^{-1}(w) \in (t^*)^c$ as f is continuous and $x \in X$ such that $f(x) = p$ as f is one-one and onto. Hence $f^{-1}(w)(x) = w(f(x)) = w(p) < 1$. Since (X, t^*) is α - SFR (i) , for $\alpha \in I_1$, then there exist $u, v \in t^*$ such that $u(x) = 1$, $v(y) = 1$, $y \in \{ f^{-1}(w) \}^{-1}\{1\}$ and $u \cap v \leq \alpha$. This implies that

$$f(u)(p) = \{ \text{Sup } u(x) \ : f(x) = p \} = 1,$$

and $f(v) f(y) = \{ \text{Sup } v(y) \} = 1$ as $f(f^{-1}(w)) \subseteq w \Rightarrow f(y) \in w^{-1}\{1\}$

and $f(u \cap v) \leq \alpha$ as $u \cap v \leq \alpha \Rightarrow f(u) \cap f(v) \leq \alpha$.

Now, it is clear that for every $f(u), f(v) \in s^*$ such that $f(u)(x) = 1$, $f(v)(f(y)) = 1$, $f(y) \in w^{-1}\{1\}$ and $f(u) \cap f(v) \leq \alpha$. Hence (Y, s^*) is α - SFR (i).

Similarly (b), (c) and (d) can be proved.

Theorem 5.14: Let (X, t^*) and (Y, s^*) be two supra fuzzy topological spaces and $f : X \longrightarrow Y$ be a continuous, one-one, onto and closed map then,

(a) (Y, s^*) is α - SFR (i) implies (X, t^*) is α - SFR (i).

(b) (Y, s^*) is α - SFR (ii) implies (X, t^*) is α - SFR (ii).

(c) (Y, s^*) is α - SFR (iii) implies (X, t^*) is α - SFR (iii).

(d) (Y, s^*) is α - SFR (iv) implies (X, t^*) is α - SFR (iv).

Proof: Suppose (Y, s^*) be α - SFR (i). For $w \in (t^*)^c$ and $x \in X$ with $w(x) < 1$, then $f(w) \in (s^*)^c$ as f is closed and we find $p \in Y$ such that $f(x) = p$ as f is one-one. Now we have $f(w)(p) = \{\text{Sup } w(x) \ : f(x) = p\} < 1$. Since (Y, s^*) is α - SFR (i), for $\alpha \in I_1$, then there exist $u, v \in s^*$ such that $u(f(x)) = 1$, $v(y) = 1$, $y \in (f(w))^{-1}\{1\}$ and $u \cap v \leq \alpha$. This implies that $f^{-1}(u), f^{-1}(v) \in t^*$ as f is continuous and $u, v \in s^*$. Now $f^{-1}(u)(x) = u(f(x)) = u(p) = 1$ and $f^{-1}(v)(q) = v(f(q)) = v(y) = 1$ as $f(q) = y$, $y \in (f(w))^{-1}\{1\}$, i.e., $f(p) \in (f(w))^{-1}\{1\} \Rightarrow q \in w^{-1}\{1\}$ and $f^{-1}(u) \cap f^{-1}(v) = f^{-1}(u \cap v) \leq \alpha$ as $u \cap v \leq \alpha$. Now we observe that $\exists\ f^{-1}(u), f^{-1}(v) \in t^*$ such that $f^{-1}(u)(x) = 1$, $f^{-1}(v)(q) = 1$, $q \in w^{-1}\{1\}$and $f^{-1}(u) \cap f^{-1}(v) \leq \alpha$. Hence (X, t^*) is α -SFR (i).

Similarly, (b), (c) and (d) can be proved.

CHAPTER 6

Normal Spaces in Supra Fuzzy Topology

6. 1 Introduction

In this chapter, we introduce and study normal property in supra fuzzy topological spaces and obtain their several features.

Definition 6.2: Let (X, t^*) be a supra fuzzy topological space and $\alpha \in I_1$. Then

(a) (X, t^*) is α - SFN(i) space if and only if for all $w, w^* \in (t^*)^c$ with $w \cap w^* \leq \alpha$, there exist $u, u^* \in t^*$ such that $u(x) = 1$, for all $x \in w^{-1}\{1\}$, $u^*(y) = 1$, for all $y \in w^{*-1}\{1\}$ and $u \cap u^* = 0$.

(b) (X, t^*) is α - SFN (ii) space if and only if for all $w, w^* \in (t^*)^c$ with $w \cap w^* = 0$, there exist $u, u^* \in t^*$ such that $u(x) = 1$, for all $x \in w^{-1}\{1\}$, $u^*(y) = 1$, for all $y \in w^{*-1}\{1\}$ and $u \cap u^* \leq \alpha$.

(c) (X, t^*) is a α - SFN(iii) space if and only if for all $w, w^* \in (t^*)^c$ with $w \cap w^* \leq \alpha$, there exist $u, u^*, \in t^*$ such that $u(x) = 1$ for all $x \in w^{-1}\{1\}$, $u^*(y) = 1$, for all $y \in w^{*-1}\{1\}$ and $u \cap u^* \leq \alpha$.

Theorem 6.3: The following implications are true:

(a) (X, t^*) is 0 - SFN (i) if and only if (X, t^*) is 0 - SFN (ii).

(b) (X, t^*) is 0 - SFN (ii) if and only if (X, t^*) is 0 -SFN (iii).

Proof: Suppose (X, t^*) be α -SFN (i). We have to show that (X, t^*) is 0 -SFN (ii). Let $w, w^* \in (t^*)^c$ with $w \cap w^* = 0$. This implies that $w \cap w^* \leq 0$. Since (X, t^*) is 0 - SFN(i), for $\alpha \in I_1$, there exist $u, u^* \in t^*$ such that $u(x) = 1$, $\forall\ x \in w^{-1}\{1\}$, $u^*(y) = 1$, $\forall\ y \in w^{*\,-1}\{1\}$ and $u \cap u^* = 0$ .Then we can write it as $u \cap u^* \leq 0$. Hence by definition, (X, t^*) is α - SFN (ii).

Conversely, suppose that (X, t^*) is 0 - SFN (ii). We have to prove that (X, t^*) is - FN (i). Let $w, w^* \in (t^*)^c$ with $w \cap w^* \leq 0$, i.e., $w \cap w^* = 0$. Since (X, t^*) is 0 –FN(ii) , there exist $u, u^* \in t^*$ such that $u(x) = 1$, $\forall\ x \in w^{-1}\{1\}$, $u^*(y) = 1$, $\forall\ y \in w^{*\,-1}$, and $u \cap u^* \leq 0$, i. e., $u \cap u^* = 0$. Hence by definition, (X, t^*) is 0 – SFN (i).

Next, suppose that (X, t^*) is 0 – SFN (ii). We have to prove that (X, t^*) is 0 – SFN (iii). Let $w, w^* \in (t^*)^c$ with $w \cap w^* \leq 0$, i.e., $w \cap w^* = 0$. Since (X, t^*) is 0 – SFN (ii), there exist $u, u^* \in t^*$ such that $u(x) = 1$, $\forall\ x \in w^{-1}\{1\}$, $u^*(y) = 1$, $\forall\ y \in w^{*\,-1}\{1\}$ and $u \cap u^* \leq 0$. Hence by definition, (X, t^*) is α-SFN (iii).

Conversely, suppose that (X, t^*) is 0 –SFN (iii). We have to prove that (X, t^*) is 0 – SFN (ii). Let $w, w^* \in (t^*)^c$ with $w \cap w^* = 0$, i.e., $w \cap w^* \leq 0$. Since (X, t^*) is 0 – SFN (iii), there exist $u, u^* \in t^*$ such that $u(x) = 1$, $\forall\ x \in w^{-1}\{1\}$, $u^*(y) = 1$, $\forall\ y \in w^{*\,-1}\{1\}$ and $u \cap u^* \leq 0$. Hence by definition (X, t^*) is α-SFN (ii).

Theorem 6.4: Let (X, t^*) be a supra topological space and $\alpha, \beta \in t^*$ with $0 \leq \alpha \leq \beta < 1$, then

(a) (X, t^*) is β - SFN (i) implies (X, t^*) is α - SFN (i).

(b) (X, t^*) is α - SFN (ii) implies (X, t^*) is β - SFN (ii).

Proof: First, suppose that (X, t^*) is β - SFN (i). We have to prove that (X, t^*) is α -SFN (i). Let $w, w^* \in (t^*)^c$ with $w \cap w^* \leq \alpha$. Since $0 \leq \alpha \leq \beta < 1$, then $w \cap w^* \leq \beta$. Since (X, t^*) is β - SFN(i), for $\beta \in I_1$, there exist $u, u^* \in t^*$ such that $u(x) = 1, \forall x \in w^{-1}\{1\}$, $u^*(y) = 1$, $\forall y \in w^{*-1}\{1\}$ and $u \cap u^* = 0$. Thus by definition, (X, t^*) is a α - SFN (i).

Next, suppose that (X, t^*) is α - SFN (ii). We have to prove that (X, t^*) is β - SFN (ii). Let $w, w^* \in (t^*)^c$ with $w \cap w^* = 0$. Since (X, t^*) is α -SFN(ii), there exist $u, u^* \in t^*$ such that $u(x) = 1, x \in w^{-1}\{1\}$, $u^*(y) = 1$, $y \in w^{*-1}\{1\}$ and $u \cap u^* \leq \alpha$. Since $0 \leq \alpha \leq \beta < 1$, then $u \cap u^* \leq \beta$. Then we have (X, t^*) is β - SFN (ii).

Theorem 6.5: Let (X, t^*) be a supra fuzzy topological space, and $I_\alpha(t^*) = \{ u^{-1}(\alpha, 1] : u \in t^* \}$, then (X, t^*) is 0 – SFN(iii) space implies $(X, I_0(t^*))$ is Normal space.

Proof: Suppose that (X, t^*) be a α - FN (iii) space. We have to prove that $(X, I_0(t^*))$ is Normal space. Let V, V^* be closed set in $I_0(t^*)$, and $V \cap V^* = \phi$. Then $V^c, V^{*c} \in I_0(t^*)$ and $(V \cap V^*)^c = V^c \cup V^{*c} = X$. Since $V^c, V^{*c} \in I_0(t^*)$, then there exist $u, u^* \in t^*$ such that $V^c = u^{-1}(0, 1]$ and $V^{*c} = u^{*-1}(0, 1]$ and $u^{-1}(0, 1] \cup u^{*-1}(0, 1] = V^c \cup V^{*c} = X$. Hence $(u \cup u^*)^{-1}(0, 1] = X$. Now we find $u^c, u^{*c} \in (t^*)^c$ such that $((u \cup u^*)^{-1}(0, 1])^c = \phi$. This implies that $(u \cup u^*)^c = u^c \cap u^{*c} = 0$. Since (X, t^*) is 0 - SFN(iii), there exist $v, v^* \in t^*$ such that $v \geq 1_{(u^c)^{-1}\{1\}}$, $v^* \geq 1_{((u^*)^c)^{-1}\{1\}}$, $v \cap v^* = 0$. But from the definition of $I_0(t^*)$, $v^{-1}(0, 1]$, $v^{*-1}(0, 1] \in I_0(t^*)$, and we get $v^{-1}(0, 1] \supseteq 1_{(u^c)^{-1}\{1\}}(0, 1]$, $v^{*-1}(0, 1] \supseteq 1_{((u^*)^c)^{-1}\{1\}}(0, 1]$, $(v \cap v^*)^{-1}(0, 1] = \phi$. Put $W = v^{-1}(0, 1]$, $W^* = v^{*-1}(0, 1]$ in

$I_0(t^*)$. Then finally we find, $W \supseteq V$, $W^* \supseteq V^*$ and $W \cap W^* = \phi$. Hence by definition, $(X, I_0(t^*))$ is Normal space.

Theorem 6.6: Suppose (X, T^*) be supra topological space, then consider the following statements:

(a) (X, T^*) be a Normal space

(b) $(X, \omega(T^*))$ be α - SFN (i) space.

(c) $(X, \omega(T^*))$ be α - SFN (ii) space.

(d) $(X, \omega(T^*))$ be α - SFN (iii) space.

Then (a) implies (b); (b) implies (c) and (c) $\Rightarrow$ (a).

Proof: First suppose that (X, T^*) be a Normal space. We have to prove that $(X, w(T^*))$ be α - SFN (i) space. Let w, w^* be closed in $w(T^*)$ and $w \cap w^* \leq \alpha$. Then we have $w^c, w^{*c} \in w(T^*)$ and $(w \cap w^*)^c \geq 1 - \alpha > 0$. But from the definition of $w(T^*)$, $(w^c)^{-1}(0, 1], (w^{*c})^{-1}(0, 1] \in T^*$. Now we see that $((w \cap w^*)^c)^{-1}(0, 1] = X$, and we also see that $((w^c)^{-1}(0, 1])^c = w^{-1}\{1\}$ and $((w^{*c})^{-1}(0, 1])^c = (w^*)^{-1}\{1\}$ be closed in T^*. Now $(((w \cap w^*)^c)^{-1}(0, 1])^c = (w \cap w^*)^{-1}\{1\} = w^{-1}\{1\} \cap w^{*-1}\{1\} = \phi$. Since (X, t^*) is supra fuzzy Normal, then there exist $V, V^* \in T^*$ such that $V \supseteq w^{-1}\{1\}$, $V^* \supseteq w^{*-1}\{1\}$ and $V \cap V^* = \phi$. But from the definition of $w(T^*)$, $1_V, 1_{V^*} \in w(T)$, $1_V \geq 1_{w^{-1}\{1\}}$, $1_{V^*} \geq 1_{(w^*)^{-1}\{1\}}$ and $1_{V \cap V^*} = 0$. Hence $(X, w(T^*))$ is α -SFN (i) space.

We can easily show that (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

We, therefore, prove that (d) $\Rightarrow$ (a).

Suppose $(X, w(T^*))$ is α - SFN (iii). We have to prove that (X, T^*) is Normal space. Let $V, V^* \in (T^*)^c$ and $V \cap V^c = \phi$. Then we have $1_V, 1_{V^*}$ be closed in $w(T^*)$ and $1_{V \cap V^*} = 0$.

Since $(X, w(T^*))$ is α - SFN (iii), then there exist $u, u^* \in w(T^*)$ such that $u \geq 1_V$, $u^* \geq 1_{V^*}$ and $u \cap u^* \leq \alpha$. But from the definition of $w(T^*)$, $u^{-1}(\alpha, 1]$, $u^{*-1}(\alpha, 1] \in T^*$ an $u^{-1}(\alpha, 1] \supseteq (1_V)^{-1}(\alpha, 1] = V$, $u^{*-1}(\alpha, 1] \supseteq (1_{V^*})^{-1}(\alpha, 1] = V^*$ and $u^{-1}(\alpha, 1] \cap u^{*-1}(\alpha, 1] = (u \cap u^*)^{-1}(\alpha, 1] = \phi$. Hence by definition, (X, T^*) is Normal space.

Thus it is seen that α - SFN (p) is a good extension of its topological counter part. (p = i, ii, iii).

Theorem6.7: Let (X, t^*) and (Y, s^*) be two supra fuzzy topological spaces and $f: X \longrightarrow Y$ be a continuous, one-one, onto and open map then,

(a) (X, t^*) is α -SFN (i) implies (Y, s^*) is α - SFN(i).

(b) (X, t^*) is α -SFN (ii) implies (Y, s^*) is α - SFN (ii).

(c) (X, t^*) is α -SFN(iii) implies (Y, s^*) is α - SFN(iii).

Proof: Suppose (X, t^*) be α - SFN (i). We have to prove that (Y, s^*) is α -SFN (i). Let $w, w^* \in (s^*)^c$ with $w \cap w^* \leq \alpha$ then $f^{-1}(w)$, $f^{-1}(w^*) \in (t^*)^c$ as f is continuous. Now $f^{-1}(w \cap w^*) \leq \alpha$ as $w \cap w^* \leq \alpha \Rightarrow f^{-1}(w) \cap f^{-1}(w^*) \leq \alpha$. Since (X, t^*) is α - SFN(i), for $\alpha \in I_1$, then there $u, u^* \in t^*$ such that $u(x) = 1$, $x \in (f^{-1}(w))^{-1}\{1\}$, $u^*(y) = 1$, $y \in (f^{-1}(w^*))^{-1}\{1\}$ and $u \cap u^* = 0$. This implies that $f(u)$, $f(v) \in s$ as f is open. Now $f(u)(p) = \{\text{Sup } u(x) : f(x) = p\}$, $f^{-1}(p) \in (f^{-1}(w))^{-1}\{1\}$, i.e., $f(u)(p) = 1$, $p \in w^{-1}\{1\}$ and $f(u^*)(q) = \{\text{Sup } u^*(y); f(y) = q\}$, $f^{-1}(q) \in (f^{-1}(w))^{-1}\{1\}$, i.e., $f(u^*)(q) = 1$, $q \in w^{*-1}\{1\}$ and $f(u) \cap f(u^*) = f(u \cap u^*) = 0$ as $u \cap u^* = 0$.

Now it is clear that there are $f(u)$, $f(u^*) \in s^*$ such that $f(u)(p) = 1$, $p \in w^{-1}\{1\}$, $f(u^*)(q) = 1$, $q \in w^{*-1}\{1\}$ and $f(u) \cap f(u^*) = 0$. Hence (Y, s^*) is α - SFN (i).

Similarly, (b) and (c) can be proved.

Theorem 6.8: Let (X, t^*) and (Y, s^*) be two supra fuzzy topological spaces and $f : X \longrightarrow Y$ be continuous, one-one, onto and closed map then,

(a) (Y, s) is α - SFN(i) implies (X, t^*) is α - SFN(i).

(b) (Y, s^*) is α - SFN(ii) implies (X, t^*) is α - SFN(ii).

(c) (Y, s^*) is α - SFN (iii) implies (X, t^*) is α - SFN (iii).

Proof: Suppose (Y, s^*) be α - SFN (i). We have to prove that (X, t^*) is α - SFN(i). Let $w, w^* \in (t^*)^c$ with $w \cap w^* \leq \alpha$, then $f(w), f(w^*) \in s^c$ as f is closed and $f(w) \cap f(w^*) = f(w \cap w^*) \leq \alpha$, as $w \cap w^* \leq \alpha$. Since (Y, s^*) is α -SFN(i), for $\alpha \in I_1$, then there exist $u, u^* \in s^*$ such that $u(x) = 1,\ x \in (f(w))^{-1}\{1, u^*(y) = 1,\ y \in (f(w^*))^{-1}\{1\}$ and $u \cap u^* = 0$. This implies that $f^{-1}(u), f^{-1}(u^*) \in t^*$ as f is continuous and $u, u^* \in s^*$.

Now $f^{-1}(u)(p) = u(f(p)) = u(x) = 1$ as $f(p) = x \in (f(w))^{-1}\{1\}$

i.e., $f^{-1}(u)(p) = 1, p \in w^{-1}\{1\}$

$f^{-1}(u^*)(q) = u^*(f(q)) = u^*(y) = 1$ as $f(q) = y \in (f(w^*))^{-1}\{1\}$

i.e., $f^{-1}(u^*)(q) = 1, q \in w^{*-1}\{1\}$

and $f^{-1}(u) \cap f^{-1}(u^*) = f^{-1}(u \cap u^*) = 0$.

Then we have there exist $f^{-1}(u), f^{-1}(u^*) \in t^*$ such that $f^{-1}(u)(p) = 1, p \in w^{-1}\{1\}$, $f^{-1}(u^*)(q) = 1, q \in w^{*-1}\{1\}$ and $f^{-1}(u) \cap f^{-1}(u^*) = 0$. Hence (X, t^*) is α - SFN (i).

Similarly, (b) and (c) can be proved.

CHAPTER 7

R_0 and R_1 Spaces in Supra Fuzzy Topology

7. 1 Introduction

We introduce and study some R_0 and R_1 properties in supra fuzzy topological spaces and obtain their several features in this chapter.

Definition 7.2: Let (X, t^*) be a supra fuzzy topological space and $\alpha \in I_1$. Then

(a) (X, t^*) is an $\alpha - R_0$ (i) space if and only if for all $x, y \in X$ with $x \neq y$, whenever there exist $u \in t^*$ with $u(x) = 1$ and $u(y) \leq \alpha$, then there exist $v \in t^*$ with $v(x) \leq \alpha$ and $v(y) = 1$.

(b) (X, t^*) is an $\alpha - R_0$ (ii) space if and only if for all $x, y \in X$, $x \neq y$, whenever $u \in t^*$ with $u(x) = 0$ and $u(y) > \alpha$, then there exists $v \in t^*$ with $v(x) > \alpha$ and $v(y) = 0$.

(c) (X, t^*) is an α-R_0 (iii) space if and only if for all $x, y \in X$ with $x \neq y$, whenever there exists $u \in t^*$ with $0 \leq u(x) \leq \alpha < u(y) \leq 1$, then there exists $v \in t^*$ with $0 \leq v(y) \leq \alpha < v(x) \leq 1$.

(d) (X, t^*) is an R_0 (iv) space if and only if for all $x, y \in X$ with $x \neq y$, whenever there exists $u \in t^*$ with $u(x) < u(y)$, then there exists $v \in t^*$ with $v(x) > v(y)$.

The following examples show that $\alpha - R_0$ (i), α- R_0 (ii), $\alpha - R_0$ (iii) and R_0 (iv) are all independent.

Example 7.3: Let $X = \{x, y\}$ and $u, v \in I^X$ are defined by $u(x) = 1$, $u(y) = 0$ and $v(x) = 0.45$, $v(y) = 1$. Consider the supra fuzzy topology t^* on X generated by {0, u, v, 1, Constants}. Then by definition, for $\alpha = 0.55$, (X, t^*) is $\alpha - R_0$(i), but (X, t^*) is not $\alpha - R_0$ (ii) .

Example 7.4: Let X = {x, y} and u, v ∈ I^X are defined by u (x) = 0, u (y) =1 and v (x) = 0.73, v (y) = 0. Consider the supra fuzzy topology t^* on X generated by {0, u, v, 1, Constants}. Then by definition, for α = 0.63 , (X, t^*) is α –R_0 (ii) , but (X , t^*) is not α –R_0 (i).

Example 7.5: Let X = { x , y } and u, v ∈ I^X are defined by u (x) =1 , u (y) = 0 and v (x) = 0.32 , v (y) = 0.69. Consider the supra fuzzy topology generated by {0, u, v, 1, Constant}. Then by definition, for α = 0.52, (X, t^*) is α-R_0 (iii), but (X, t^*) is not α –R_0 (i) and (X, t^*) is not α – R_0(ii).

Example 7.6: Let X = { x , y } and u, v ∈ I^X are defined by u (x) = 1, u (y) = 0 and v (x) = 0.24 , v (y) = 0.48. Consider the supra fuzzy topology t^* on X generated by {0, u, v, 1, Constants}. Then by definition , for α= 0.53, (X , t^*) is α –R_0 (iv), but (X, t^*) is not α – R_0 (i), (X, t^*) is not α –R_0 (ii), and (X, t^*) is not α – R_0 (iii).

Example 7.7: Let X = { x ,y , z} and u , v, w ∈ I^X are defined by u (x) = 1, u (y) =1, u (z) = 0 and v (x) = 0, v (y) = 0, v (z) =1 and w (x) = 0.92 , w (y) = 0.52 , w (z) = 0. Consider the fuzzy topology t^* on X generated by { 0 , u ,v , w , 1, Constants}. Then for α = 0.63, it can easily shown that (X , t^*) is α –R_0 (i) and (X , t^*) is α –R_0 (ii). But we observe (X , t^*) is not α –R_0 (iii), and (X , t^*) is not R_0 (iv), Since w (x) > α ≥ w (y) but there does not exist q ∈ t^* such that q (x) ≤ α < q (y).

Example 7.8: Let X = { x , y , z } and u , v ∈ I^X are defined by u (x) = 0.83, u (y) = 0.41, u (z) =0.36, and v(x) = 0.36, v (y) = 0.83 , v (z) =0.24. Consider the supra fuzzy topology t^* on X generated by {0, u, v, 1, Constants}. Then by definition, for α =0.5, (X , t^*) is α – R_0(iii), but (X , t^*) is not R_0 (iv), since u (y) > u (z) but we have no q ∈ t^* such that q (y) < q(z) .

Theorem 7.9: Prove that (X, t^*) is 0- R_0 (ii) if and only if (X, t^*) is 0- R_0 (iii).

Proof: The proof is trivial.

Theorem 7.10: Let (X, t^*) be a supra fuzzy topological space and $I_\alpha(t^*) = \{u^{-1}(\alpha, 1] \ / u \in t^*\}$. Then the following is true:

(a) (X, t^*) is $\alpha - R_0$ (iii) space if and only if $(X, I_\alpha(t^*))$ is R_0 space.

(b) if (X, t^*) is $\alpha - R_0$ (i) space, then $(X, I_\alpha(t^*))$ is not R_0 space and conversely.

(c) if (X, t^*) is α - R_0 (ii) space then $(X, I_\alpha(t^*))$ is not R_0 space and conversely.

(d) if (X, t^*) is R_0(iv), then $(X, I_\alpha(t^*))$ is not R_0 space and conversely.

Proof: Let (X, t^*) be $\alpha - R_0$ (iii). We have to prove that $(X, I_\alpha(t^*))$ is R_0. Let $x, y \in X$ with $x \neq y$ and $M \in I_\alpha(t^*)$ with $x \in M, y \notin M$ or $x \notin M$, $y \in M$. Suppose that $x \in M$, $y \notin M$. We can write, $M = u^{-1}(\alpha, 1]$, for some $u \in t^*$. Then we have $u(x) > \alpha$, $(y) \leq \alpha$, i.e., $0 \leq u(y) \leq \alpha < u(x) \leq 1$. Since (X, t^*) is $\alpha - R_0$(iii), $\alpha \in I_1$,then there exists $v \in t^*$ such that $0 \leq v(x) \leq \alpha < v(y) \leq 1$, i.e., $v(x) \leq \alpha$, $v(y) > \alpha$. It follows that $x \notin v^{-1}(\alpha, 1]$, $y \in v^{-1}(\alpha, 1]$ and also $v^{-1}(\alpha, 1] \in I_\alpha(t^*)$. Thus $(X, I_\alpha(t^*))$ is R_0. Conversely, suppose that $(X, I_\alpha(t^*))$ is R_0. We have to prove that (X, t^*) is $\alpha - R_0$ (iii). Let $x, y \in X$ with $x \neq y$ and $u \in t^*$ with $0 \leq u(x) \leq \alpha < u(y) \leq 1$, i.e., $u(x) \leq \alpha$, $u(y) > \alpha$, it follows that $x \notin u^{-1}(\alpha, 1]$, $y \in u^{-1}(\alpha, 1]$, and $u^{-1}(\alpha, 1] \in I_\alpha(t)$, for every $u \in t^*$. Since $(X, I_\alpha(t^*))$ is R_0, then there exists $M \in I_\alpha(t^*)$ such that $x \in M, y \notin M$. We can write $M = v^{-1}(\alpha, 1]$, where $v \in t^*$, it follows that $v(x) > \alpha$, $v(y) \leq \alpha$, i.e., $0 \leq v(y) \leq \alpha < v(x) \leq 1$. Thus (X, t^*) is $\alpha - R_0$ (t^*), i.e., (a) is proved.

Example 7.11: Let $X = \{x, y, z\}$ and $u, v \in I^X$ are defined by $u(x) = 1$, $u(y) = 0$, $u(z) = 0.8$ and $v(x) = 0$, $v(y) = 1$, $v(z) = 0.7$. Consider the supra fuzzy topology t^* on X generated by $\{0, u, v, 1, \text{Constants}\}$. Then for $\alpha = 0.6$, we have, (X, t^*) is

$\alpha - R_0$ (i) . Now, $I_\alpha(t^*) = \{X, \Phi, \{x, z\}, \{y, z\}, \{z\}\}$. It is observed that (X, t^*) is not R_0 space, since $y, z \in X$, $y \neq z$ and $\{x, z\} \in I_\alpha(t^*)$, with $z \in \{x, z\}$, $y \notin \{x, z\}$, but no such $U \in I_\alpha(t^*)$ with $x \notin U, y \in U$.

Example 7.12: Let $X = \{x, y, z\}$ and $u, v \in I^X$ are defined by $u(x) = 0.3$, $u(y) = 0$, $u(z) = 0.8$, and $v(x) = 0.8$, $v(y) = 1$, $v(z) = 0$. Consider the supra fuzzy topology t^* on X generated by {0, u, v, 1, Constants}. Then, for $\alpha = 0.5$, we have, (X, t^*) is $\alpha - R_0$ (ii) and (X, t^*) is also R_0 (iv) . Now $I_\alpha(t^*) = \{X, \Phi, \{z\}, \{y\}, \{y, z\}\}$. It is observed that $(X, I_\alpha(t^*))$ is not R_0 space, since $x, y \in X, x \neq y$ and $\{y\} \in I_\alpha(t^*)$ with $x \notin \{y\}$, $y \in \{y\}$, but no such $U \in I_\alpha(t^*)$ with $x \in U, y \notin U$.

Example 7.13: Let $X = \{x, y\}$ and $u, v, w \in I^X$ are defined by $u(x) = 1$, $u(y) = 0$, $v(x) = 0.4$, $v(y) = 0.9$, $w(x) = 0.7$, $w(y) = 0.3$. Consider the supra fuzzy topology t^* on X generated by { 0, u, v, w, 1, Constants }. Then for $\alpha = 0.6$, we have, (X, t^*) is not $\alpha - R_0$ (i) and (X, t^*) is not $\alpha - R_0$ (ii). Now, $I_\alpha(t^*) = \{X, \Phi, \{x\}, \{y\}\}$. Then we see that $I_\alpha(t^*)$ is a topology on X and $(X, I_\alpha(t^*))$ is R_0.

Example 7.14: Let $X = \{x, y\}$ and $u, v \in I^X$ are defined by $u(x) = 0.4$, $u(y) = 0.5$, $v(x) = 0.3$, and $v(y) = 0.4$. Consider the supra fuzzy topology t^* on X generated by {0, u, v, 1, Constant}. Then, for $\alpha = 0.6$, we have (X, t^*) is not $\alpha - R_0$ (iv). Now, $I_\alpha(t^*) = \{X, \Phi\}$. Then $I_\alpha(t^*)$ is a topology on X and $(X, I_\alpha(t^*))$ is R_0.

Hence the proof is complete.

Theorem 7.15: Let (X, T^*) be a supra topological space. Then (X, T^*) is R_0, if and only if $(X, w(T^*))$ is $\alpha - R_0$ (p), where p = i , ii , iii , iv .

Proof: Let $(X, w(T^*))$ be $\alpha - R_0$ (i). Let $x, y \in X$ with $x \neq y$ and $U \in T^*$ with $x \in U$, $y \notin U$. But $1_U \in w(T^*)$ and $1_U(x) = 1$, $1_U(y) = 0$. Now, we have $1_U \in w(T^*)$ with

$1_U(x) = 1$, $1_U(y) \leq \alpha$. Since $(X, w(T^*))$ is $\alpha - R_0$ (i), there exists $v \in w(T^*)$ such that $v(x) \leq \alpha$, $v(y) = 1$. Then $x \notin v^{-1}(\alpha, 1]$, $y \in v^{-1}(\alpha, 1]$ as $v(x) \leq \alpha$, $v(y) = 1$ and also there exists $v^{-1}(\alpha, 1] \in T^*$. Thus (X, T^*) is R_0 – space.

Conversely, suppose that (X, T^*) be a R_0 –space. We have to prove that $(X, w(T^*))$ is $\alpha - R_0$(i). Let $x, y \in X$ with $x \neq y$ and there exists $u \in w(T^*)$ such that $u(x) = 1$, $u(y) \leq \alpha$. Then $x \in u^{-1}(\alpha, 1]$, $y \notin u^{-1}(\alpha, 1]$ as $u(x) = 1$, $u(y) \leq \alpha$. Hence $u^{-1}(\alpha, 1] \in T^*$. Since (X, T^*) is R_0, then there exists $V \in T^*$ such that $x \notin V$, $y \in V$, but $1_V \in w(T^*)$ and $1_V(x) = 0$, $1_V(y) = 1$, i.e., there exists $1_V \in w(T^*)$ such that $1_V(x) \leq \alpha$, $1_V(y) = 1$

Thus $(X, w(T^*))$ is $\alpha - R_0$(i).

Hence (X, T^*) is R_0 if and only if $(X, w(T^*))$ is $\alpha - R_0$(i).

In the same way, we can prove that

(a) (X, T^*) is R_0 if and only if $(X, w(T^*))$ is $\alpha - R_0$(ii).

(b) (X, T^*) is R_0 if and only if $(X, w(T^*))$ is $\alpha - R_0$(iii).

(c) (X, T^*) is R_0 if and only if $(X, w(T^*))$ is R_0(iv)

Thus it is seen that $\alpha - R_0$ (p) is a good extension of its topological counter part (p = i, ii, iii, iv).

Theorem 7.16: Let (X, t^*) be a supra fuzzy topological space and $A \subseteq X$, $t^*_A = \{u/A : u \in t^*\}$, then

(a) (X, t^*) is an $\alpha - R_0$ (i) if and only if (A, t^*_A) is an $\alpha - R_0$ (i) .

(b) (X, t^*) is an $\alpha - R_0$ (ii) if and only if (A, t^*_A) is an $\alpha - R_0$ (ii).

(c) (X, t^*) is an $\alpha - R_0$ (iii) if and only if (A, t^*_A) is an $\alpha - R_0$ (iii)

(d) (X, t^*) is an R_0(iv) if and only if (A, t^*_A) is an R_0 (iv).

Proof: Suppose that (X, t^*) is $\alpha - R_0$ (i). Then for $x, y \in A$, with $x \neq y$ and $u \in t^*_A$ such that $u(x) = 1$, $u(y) \leq \alpha$, then also $x, y \in X$, $x \neq y$. But we can write $u = w/A$, where $w \in t^*$ and hence $w(x) = 1$, $w(y) \leq \alpha$. Since (X, t^*) is $\alpha - R_0$ (i), then there exists $m \in t^*$ such that $m(x) \leq \alpha$, $m(y) = 1$. But from the definition $m/A \in t^*_A$, for every $m \in t^*$ and $m/A(x) \leq \alpha$, $m/A(y) = 1$. Thus (A, t^*_A) is $\alpha - R_0$ (i) . i.e., (a) proved.

Similarly (b), (c) and (d) can be proved.

Theorem 7.17: Given (X_i, t^*_i), $i \in \Lambda$ be supra fuzzy topological spaces and $X = \prod_{i \in \Lambda} X_i$ and t^* be a product supra fuzzy topology on X. Then

(a) $\forall i \in \Lambda$, (X_i, t^*_i) is $\alpha - R_0$ (i) if and only if (X, t^*) is $\alpha - R_0$ (i) .

(b) $\forall i \in \Lambda$, (X_i, t^*_i) is $\alpha - R_0$ (ii) if and only if (X, t^*) is $\alpha - R_0$ (ii) .

(c) $\forall i \in \Lambda$, (X_i, t^*_i) is $\alpha - R_0$ (iii) if and only if (X, t^*) is $\alpha - R_0$ (iii) .

(d) $\forall i \in \Lambda$, (X_i, t^*_i) is R_0 (iv) if and only if (X, t^*) is R_0 (iv) .

Proof: Let (X_i, t^*_i), $i \in \Lambda$ be $\alpha - R_0$ (i) . We have to prove that (X, t^*) is $\alpha - R_0$ (i). Let $x, y \in X$, with $x \neq y$ and $u \in t^*$ such that $u(x) = 1$, $u(y) \leq \alpha$. But we have $u(x) = \min\{ u_i(x_i) : i \in \Lambda\}$ and $u(y) = \min\{ u_i(y_i) : i \in \Lambda \}$ and hence we can find an $u_i \in t^*_i$ and $x_i \neq y_i$ such that $u_i(x_i) = 1$ and $u_i(y_i) \leq \alpha$. Since (X_i, t^*_i), $i \in \Lambda$ is α- R_0 (i) , $\alpha \in I_1$, then there exist $v_i \in t^*_i$, such that $v_i(x_i) \leq \alpha$, $v_i(y_i) = 1$. But $\pi_i(x) = x_i$ and $\pi_i(y) = y_i$ and hence $v_i(\pi_i(x) \leq \alpha$, $v_i(\pi_i(y)) = 1$. It follows that there exists $u_i o \pi_i \in t^*$ such that $(v_i o \pi_i)(x) \leq \alpha$, $(v_i o \pi_i)(y) = 1$. Thus (X, t^*) is $\alpha - R_0$ (i).

Conversely, suppose that (X, t^*) is $\alpha - R_0$ (i). We have to prove that (x_i, t^*_i), $i \in \Lambda$, is $\alpha - R_0$ (i). Let for some $i \in \Lambda$, a_i be a fixed element in X_i, suppose that $A_i = \{ x \in X = \prod_{i \in \Lambda} X_i \ / \ x_j = a_j$ for some $i \neq j \}$. So that A_i is the subset of X, and this implies that $(A_i, t^*_{A_i})$ is also the subspace of (X, t^*). Since (X, t^*) is $\alpha - R_0$(i) , then

$(A_i, t^*_{A_i})$ is also $\alpha - R_0$ (i) and A_i is a homeomorphic image of X_i. Thus (X_i, t^*_i) is $\alpha - R_0$ (i), i.e., (a) is proved.

Similarly, (b), (c) and (d) can be proved.

Definition 7.18: Let (X, t^*) be a fuzzy topological space and $\alpha \in I_1$. Then

(a) (X, t^*) is said to be α–R_1(i) if and only if for all $x, y \in X$ with $x \neq y$ whenever there exists $w \in t^*$ with $w(x) \neq w(y)$, then there exist $u, v \in t^*$ such that $u(x) = 1 = v(y)$ and $u \cap v \leq \alpha$.

(b) (X, t^*) is said to be α–R_1(ii) if and only if for all $x, y \in X$ with $x \neq y$ whenever there exists $w \in t^*$ with $w(x) \neq w(y)$, then there exist $u, v \in t^*$ such that $u(x) = v(y) = 1$ and $u \cap v \leq \alpha$

(c) (X, t^*) is said to be $\alpha - R_1$(iii) if and only if for all $x, y \in X$ with $x \neq y$ whenever there exists $w \in t^*$ with $w(x) \neq w(y)$ then there exist $u, v \in t^*$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v = 0$.

(d) (X, t^*) is said to be $\alpha - R_1$(iv) if and only if for all $x, y \in X$ with $x \neq y$ whenever there exists $w \in t^*$ with $w(x) \neq w(y)$ then there exist $u, v \in t^*$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v \leq \alpha$.

Lemma 7.19: The following implication is true:

(a) (X, t^*) is $\alpha - R_1$ (i) implies $\alpha - R_1$(ii) implies $\alpha - R_1$(iii) but $\alpha - R_1$(iv) does not imply $\alpha - R_1$(iii). So, not imply $\alpha - R_1$(i) and $\alpha - R_1$(ii).

(b) (X, t^*) is $\alpha - R_1$ (i) does not imply $\alpha - R_1$(iv) and also $\alpha - R_1$(ii) and $\alpha - R_1$(iii) does not imply $\alpha - R_1$(iv).

Proof: The proof is trivial.

Theorem 7.20: Suppose that (X, t^*) is a supra fuzzy topological space and $0 \leq \alpha \leq \beta < 1$, then

(a) (X, t^*) is $\alpha - R_1$(ii) implies (X, t^*) is β - R_1 (ii).

(b) (X, t^*) is β - R_1 (iii) implies (X, t^*) is $\alpha - R_1$(iii).

(c) (X, t^*) is 0 - R_1 (iii) if and only if (X, t^*) is $0 - R_1$(iv).

Proof: (a) Suppose that (X, t^*) is a α - R_1 (ii) space. We have to prove that (X, t^*) is a β- R_1(ii) space. Let $x, y \in X, x \neq y$ and $w \in t^*$ such that $w(x) \neq w(y)$. Since (X, t^*) is α - R_1 (ii), for some $\alpha \in I_1$, there exist $u, v \in t^*$, such that $u(x) = 1 = v(y)$ and $u \cap v \leq \alpha$. Since $0 \leq \alpha \leq \beta < 1$. Thus, we have $u \cap v \leq \beta$. Hence (X, t^*) is a β - R_1 (ii) space.

Similarly, we can prove that.

(X, t^*) is β -R_1(iii) space implies (X, t^*) is α - R_1 (iii) space.

(X, t^*) is 0- R_1 (iii) space if and only if (X, t^*) is 0- R_1 (iv) space.

Theorem 7.21: Let (X, t^*) be a supra fuzzy topological space, and $I_\alpha(t^*) = \{ u^{-1}(\alpha, 1) \,/\, u \in t^* \}$, then

(a) (X, t^*) is an α - R_1 (ii) implies $(X, I_\alpha(t^*))$ is a R_1.

(b) (X, t^*) is an α - R_1 (iii) implies $(X, I_\alpha(t^*))$ is a R_1.

(c) (X, t^*) is an α - R_1 (iv) implies $(X, I_\alpha(t^*))$ is a R_1.

Proof: (c) Suppose that (X, t^*) is α - R_1 (iv) space. We have to prove that $(X, I_\alpha(t^*))$ is a R_1 space. Let $x, y \in X$, with $x \neq y$ and $M \in I_\alpha(t^*)$ such that $x \in m$, $y \notin M$ or $x \notin M$, $y \in M$. Suppose that $x \in M$, $y \notin M$. We can write $M = w^{-1}(\alpha, 1]$, where $w \in t^*$. Now we have $w(x) > \alpha$, $w(y) \leq \alpha$, i.e., $w(x) \neq w(y)$. Since (X, t^*) is α -R_1(iv), $\alpha \in I_1$, then there exist $u, v \in t^*$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v \leq \alpha$. It follows that there exist $u^{-1}(\alpha, 1], v^{-1}(\alpha, 1] \in I_\alpha(t^*)$ with $x \in u^{-1}(\alpha, 1]$, $y \in v^{-1}(\alpha, 1]$ as $u(x) > \alpha$, $v(y) > \alpha$. and $u^{-1}(\alpha, 1] \cap v^{-1}(\alpha, 1] = \phi$, since $u \cap v \leq \alpha$. Thus we have, (X, t^*) is R_1.

Similarly, (a) and (b) can be proved.

Theorem 7.22: Suppose that (X, T^*) be a supra topological space. Consider the following statements:

(1) (X, T^*) be a R_1 space.

(2) $(X, w(T^*))$ be an α - R_1(i) space.

(3) $(X, w(T^*))$ be an α - R_1(ii) space.

(4) $(X, w(T^*))$ be an α - R_1(iii) space.

(5) $(X, w(T^*))$ be an α - R_1(iv) space.

Then

(a) (1) => (2) => (3) => (5)=> (1)

(b) (1) => (2) => (4) => (5)=> (1).

Proof: First, suppose that (X, T^*) be a R_1 space. We have to prove that $(X, w(T^*))$ be a α -R_1 (i) space. Let $x, y \in X$ with $x \neq y$ and $m \in w(T^*)$ such that $m(x) \neq m(y)$ i.e., either $m(x) < m(y)$ or $m(x) > m(y)$. Suppose that $m(x) < m(y)$. Then it is clear that $m^{-1}(m(x), 1] \in T^*$ as $m \in w(T^*)$ and $x \notin m^{-1}(m(x), 1]$, $y \in m^{-1}(m(x), 1]$. Since (X, T^*) is R_1 space then there exist $U, V \in T^*$ such that $x \in U$, $y \in V$ and $U \cap V = \phi$. Since $1_U, 1_V$ are lower semi continuous function from X into I, then 1_U, $1_V \in w(T^*)$, and it is clear that $1_U(x) = 1$, $1_V(y) = 1$ and $1_U \cap 1_V = 0$, i.e., $(X, w(T^*))$ is α - R_1 (i) space. Hence $(X, w(T^*))$ is also an α - R_1 (ii) space.

We have already proved (3) => (5) and (4) => (5).

Finally, we shall prove that (5) => (1).

Suppose that $(X, w(T^*))$ is α - R_1 (iv). We have to prove that (X, T^*) is R_1 space. Let $x, y \in X$ with $x \neq y$ and $M \in T^*$ such that $x \in M, y \notin M$ or $x \notin M, y \in M$. Suppose that $x \in M$, $y \notin M$. But 1_M is lower semi continuous function from X into I, so $1_M \in w(T^*)$ and $1_M(x) = 1$, $1_M(y) = 0$, i.e., $1_M(x) \neq 1_M(y)$. Since $(X, w(T^*))$ is α -R_1(iii), $\alpha \in I_1$ then $\exists\, u, v \in w(T^*)$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v \leq \alpha$. Now it is observed that $u^{-1}(\alpha, 1]$, $v^{-1}(\alpha, 1] \in T^*$ such that $x \in u^{-1}(\alpha, 1]$, $y \in v^{-1}(\alpha, 1]$ and $u^{-1}(\alpha, 1] \cap v^{-1}(\alpha, 1] = \phi$. Thus (X, T^*) is R_1 space.

The proof is now complete.

Thus it is seen that α - R_1(p) is a good extension of its topological counter part (p = i , ii , iii,).

Theorem 7.23: Let (X, t^*) be a supra fuzzy topological space, $A \subseteq X$ and $t^*_A = \{ u/A : u \in t^* \}$, then

(a) (X, t^*) is α - R_1 (i) implies (A, t^*_A) is α - R_1 (i).

(b) (X, t^*) is α - R_1 (ii) implies (A, t^*_A) is α - R_1 (ii).

(c) (X, t^*) is α - R_1 (iii) implies (A, t^*_A) is α - R_1 (iii).

(d) (X, t^*) is α - R_1 (iv) implies (A, t^*_A) is α - R_1 (iv).

Proof: (d) Suppose that (X, t^*) is α - R_1 (iv). We have to prove that (A, t^*_A) is α -R_1 (iv). Let $x, y \in A$ with $x \neq y$ then $x, y \in X$ and $x \neq y$. Consider $m \in t^*_A$ with $m(x) \neq m(y)$. Then m can be written as w/A, where $w \in t^*$ and hence $w(x) \neq w(y)$. Since (X, t^*) is α - R_1 (iv), $\alpha \in I_1$, then there exist $u, v \in t^*$ such that $u(x) > \alpha$, $v(y) > \alpha$ and $u \cap v \leq \alpha$. But we know $u/A \in t^*_A$, for every $u \in t^*$. Now we observed that

$u/A\,(x) > \alpha$, $v/A\,(y) > \alpha$ and $u/A \cap v/A \leq \alpha$, since $u \cap v \leq \alpha$. Thus (A, t^*_A) is α-R_1 (iv).

Similarly (a), (b) and (c) can be proved.

Theorem 7.24: Given (X_i, t^*_i), $i \in \Lambda$ be supra fuzzy topological spaces and $X = \Pi_{i\in\Lambda} X_i$ and t^* be the product supra topology on X, then

(a) $\forall\, i \in \Lambda$, (X_I, t^*_i) is α - R_1(i) if and only if (X, t^*) is α-R_1 (i).

(b) $\forall\, i \in \Lambda$, (X_I, t^*_i) is α - R_1(ii) if and only if (X, t^*) is α-R_1 (ii).

(c) $\forall\, i \in \Lambda$, (X_I, t^*_i) is α - R_1(iii) if and only if (X, t^*) is α-R_1 (iii).

(d) $\forall\, i \in \Lambda$, (X_I, t^*_i) is α - R_1(iv) if and only if (X, t^*) is α-R_1 (iv).

Proof: Suppose that $\forall\, i \in \Lambda$, (X_I, t^*_i) is α - R_1 (iv). We have to prove that (X, t^*) is α-R_1 (iv). Let $x, y \in X$ with $x \neq y$ and $w \in t^*$ with $w(x) \neq w(y)$. But we have $w(x) = \min\{w_i(x_i) : i \in \Lambda\}$, $w(y) = \min\{w_i(y_i) : i \in \Lambda\}$. Hence we can find at least one $w_i \in t^*_i$ and $x_i, y_i \in X_i$, with $w_i(x_i) \neq w_i(y_i)$. Since (X_i, t^*_i), $i \in \Lambda$ is $\alpha\, R_1$ (iv), $\alpha \in I_1$, then there exist $u_i, v_i \in t^*_i$ such that $u_i(x_i) > \alpha$, $v_i(y_i) > \alpha$ and $u_i \cap v_i \leq \alpha$. But we have $\pi_i(x) = x_i$ and $\pi_i(y) = y_i$ and hence $u_i(\pi_i(x)) > \alpha$, $v_i(\pi_i(y)) > \alpha$. It follows that there exist $u_i \circ \pi_i$, $v_i \circ \pi_i \in t^*$ such that $(u_i \circ \pi_i)(x) > \alpha$, $(v_i \circ \pi_i)(y) > \alpha$ and $(u_i \circ \pi_i) \cap (v_i \circ \pi_i) \leq \alpha$. Thus (X, t^*) is α-R_1 (iv).

Conversely, suppose that (X, t^*) is α - R_1 (iii). We have to prove that (X_i, t^*_i), $i \in \Lambda$, is α-R_1(iv). Let for some $i \in \Lambda$, a_i be a fixed element in X_i, suppose that $A_i = \{x \in X = \Pi_{i\in\Lambda} X_i \,/\, x_j = a_j$ for some $i \neq j\}$. So that A_i is a subset of X, and hence (A_i, t^*_{Ai}) is also a subspace of (X, t^*). Since (X, t^*) is α - R_1 (iv), then (A_i, t^*_{Ai}) is also α - R_1 (iii). Now we have A_i is homomorphic image of X_i. Hence (X_i, t^*_i) is α - R_1 (iv) i.e., (c) is proved.

Similarly, (a), (b) and (c) can be proved.

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