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Semilattices with a Partial Binary Operation

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Semilattices with a Partial Binary Operation

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by

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Statement of Supervisors

It is certified and declared with due confidence that the thesis entitled "**Semilattices with a Partial Binary Operation**" submitted by *Shamsun Naher Begum* contains the fulfillment of all the requirements for the degree of **Doctor of Philosophy** in Mathematics at University of Rajshahi has been completed under our supervision. We believe that this research work is an original one. It is doubtless to ascertain that this has not been submitted elsewhere for any degree. Except where the proper reference is made in the text of the thesis, it contains no materials published elsewhere. Nobody's work has been used in the main text of this thesis without due acknowledgement.

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Statement of Authorship

Except where reference is made in the text of the thesis, this thesis contains no material published elsewhere or extracted in whole or in part from a thesis by which I have qualified for or been awarded another degree or diploma.

No other person's work has been used without due acknowledgement in the main text of the thesis.

This thesis has not been submitted for the award of any degree or diploma in any other tertiary institution.

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Abstracts

This thesis centres on a class of certain pseudocomplemented partial lattices. This partial lattice is said to be a JP-semilattice.

An algebraic structure $\mathbf{S} = \langle S, \wedge, \vee \rangle$, where $\langle S, \wedge \rangle$ is a semilattice and $\vee$ is a partial binary operation on S, is said to be a **JP-Semilattice** if for all $x, y, z \in S$ the following axioms hold.

(i) $x \vee x$ exists and $x \vee x = x$;

(ii) $x \vee y$ exists implies $y \vee x$ exists and $x \vee y = y \vee x$;

(iii) $x \vee y$, $y \vee z$ and $(x \vee y) \vee z$ exists implies $x \vee (y \vee z)$ exists and $(x \vee y) \vee z = x \vee (y \vee z)$;

(iv) $x \vee y$ exists implies $x = x \wedge (x \vee y)$;

(v) $x \vee (x \wedge y)$ exists and $x = x \vee (x \wedge y)$;

(vi) $y \vee z$ exists implies $(x \wedge y) \vee (x \wedge z)$ exists.

In Chapter 1 we give a background of JP-semilattices. We prove that the set of all ideals of a JP-semilattice is a lattice. Unfortunately, the description of join of two ideals of a JP-semilattice is not good enough like as ideals of a lattice. We close this chapter by giving a relation between JP-homomorphism and order-preserving map.

In Chapter 2 we define modular and distributive JP-semilattices. We show that every distributive JP-semilattice is modular but the converse is not necessarily true. We prove that a JP-semilattice is non-modular if and only if it has a sublattice isomorphic to the pentagonal lattice. Here we also study the ideals lattice of a modular (distributive) JP-semilattice. We have given some characterizations of modular and distributive JP-semilattice using the ideals lattice. We also give the Stone's Separation Theorem for distributive JP-semilattices. We also prove that if I is an ideal and F is a filter of a distributive JP-semilattice disjoint from I, then there is a minimal prime ideal containing I and disjoint from F.

In Chapter 3 we study the congruences of a JP-semilattice. We describe the smallest and largest JP-congruences containing an ideal as a class. Here we characterize a distributive JP-semilattice by JP-homomorphism and JP-congruence. We prove the Homomorphism Theorem for JP-semilattices. We have given a description of the smallest JP-congruence containing a filter as a class. The quotient of JP-congruence containing a filter as a class is not necessarily a lattice. We impose a condition on the filter, which we call strong filter, to make the quotient JP-semilattice a lattice. Then we study the quotient lattices. Here we give a representation of the set of the prime ideals of the quotient lattice.

Cornish [9] studied congruence kernels and cokernels for pseudocomplemented distributive lattices and Blyth [2] studied congruence kernels and cokernels of a pseudocomplemented semilattice. In Chapter 4 we study the congruence kernels and cokernels for distributive pseudocomplemented JP-semilattices. A pseudocomplemented JP-semilattice will be called a PJP-semilattice. We give a description of a PJP-congruence containing an ideal as a class. We give several characterizations of kernel ideals of a distributive PJP-semilattice. Then we introduce the *-ideals for PJP-semilattices. We give a characterization of *-ideal. We describe the Glivenko congruence for PJP-semilattices. In this chapter we also describe the cokernels, Boolean congruence, *-filter and D-filter. We prove that every D-filter is a *-filter of a distributive PJP-semilattice if and only if the smallest PJP-congruence containing D as a class is a boolean congruence.

In Chapter 5 we introduce the notion of Stone JP-semilattice like a Stone lattice. First we give a very useful characterization of Stone JP-semilattices. Then we give a nice characterization of kernel ideals of Stone JP-semilattices. We describe the join of two kernel ideals of a Stone JP-semilattice which is very easier than the description of the join of two ideals for a distributive JP-semilattice. This description makes the world of kernel ideals of Stone JP-semilattices so easier. We prove that the set of all kernel ideals of a Stone JP-semilattice is a complete lattice and it is isomorphic to the set of all *-filters of the Stone JP-semilattice. Kernel homomorphism is introduced for Stone JP-semilattices. We also introduce a new notion of strong PJP-semilattice homomorphism. We give some results for PJP-semilattice homomorphisms.

In Chapter 6 we study the JP-semilattices such that the underlying semilattice is a distributive semilattice. We call these semilattices are JP distributive semilattices. Every JP distributive semilattice is a distributive JP-semilattice. Although we have the Stone's Separation Theorem for a distributive JP-semilattice, we also

prove the Stone's Separation Theorem for JP distributive semilattice. We show a different technique to prove the theorem. Next we discuss the JP Stone semilattices. A JP Stone semilattice is a Stone JP-semilattice described in Chapter 5 such that the underlying semilattice is a distributive semilattice. We have a great advantage here that in a JP Stone semilattice **S**, for any $x, y \in S$ we have $x \vee y^*$ always exists. This observation turns that we have a straightforward generalization of a famous result of C.C. Chen and G.Grätzer [**3**, Theorem 14.5] on Stone lattices. Then we characterize the minimal prime ideals of a JP Stone semilattice. Finally we have some characterizations of kernel ideals of a JP Stone semilattices.

CHAPTER 1

JP-Semilattices

1.1. Introduction

Partial lattices have been studied by many authors. For examples Grätzer and Lakser [**18, 19**], Nieminen [**24**], Cornish and Hickman [**12**], Hickman [**21**], Cornish and Noor [**13**], Noor and Cornish [**25**] etc. For the basic concepts and the background materials in partial lattices and lattices, we refer the reader to Grätzer [**16, 17**]. In this chapter we define a JP-semilattice and we give basic algebraic concepts of the JP-semilattices.

A meet semilattice with a partial binary operation satisfying some axioms is said to be a JP-semilattice. In Section 1.2, we discuss the background of partial lattices. This section is on the basis of Grätzer [**16, 17**].

In Section 1.3 we define JP-semilattice and we discuss down-sets and ideals of a JP-semilattice. We give some properties of ideals of a JP-semilattice.

In Section 1.4, we discuss order preserving maps and JP-homomorphisms. We show a relation between order-isomorphisms and JP-isomorphisms.

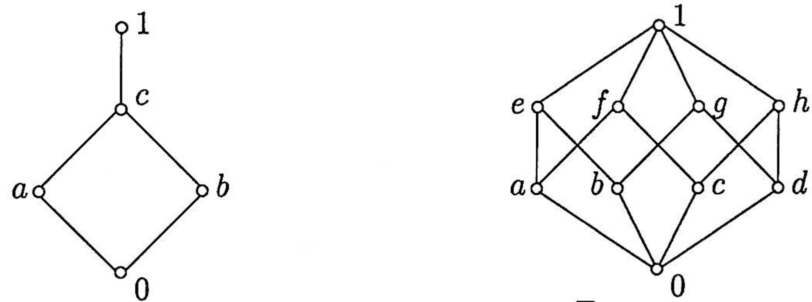
1.2. Partial Lattices

Let $\langle L; \wedge, \vee\rangle$ be a lattice, $H \subseteq L$, and $\wedge$ and $\vee$ on L are restrictions to H as follows:

For any $x, y, z \in H$, if $x \wedge y = z$ (dually, $x \vee y = z$), then we say that in H, $x \wedge y$ (dually, $x \vee y$) is defined and $x \wedge y = z$ (dually $x \vee y = z$); if for $x, y \in H$, $x \wedge y$ (dually, $x \vee y$) $\notin H$, then we say that $x \wedge y$ (dually, $x \vee y$) is not defined in H.

Thus $\langle H; \wedge, \vee\rangle$ is a set with two binary operations $\wedge$ and $\vee$. By [**16, 17**], $\langle H; \wedge, \vee\rangle$ is called a **partial lattice** or a **relative sublattice** of L. Clearly, every subset of a lattice determines a partial lattice.

Examples of a partial lattice and a non-partial lattice. Let $\mathbf{P}_1 = \langle P_1; \wedge, \vee\rangle$ be a lattice given in the following Figure 1.1, and let $H = \{0, a, b, 1\} \subseteq P_1$. Then $\mathbf{H} = \langle H; \wedge, \vee\rangle$ is a partial lattice and a relative sublattice of $\mathbf{P}_1$.



$\mathbf{P}_1$ $\mathbf{P}_2$

FIGURE 1.1. Partial and Non-partial lattices

Observe that $\sup\{a, b\} = 1 \in H$ but $a \vee b$ is not defined in H because $a \vee b \notin H$.

Now let $H = \{0, a, b, c, d, e, f, g, h, 1\}$ and consider the lattice $\mathbf{P}_2$ given in Figure 1.1. Define $\wedge$ and $\vee$ on H given by, for all $x, y \in H$,

(i) $x \wedge y = z \in H$ if and only if $x \wedge y = z \in P_2$, and

(ii) $x \vee y = z \in H$ if and only if

either $x \leqslant y$ in P_2 and $y = z$, or $y \leqslant x$ in P_2 and $x = z$;

or if $\{x, y\} = \{a, c\}$, and $z = f$;

or if $\{x, y\} = \{b, d\}$, and $z = g$;

or if $\{x, y\} = \{f, g\}$, and $z = 1$;

We claim that $\mathbf{H} = \langle H; \wedge, \vee \rangle$ is not a partial lattice. If possible suppose that there exists a lattice $\mathbf{L}$ with $H \subseteq L$ such that $\mathbf{H}$ is a relative sublattice of $\mathbf{L}$. Then $(a \vee c) \vee (b \vee d) = 1 \in L$, and thus $\sup\{a, b, c, d\} = 1$. Since $a, b \leqslant e$ and $c, d \leqslant h$ and $e, h \leqslant 1$ we have $\sup\{e, h\} = 1 \in L$. But $e, h, 1 \in H$ implies $e \vee h$ is defined in H (and $e \vee h = 1$ in H), which is a contradiction of the definition of $\vee$ on H. Hence $\mathbf{H} = \langle H; \wedge, \vee \rangle$ is not a partial lattice.

The following lemma on partial lattice is due to Grätzer [**16**, Lemma 13, pp-48].

Lemma 1.2.1 *Let $\langle H; \wedge, \vee \rangle$ be a partial lattice. For $x, y, z \in H$, we have*

(i) *$x \wedge x$ exists and $x \wedge x = x$;*

(ii) *if $x \wedge y$ exists, then $y \wedge x$ exists and $x \wedge y = y \wedge x$;*

(iii) *if $x \wedge y$, $y \wedge z$ and $(x \wedge y) \wedge z$ exist, then $x \wedge (y \wedge z)$ exists and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$;*

(iv) *if $x \wedge y$ exists, then $x \vee (x \wedge y)$ exists, and $x = x \vee (x \wedge y)$;*

□

By the dual arguments of the above lemma, the following result is also due to Grätzer [**16**, Lemma 13′, pp-49].

Lemma 1.2.2 *Let $\langle H; \wedge, \vee\rangle$ be a partial lattice. For $x, y, z \in H$, we have*

(i) *$x \vee x$ exists and $x \vee x = x$;*

(ii) *if $x \vee y$ exists, then $y \vee x$ exists and $x \vee y = y \vee x$;*

(iii) *if $x \vee y$, $y \vee z$ and $(x \vee y) \vee z$ exist, then $x \vee (y \vee z)$ exists and $(x \vee y) \vee z = x \vee (y \vee z)$;*

(iv) *if $x \vee y$ exists, then $x \wedge (x \vee y)$ exists, and $x = x \wedge (x \vee y)$;*

□

A structure $\mathbf{S} = \langle S, \wedge, \vee\rangle$ with two partial binary operations $\wedge$ and $\vee$ on S satisfying the conditions of Lemma 1.2.1 and Lemma 1.2.2 is said to be a **weak partial lattice.**

Theorem 1.2.3 *Every partial lattice is a weak partial lattice but the converse is not true.*

Proof. By the definition, every partial lattice is a weak partial lattice. To prove the converse, if we consider $\mathbf{H}$ which has been constructed in the example of a partial and non-partial lattices above, then it is a routine work to show that $\mathbf{H}$ satisfy all the conditions of Lemma 1.2.1 and Lemma 1.2.2. Thus $\mathbf{H}$ is a weak partial lattice but we already have claimed that it is not a partial lattice. □

Now we have the following result for a weak partial lattice.

Lemma 1.2.4 *Let $\langle L; \vee, \wedge \rangle$ be a weak partial lattice. Then $x \wedge y$ exists and $x \wedge y = x$ if and only if $x \vee y$ exists and $x \vee y = y$.*

Proof. Suppose $x \wedge y$ exists and $x \wedge y = x$. Then by Lemma 1.2.1 (ii), we have $y \wedge x$ exists and $y \wedge x = x \wedge y = x$. Hence by Lemma 1.2.1 (iv), we have $y \vee (y \wedge x)$ exists and $y \vee (y \wedge x) = y$. This implies $y \vee x$ exists and $y \vee x = y$. Thus by Lemma 1.2.2 (ii), we have $x \vee y$ exists and $x \vee y = y \vee x = y$. The converse is true by the dual argument. □

The proof of the following lemma was omitted in **[17]** as it is a bit longer. Here we give the proof of the lemma.

Lemma 1.2.5 *Let $\langle L; \vee, \wedge \rangle$ be a weak partial lattice. Define a binary relation $\leqslant$ on L by*

$$x \leqslant y \text{ if and only if } x \wedge y \text{ exists and } x \wedge y = x.$$

Then $\leqslant$ is a partial ordering relation. Moreover if $x \wedge y$ exists, then $x \wedge y = \inf\{x, y\}$ and if $x \vee y$ exists, then $x \vee y = \sup\{x, y\}$.

Proof. Since for all $x \in L$, $x \wedge x$ exists and $x \wedge x = x$, we have $x \leqslant x$. Hence $\leqslant$ is reflexive.

Let $x \leqslant y$ and $y \leqslant x$. Then $x \wedge y$ exists, $x \wedge y = x$ and $y \wedge x$ exists, $y \wedge x = y$. Hence by the Lemma 1.2.1 (ii), we have $x \wedge y = y \wedge x$. This implies $x = y$. Thus $\leqslant$ is anti-symmetric.

Let $x \leqslant y$ and $y \leqslant z$. Then $x \wedge y$ exists and $x \wedge y = x$. Hence by the Lemma 1.2.1 (ii), $y \wedge x$ exists and $y \wedge x = x \wedge y$. Now $y \leqslant z$ implies $y \wedge z$ exists and $y \wedge z = y$. Hence $y \wedge x = (y \wedge z) \wedge x = (z \wedge y) \wedge x$ exists. Hence by the Lemma 1.2.1 (iii), $z \wedge (y \wedge x)$ exists and $z \wedge (y \wedge x) = (z \wedge y) \wedge x$. Now, $z \wedge (y \wedge x) = z \wedge x = x \wedge z$. Therefore, $x = x \wedge y = y \wedge x = z \wedge x = x \wedge z$. Hence $x \leqslant z$. Thus $\leqslant$ is transitive.

Hence $\leqslant$ is a partial ordering relation.

Let $x \wedge y$ exists. Since by the Lemma 1.2.1 (i), $x \wedge x$ exists and $x \wedge x = x$, we have $(x \wedge x) \wedge y$ exists and $x \wedge y = (x \wedge x) \wedge y = x \wedge (x \wedge y)$, by the Lemma 1.2.1 (iii). Hence $x \wedge y \leqslant x$. Similarly, $x \wedge y \leqslant y$. Thus $x \wedge y$ is a lower bound of x and y. Let c be a lower bound of x and y. Then $c \leqslant x$ and $c \leqslant y$. Hence $c \wedge x$, $c \wedge y$ exists and $c = c \wedge x = c \wedge y$. This implies $c = (c \wedge x) \wedge y$. Since $c \wedge x$ and $x \wedge y$ exist, we have by the Lemma 1.2.1 (iii), $c \wedge (x \wedge y)$ exists and $c = (c \wedge x) \wedge y = c \wedge (x \wedge y)$. Hence $c \leqslant x \wedge y$. Therefore $x \wedge y = \inf\{x, y\}$. By the dual arguments we can show that if $x \vee y$ exists, then $x \vee y = \sup\{x, y\}$. □

Let $\langle P; \leqslant \rangle$ be an ordered set. Define two partial operations $\vee$ and $\wedge$ on P as follows: $x \vee y$ is defined if $\sup\{x, y\}$ exists and $x \vee y = \sup\{x, y\}$, similarly, $x \wedge y$ is defined if $\inf\{x, y\}$ exists and $x \wedge y = \inf\{x, y\}$.

Theorem 1.2.6 $\mathbf{P} = \langle P; \vee, \wedge \rangle$ *is a weak partial lattice.*

Proof. It is easy to show that **P** satisfies all the conditions of Lemma 1.2.1 and Lemma 1.2.2. □

If $x_1, x_2, \cdots, x_n \in N$, then by $x_1 \vee x_2 \vee \cdots \vee x_n$ we mean that supremum of $x_1, x_2, \cdots, x_n$ exists and $x_1 \vee x_2 \vee \cdots \vee x_n$ is the supremum of $x_1, x_2, \cdots, x_n$. Dually, if $x_1, x_2, \cdots, x_n \in N$, then by $x_1 \wedge x_2 \wedge \cdots \wedge x_n$ we mean that infremum of $x_1, x_2, \cdots, x_n$ exists and $x_1 \wedge x_2 \wedge \cdots \wedge x_n$ is the infremum of $x_1, x_2, \cdots, x_n$.

1.3. Definition of JP-semilattices

It is very complicated to handle two binary partial operations. So we restrict our attention to a meet semilattice with one binary partial operation. A meet semilattice $\langle S; \wedge \rangle$ is a non-empty set S with an idempotent, commutative and associative binary operation $\wedge$ on S. Throughout the Thesis by a semilattice we always mean a meet semilattice. Observe that if $\mathbf{S} = \langle S; \wedge, \vee \rangle$ is a meet semilattice with a partial operation $\vee$ satisfying the axioms of Lemma 1.2.2, then the existence of $y \vee z$ does not imply the existence of $(x \wedge y) \vee (x \wedge z)$ for any $x, y, z \in S$. For example, consider the following Figure 1.2. Here $b \vee c$ exists, but

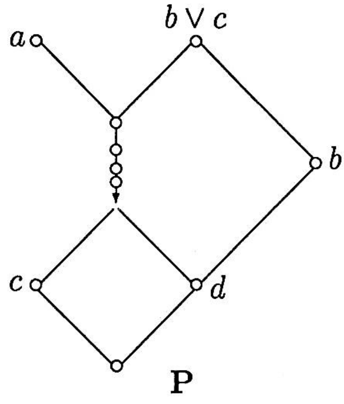


FIGURE 1.2. a non-JP-semilattice

$(a \wedge b) \vee (a \wedge c)$ does not. Also remark that the existence of $x \vee y \vee z$ in S does not imply the existence of $x \vee y$ for any $x, y, z \in S$.

An algebraic structure $\mathbf{S} = \langle S, \wedge, \vee \rangle$ where $\langle S, \wedge \rangle$ is a semilattice and $\vee$ is a partial binary operation on S is said to be a **join partial semilattice** (or simply **JP-Semilattice**) if for all $x, y, z \in S$,

(i) $x \vee x$ exists and $x \vee x = x$;

(ii) $x \vee y$ exists implies $y \vee x$ exists and $x \vee y = y \vee x$;

(iii) $x \vee y$, $y \vee z$ and $(x \vee y) \vee z$ exists implies $x \vee (y \vee z)$ exists and $(x \vee y) \vee z = x \vee (y \vee z)$;

(iv) $x \vee y$ exists implies $x = x \wedge (x \vee y)$;

(v) $x \vee (x \wedge y)$ exists and $x = x \vee (x \wedge y)$;

(vi) $y \vee z$ exists implies $(x \wedge y) \vee (x \wedge z)$ exists.

Every JP-Semilattice is clearly a weak partial lattice as it satisfies all the conditions of Lemma 1.2.1 and Lemma 1.2.2. But the converse is not necessarily true, for example, the semilattice **P** given in Figure 1.2 is a weak partial lattice but not a JP-semilattice. This example also shows that every semilattice need not be a JP-semilattice but by the definition of a JP-semilattice, every JP-semilattice is a semilattice. Thus the class of all JP-semilattices is a subclass of semilattices and also a subclass of the class of weak partial lattices.

1.3.1. Down-sets and ideals of JP-semilattices.

Down-sets. Let **P** be an ordered set. A subset A of **P** is said to be a **down-set** if

$$x \in A \text{ and } y \leqslant x \text{ implies } y \in A.$$

The set of all down-sets of an ordered set **P** is denoted by $\mathcal{O}(\mathbf{P})$. Clearly, $\emptyset, P \in \mathcal{O}(\mathbf{P})$. It is evident that $\mathcal{O}(\mathbf{P})$ is a bounded complete distributive lattice for any ordered set **P**, when partially ordered by set inclusion. The meet and join in $\mathcal{O}(\mathbf{P})$ are given by set-theoretic intersection and union respectively.

Lemma 1.3.1 *Let* **S** *be a JP-semilattice and* $\emptyset \neq K \subseteq S$. *Define* $K_0 = K$ *and for* $n \geqslant 1$,

$$K_n = \{x \in S \mid x \leqslant y \vee z \text{ for } y, z \in K_{n-1}\}.$$

Then for each $n \geqslant 1$, K_n *is a down-set and*

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots.$$

Proof. Let $x \in K_n$ for some $n \geqslant 1$ and $y \in S$ with $y \leqslant x$. Then $x \leqslant p \vee q$ for some $p, q \in K_{n-1}$. Hence $y \leqslant p \vee q$ for some $p, q \in K_{n-1}$. Therefore, $y \in K_n$ and hence K_n is a down-set.

Let $x \in K_n$ for some $n \geqslant 0$. Then $x \leqslant x \vee x$ implies $x \in K_{n+1}$. Hence $K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$. □

Ideals. A non-empty down-set I of a JP-semilattice S is said to be an **ideal** of S if

$$x, y \in I \text{ and } x \vee y \text{ exists, implies } x \vee y \in I.$$

Let $\mathbf{S} = \langle S; \wedge, \vee \rangle$ be a JP-semilattice and $A \subseteq S$. A structure $\mathbf{A} = \langle A; \wedge, \vee \rangle$ is said to be a **subJP-semilattice** of **S** if **A** itself is a JP-semilattice where $\wedge$ and $\vee$ in **A** are restrictions of $\wedge$ and $\vee$ in **S**.

Theorem 1.3.2 *Every ideal of a JP-semilattice is a subJP-semilattice.*

Proof. Let I be an ideal of a JP-semilattice **S**. Let $x, y \in I$. Since $x \wedge y \leqslant x$, we have $x \wedge y \in I$. If $x \vee y$ exists, then by the definition of an ideal $x \vee y \in I$. Hence I is a subJP-semilattice. □

The set of all ideals of a JP-semilattice S will be denoted by $\mathcal{I}(S)$. For any non-empty subset K of a JP-semilattice S, the smallest ideal containing K is denoted by $(K]$ and is called the **ideal generated by** K. If $K = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. For $a \in S$, the ideal $(a]$ is called the **princpal ideal** generated by a.

The following results give us the description of principal ideals and ideals generated by a subset of a JP-semilattice.

Theorem 1.3.3 *Let* S *be a JP-semilattice and* $\emptyset \neq K \subseteq S$. *Then*

(i) $(K] = \bigcup_{n=0}^{\infty} K_n$ *where* $K_0 = K$ *and for* $n \geqslant 1$,

$$K_n = \{x \in S \mid x \leqslant y \vee z \text{ for } y, z \in K_{n-1}\}$$

(ii) *For* $a \in S$ *we have* $(a] = \{x \in S \mid x \leqslant a\}$.

Proof. (i) Trivially, $\bigcup_{n=0}^{\infty} K_n$ is non-empty as it contains K. Let $x \in \bigcup_{n=0}^{\infty} K_n$ and $y \in S$ with $y \leqslant x$. If $x \in K_n$ for some $n \geqslant 1$, then $y \in K_n$ as K_n is a down-set (by Lemma 1.3.1). Hence $y \in \bigcup_{n=0}^{\infty} K_n$. If $x \in K = K_0$, then $x \in K_1$ as $K_0 \subseteq K_1$ (by Lemma 1.3.1). Hence $y \in K_1$. This implies $y \in \bigcup_{n=0}^{\infty} K_n$. Thus $\bigcup_{n=0}^{\infty} K_n$ is a down-set.

Let $x, y \in \bigcup_{n=0}^{\infty} K_n$ such that $x \vee y$ exists. Then $x, y \in K_n$ for some $n \geqslant 0$ as $K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots$. Since $x \vee y \leqslant x \vee y$, we have $x \vee y \in K_{n+1}$. Hence $x \vee y \in \bigcup_{n=0}^{\infty} K_n$. Therefore, $\bigcup_{n=0}^{\infty} K_n$ is an ideal of S.

Let I be an ideal containing $K = K_0$. We use the mathematical induction to show that for each $n \geqslant 0$, $K_n \subset I$. Let $K_n \subseteq I$ for some $n \geqslant 1$ and let $x \in K_{n+1}$. Then $x \leqslant y \vee z$ for some $y, z \in K_n$ and hence $y \vee z \in I$ as I is an ideal. Therefore, $x \in I$. Hence for all $n \geqslant 0$, $K_n \subseteq I$. Thus $\bigcup_{n=0}^{\infty} K_n$ is the smallest ideal containing K. Hence $(K] = \bigcup_{n=0}^{\infty} K_n$. □

The following result give us the description of the join of two ideals of a JP-semilattice.

Theorem 1.3.4 *Let I and J be two ideals of a JP-semilattice S. Then*

$$I \vee J = (I \cup J] = \bigcup_{n=0}^{\infty} A_n$$

where $A_0 = I \cup J$ and for $n \geqslant 1$,

$$A_n = \{x \in S \mid x \leqslant y \vee z \text{ for } y, z \in A_{n-1}\}$$

Proof. Suppose $K = \bigcup_{n=0}^{\infty} A_n$. Let $x \in K$ and $y \leqslant x$. Then $x \in A_n$ for some $n = 0, 1, 2, \cdots$. If $n = 0$, then either $x \in I$ or $x \in J$ and hence either $y \in I$ or $y \in J$ as I and J are ideals. Thus, $y \in K$. If $n \geqslant 1$, then $x \leqslant p \vee q$ and $p, q \in A_{n-1}$ and hence $y \leqslant p \vee q$ where $p, q \in A_{n-1}$. Thus $y \in A_n$ and hence $y \in K$. Now let $x, y \in K$ and $x \vee y$ exists. Since $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$, we have $x, y \in A_n$ for some n and hence $x \vee y \in A_{n+1}$. Thus $x \vee y \in K$. Hence K is an ideal. Clearly, K is containing I and J. Let H be any ideal containing I and J. Clearly $A_0 \subseteq H$. We use the mathematical induction to show $K \subseteq H$. Let $A_n \subseteq H$ for some n and let $x \in A_{n+1}$. Then $x \leqslant y \vee z$ where $y, z \in A_n$. This implies $y, z \in H$ and hence $y \vee z \in H$ as H is an ideal. Thus $x \in H$. Hence $A_{n+1} \subseteq H$. Thus for any $n \geqslant 0$, we have $A_n \subseteq H$. Hence $K \subseteq H$. Therefore $K = I \vee J$. □

A routine work shows that $\mathcal{I}(\mathrm{S})$, the set of all ideals of a JP-semilattice S is an algebraic lattice.

Remark. For any ideals I and J of a JP-semilattice S, the description of $I \vee J$ is not so easy like the joins in semilattices or lattices. Even $I \vee J$ can not be written as $\{x \leqslant y \vee z \mid y \in I, z \in J$ whenever $y \vee z$ exists $\}$. For example, consider the JP-semilattice **B** given in the Figure 1.3. Suppose $I = (a]$ and $J = (b]$. Then

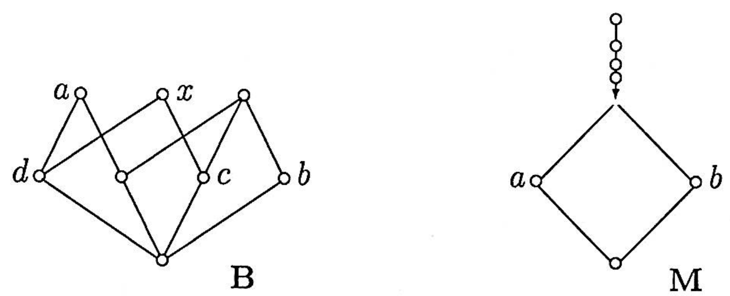


FIGURE 1.3

$x \in I \vee J$, but $x \nleqslant i \vee j$ for any $i \in I$ and $j \in J$. This observation shows that there are difficulties in studying the lattice $\mathcal{I}(S)$.

Now we turn our attention to principal ideals of a JP-semilattice. It is easy to show that the join of two principal ideals need not be principal. For example, consider the JP-semilattice **M** given in the Figure 1.3. Here $(a] \vee (b]$ is not principal. We have the following useful results.

Lemma 1.3.5 *Let S be a JP-semilattice. If $x \vee y$ exists, then $(x \vee y] = (x] \vee (y]$.*

Proof. We have $x, y \in (x] \cup (y]$. Hence $x \vee y \in (x] \vee (y]$. Thus $(x \vee y] \subseteq (x] \vee (y]$. The reverse inclusion is trivial. Hence $(x \vee y] = (x] \vee (y]$. □

Theorem 1.3.6 *Let S be a JP-semilattice. For any $x, y \in S$, we have $(x] \vee (y]$ is a principal ideal if and only if $x \vee y$ exists.*

Proof. If $x \vee y$ exists, then by the above lemma $(x] \vee (y] = (x \vee y]$ and hence $(x] \vee (y]$ is a principal ideal of S.

Conversely, let $(x] \vee (y]$ be a principal ideal. Suppose $(x] \vee (y] = (c]$. Then $x, y \leqslant c$. We show that c is the least upper bound of x and y. Suppose $x, y \leqslant d$. Then $(c] = (x] \vee (y] \subseteq (d]$. Hence $c \leqslant d$. Thus $x \vee y$ exists and $x \vee y = c$. □

Let S be a JP-semilattice. The set of all principal ideals of S is denoted by $\mathcal{P}(S)$. Clearly, $S \in \mathcal{P}(S)$ if and only if the largest element $1 \in S$. Define $\mathcal{P}_s(S) = \mathcal{P}(S) \cup \{S\}$. Observe that $\mathcal{P}_s(S)$ is not a sublattice of $\mathcal{I}(S)$ (see the Figure 1.4). Let $\mathcal{X}$ be a collection of principal ideals of S such that $(x], (y] \in \mathcal{X}$

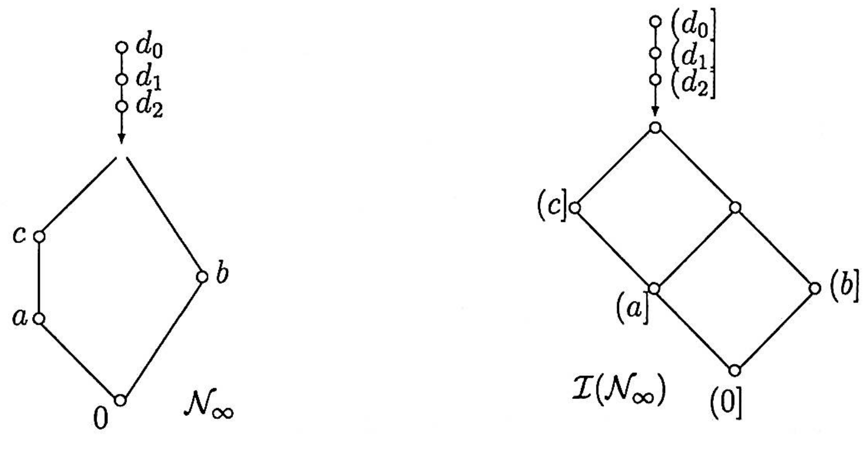


FIGURE 1.4. The ideals lattice

if $x \vee y$ exists in S. Then $\mathcal{X}$ is a sublattice of $\mathcal{I}(S)$. For example, if we consider the JP-semilattice $\mathcal{N}_\infty$ given in Figure 1.4, then $\mathcal{X}_1 = \{(0], (a], (c], (d_i]\}$ and $\mathcal{X}_2 = \{(0], (b], (d_i]\}$ for $i = 0, 1, 2, \cdots$, are the collections of such class. Clearly, for each $i = 1, 2$, we have $\mathcal{X}_i$ is a lattice. In Section 2.4 we characterize the modular and distributive JP-semilattices in terms of the set $\mathcal{X}$.

1.4. JP-homomorphisms

Let $\mathbf{P} := \langle P; \leqslant \rangle$ and $\mathbf{Q} := \langle Q; \leqslant \rangle$ be two ordered sets. A map $\varphi : P \to Q$ is said to be

(i) an **order-preserving** (or **monotone**) if

$$a \leqslant b \text{ in } P \text{ implies } \varphi(a) \leqslant \varphi(b) \text{ in } Q.$$

(ii) an **order-embedding** if

$$a \leqslant b \text{ in } P \text{ if and only if } \varphi(a) \leqslant \varphi(b) \text{ in } Q.$$

(iii) an **order-isomorphism** if φ is an onto order-embedding.

Let $\mathbf{S}$ and $\mathbf{P}$ be two JP-semilattices. A mapping $\varphi : S \to P$ is said to be a **semilattice homomorphism** if for all $x, y \in S$

$$\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y).$$

A semilattice homomorphism $\varphi : \mathbf{S} \to \mathbf{P}$ is said to be a **JP-homomorphism** if for all $x, y \in S$ with $x \vee y$ exists in S implies $\varphi(x) \vee \varphi(y)$ exists in P and

$$\varphi(x \vee y) = \varphi(x) \vee \varphi(y).$$

An one-to-one JP-homomorphism $\varphi : \mathbf{S} \to \mathbf{P}$ is said to be a **JP-embedding** if

$$x \vee y \text{ exists if and only if } \varphi(x) \vee \varphi(y) \text{ exists.}$$

A onto JP-homomorphism is called a **JP-epimorphism**. Also an onto JP-embedding φ is called a **JP-isomorphism**.

Theorem 1.4.1 *Let* A *and* B *be two JP-semilattices and let* $f : A \to B$ *be a map.*

(a) *If* f *is a JP-homomorphism, then* f *is an order-preserving map;*

(b) f *is a JP-isomorphism if and only if* f *is an order-isomorphism.*

Proof. (a) Let f be a JP-homomorphism and let $a \leqslant b$ in A. Then $a \vee b$ exists and $b = a \vee b$. Hence $f(a) \vee f(b)$ exists and $f(b) = f(a \vee b) = f(a) \vee f(b)$. This implies $f(a) \leqslant f(b)$. Thus f is an order-preserving.

(b) Let f be a JP-isomorphism. Then

$$\begin{aligned} a \leqslant b \text{ in } A &\iff a \vee b \text{ exists and } a \vee b = b \\ &\iff f(a) \vee f(b) \text{ exists and } f(b) = f(a \vee b) = f(a) \vee f(b) \\ &\iff f(a) \leqslant f(b) \text{ in } B. \end{aligned}$$

Hence f is an order-embedding. Since f is onto, f is an order-isomorphism.

Conversely, let f be an order-isomorphism. If $a \vee b$ exists, then $a, b \leqslant a \vee b$ if and only if $f(a), f(b) \leqslant f(a \vee b)$. Thus $f(a \vee b)$ is an upper bound of $f(a)$ and $f(b)$. Let c be an upper bound of $f(a)$ and $f(b)$. Since f is onto, there exists $x \in A$ such that $f(x) = c$. Now $f(a), f(b) \leqslant f(x)$ if and only if $a, b \leqslant x$. Thus $a \vee b \leqslant x$. Hence $f(a \vee b) \leqslant f(x) = c$. Therefore, $f(a \vee b)$ is the least upper bound of $f(a)$ and $f(b)$. Hence $f(a) \vee f(b)$ exists and $f(a) \vee f(b) = f(a \vee b)$. By a dual argument we can show that $f(a) \wedge f(b) = f(a \wedge b)$. Hence f is a JP-homomorphism.

Moreover, f is one-one and onto. Now let $f(a) \vee f(b)$ exists. Suppose $f(a) \vee f(b) = c$. Since f is onto, there is $x \in A$ such that $f(x) = c$. Then $f(a), f(b) \leqslant f(x)$. Since f is an order-isomorphism, so $a, b \leqslant x$. We shall show that $a \vee b$ exists and $a \vee b = x$. Let $t \in A$ such that $a, b \leqslant t$. Then $f(a) \vee f(b) \leqslant f(t)$. This implies $f(x) \leqslant f(t)$ and hence $x \leqslant t$ as f is an order-isomorphism. Hence $x = \sup\{a, b\}$. Thus $a \vee b$ exists and $a \vee b = x$. Therefore, f is a JP-embedding and so is a JP-isomorphism. □

CHAPTER 2

Modular and Distributive JP-Semilattices

2.1. Introduction

A JP-semilattice S is said to be **modular** if for all $x, y, z \in S$ with $z \leqslant x$ and $y \vee z$ exists implies

$$x \wedge (y \vee z) = (x \wedge y) \vee z.$$

A JP-semilattice S is said to be **distributive** if for all $x, y, z \in S$ with $y \vee z$ exists implies

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Two examples. Consider the JP-semilattices $\mathcal{N}_\infty$ and $\mathcal{M}_\infty$ given by the following diagrams. The JP-semilattice $\mathcal{N}_\infty$ is said to be the **JP-pentagon** and the JP-semilattice $\mathcal{M}_\infty$ is said to be the **JP-diamond**.

Claim 2.1.1 *The JP-pentagon $\mathcal{N}_\infty$ and the JP-diamond $\mathcal{M}_\infty$ are distributive JP-semilattices.*

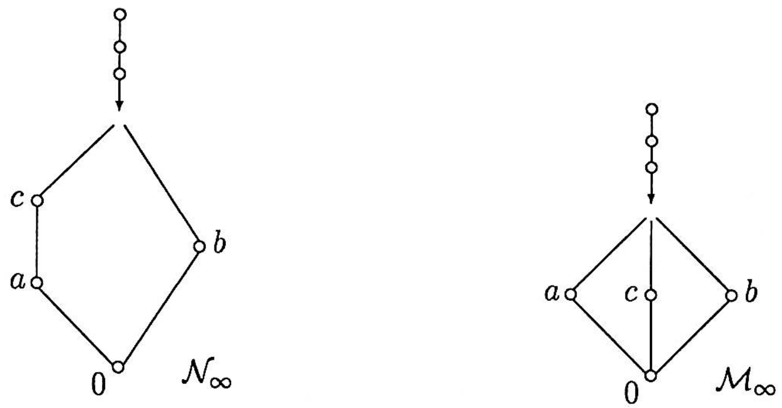


FIGURE 2.1. the JP-pentagon

Proof. In both cases, if $y \vee z$ exists, then clearly, either $y \leqslant z$ or $z \leqslant y$. Without loss of generality, let $y \leqslant z$. Then $x \wedge y \leqslant x \wedge z$. Hence

$$x \wedge (y \vee z) = x \wedge z = (x \wedge y) \vee (x \wedge z).$$

□

Clearly, the concept of modularity and the distributivity of a JP-semilattice S coincides with the concept of modularity and distributivity when S is a lattice. Thus the pentagonal lattice $\mathcal{N}_5$ (see Figure 2.2) is a non-modular and hence a non-distributive JP-semilattice, and the diamond lattice $\mathcal{M}_3$ (see Figure 2.2) is a modular but non-distributive JP-semilattice. In Section 2.2 we discuss the relations among the well known subclasses of distributive JP-semilattices. In Section 2.3 we show that every distributive JP-semilattice is modular but the converse is not necessarily true. Here we give a characterization of modular JP-semilattices. In Section 2.4 we study the lattice of ideals of modular and distributive JP-semilattices. Here we give some characterizations of modular and distributive JP-semilattices. Stone's Separation Theorem play an important role

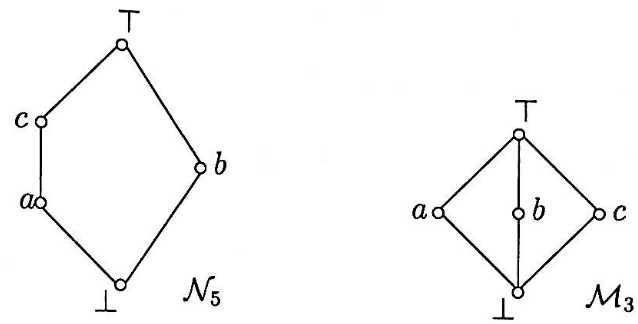


FIGURE 2.2

in Lattice Theory. In Section 2.5 we generalize the result of Stone's Separation Theorem for distributive JP-semilattices. We also extend the result of Stone's Separation Theorem for minimal prime ideals.

2.2. Subclasses of distributive JP-semilattices

A semilattice S $= \langle S; \wedge \rangle$ is said to be a **distributive semilattice** if for each $x, y, z \in S$ with $x \geqslant y \wedge z$ implies the existance of $s \geqslant y$ and $t \geqslant z$ such that $x = s \wedge t$. Rhodes [27] has proved that a smilattice is distributive if and only if it is directed above and it has no retract isomorphic to the pentagonal lattice $\mathcal{N}_5$ or the diamond lattice $\mathcal{M}_3$. Thus the JP-pentagon $\mathcal{N}_\infty$ and the JP-diamond $\mathcal{M}_\infty$ are not distributive semilattices.

In Chapter 1, Section 1.3, we already have mentioned that every semilattice need not be a JP-semilattice. But we have the following result for distributive semilattices.

Theorem 2.2.1 *Let* $\mathbf{S} = \langle S; \wedge, \vee \rangle$ *be a semilattice with a partial binary operation* $\vee$ *which satisfies the axioms (i)-(v) of the definition of JP-semilattice. If* $\langle S; \wedge \rangle$ *is a distributive semilattice, then* $\mathbf{S}$ *is a distributive JP-semilattice.*

Proof. Let $\mathbf{S} = \langle S; \wedge, \vee \rangle$ be a semilattice with a partial binary operation $\vee$ which satisfies the axioms (i)-(v) of the definition of JP-semilattice. Suppose $\langle S; \wedge \rangle$ is a distributive semilattice. Let $y \vee z$ exists for $y, z \in S$. We show that $(x \wedge y) \vee (x \wedge z)$ exists for any $x \in S$. Suppose $p = x \wedge (y \vee z)$. Then trivially, $x \wedge y \leqslant p$ and $x \wedge z \leqslant p$. Let $t \in S$ be such that $x \wedge y \leqslant t$ and $x \wedge z \leqslant t$. Since $\langle S; \wedge \rangle$ is a distributive semilattice, $t = x_1 \wedge z_1$ for some $x_1 \geqslant x$ and $z_1 \geqslant z$. Now $z_1 \geqslant x_1 \wedge z_1 = t \geqslant x \wedge y$ implies $z_1 = x_2 \wedge y_1$ for some $x_2 \geqslant x$ and $y_1 \geqslant y$. Also $y_1 \geqslant x_2 \wedge y_1 = z_1 \geqslant z$. Hence $y_1 \geqslant y \vee z$. Thus $t \geqslant t \wedge x = x_1 \wedge z_1 \wedge x = x \wedge z_1 = x \wedge x_2 \wedge y_1 = x \wedge y_1 \geqslant x \wedge (y \vee z) = p$. This implies that p is the least upper bound of $x \wedge y$ and $x \wedge z$. Hence $(x \wedge y) \vee (x \wedge z)$ exists. This tells us that $\mathbf{S}$ is a JP-semilattice. Moreover $p = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ which implies that $\mathbf{S}$ is a distributive JP-semilattice. □

The above result shows that every JP-semilattice such that the underlying semilattice is distributive is a distributive JP-semilattice. The converse is not necessarily true. For counterexamples, the JP-pentagon $\mathcal{N}_\infty$ and the JP-diamond $\mathcal{M}_\infty$ are distributive JP-semilattices but the underlying semilattices are not distributive. A JP-semilattice such that the underlying semilattice is distributive is said to be a **JP distributive semilattice**. In Chapter 6 we study the JP distributive semilattices.

Another class of semilattices with a partial binary operation has been intensively studied by Cornish and Noor [**13**]. This partial lattice has been called by **Near lattice.** A near lattice is a semilattice such that if any pair of elements has a common upper bound then it has the suppremum. Clearly the JP-pentagon $\mathcal{N}_\infty$ and the JP-diamond $\mathcal{M}_\infty$ are not near lattices because the pair a, b has a common upper bound but $a \vee b$ does not exist. A near lattice **N** is called a **distributive near lattice** if for any $x, y, z \in N$ with $y \vee z$ exists implies $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Observe that the existence of $y \vee z$ implies the existence of $(x \wedge y) \vee (x \wedge z)$ for near lattice. Every distributive near lattice need not be a distributive semilattice. For example, the near lattice **N** given in Fig 2.3 is a distributive near lattice but not a distributive semilattice. On the other hand, every distributive semilattice need not be a distributive near lattice (even need not be a near lattice). For example, the distributive semilattice **A** given in Fig 2.3 is not a near lattice.

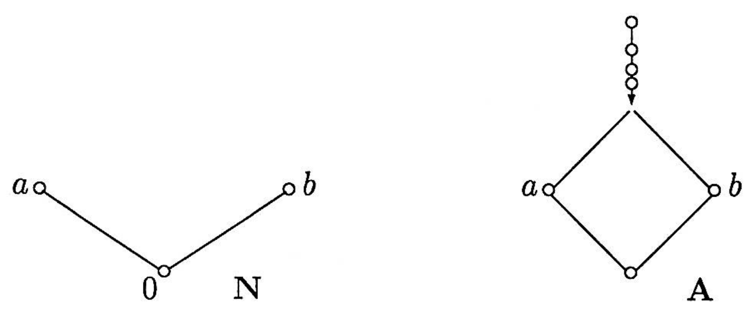


FIGURE 2.3

Thus we have the following classifications of distributive JP-semilattices (see Figure 2.4).

Here

- $\mathcal{DL}$ is the class of all distributive lattices,

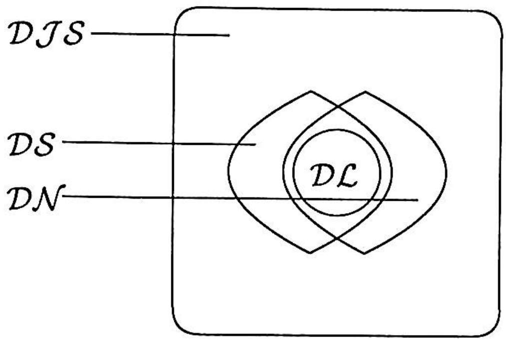


FIGURE 2.4. Classifications of distributive JP-semilattices

- $\mathcal{DN}$ is the class of all distributive near lattices,
- $\mathcal{DS}$ is the class of all distributive semilattices,
- $\mathcal{DJS}$ is the class of all distributive JP-semilattices.

2.3. Characterizations for modular and distributive JP-semilattices

Our first aim is to characterize the modular and distributive JP-semilattices like the well known characterizations for modular and distributive lattices (see Theorem 2.3.1). We refer the reader to [**16, 17, 14, 15**] for the proof of the following result.

Theorem 2.3.1 *Let* **L** *be a lattice. Then*

(a) *L is modular if and only if it has no sublattice isomorphic to the pentagonal lattice* $\mathcal{N}_5$ *(see Figure 2.2);*

(b) *L is distributive if and only if it has no sublattice isomorphic to the pentagonal lattice* $\mathcal{N}_5$ *or the diamond lattice* $\mathcal{M}_3$ *(see Figure 2.2);*

First we have the following results which we need to characterize the modular and distributive JP-semilattices.

Theorem 2.3.2 *Every distributive JP-semilattice is modular but the converse is not necessarily true.*

Proof. Let S be a distributive JP-semilattice and let $a, b, c \in S$ with $c \leqslant a$ and $b \vee c$ exists. Then $c = a \wedge c$ and hence $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) = (a \wedge b) \vee c$. Therefore S is modular. The diamond lattice $\mathcal{M}_3$ given in Figure 2.2 is a modular JP-semilattice but not distributive. □

Theorem 2.3.3 *Every subJP-semilattice of a modular (distributive) JP-semilattice is modular (distributive).*

Proof. Let **M** be a subJP-semilattice of a modular JP-semilattice **L**. Let $a, b, c \in M$ with $c \leqslant a$. If $b \vee c$ exists in M, then this holds in L. Hence $(a \wedge b) \vee c$ exists in L and $a \wedge (b \vee c) = (a \wedge b) \vee c$. Since $a \wedge (b \vee c) \in M$, we have $(a \wedge b) \vee c$ exists in M and $a \wedge (b \vee c) = (a \wedge b) \vee c$. Hence M is a modular JP-semilattice.

By a similar argument we can easily show that every subJP-semilattice of a distributive JP-semilattice is distributive. □

Now we have a characterization of modular JP-semilattices.

Theorem 2.3.4 *Let* S *be a JP-semilattice. Then* S *is non-modular if and only if it has a sublattice isomorphic to the pentagonal lattice*

Proof. Let S be non-modular. Then there exists $a, b, c \in S$ with $c \leqslant a$ such that $b \vee c$ exists and $u = (a \wedge b) \vee c < a \wedge (b \vee c) = v$. Now $v \wedge b = (a \wedge (b \vee c)) \wedge b =$

$a \wedge b$. Hence $u \wedge b \leqslant v \wedge b = a \wedge b \leqslant u$ and hence $a \wedge b \leqslant u \wedge b$. Therefore, $u \wedge b = a \wedge b = v \wedge b$.

Consequently, $b \vee c = (b \vee (a \wedge b)) \vee c = ((a \wedge b) \vee c) \vee b = u \vee b$. First we claim that $v \vee b$ exists. If not, then since $v, b \leqslant b \vee c$, there is an infinite chain $b \vee c > c_1 > c_2 > \cdots$ such that $v, b \leqslant c_i$ for each i. Now $c, b \leqslant c_i$ for each i implies $b \vee c \leqslant c_i$ for each i, which is a contradiction. Hence $v \vee b$ exists. Now $v \vee b \geqslant u \vee b = b \vee c \geqslant v, b$ implies $b \vee c \geqslant v \vee b$. Thus $v \vee b = u \vee b = b \vee c$. Therefore $\{a \wedge b, u, v, b, b \vee c\}$ form a lattice which is isomorphic to the pentagonal lattice.

Conversely, suppose S is modular. Since every subJP-semilattice of a modular lattice is modular, it does not contain the pentagonal lattice as a subJP-semilattice. □

Unfortunately we are unable to give a characterization of distributive JP-semilattices like Theorem 2.3.1. But we have the following conjecture.

Conjecture 2.3.5 *Let* L *be a lattice. Then L is distributive if and only if it has no sublattice isomorphic to the pentagonal lattice $\mathcal{N}_5$ or the diamond lattice $\mathcal{M}_3$*

2.4. Ideals of modular and distributive JP-semilattices

In this section we study the ideals of modular and distributive JP-semilattices. We already mentioned that the description of join of two ideals of a JP-semilattice is complicated. In this section we give some characterizations of modular and distributive JP-semilattices using the lattice of ideals of modular and distributive JP-semilattices. We first have the following result.

Theorem 2.4.1 *Let* S *be a JP-semilattice. If* $\mathcal{I}(S)$ *is modular, then* S *is modular, but the converse is not necessarily true.*

Proof. Let $\mathcal{I}(S)$ be modular and let $x, y, z \in S$ with $z \leqslant x$. Then $(z] \subseteq (x]$. If $y \vee z$ exists, then $(x \wedge y) \vee z$ exists and

$$\begin{aligned}
(x \wedge (y \vee z)] &= (x] \wedge (y \vee z] \\
&= (x] \wedge ((y] \vee (z]), \quad \text{by Lemma 1.3.5} \\
&= ((x] \wedge (y]) \vee (z], \quad \text{as } \mathcal{I}(\mathrm{S}) \text{ is modular} \\
&= (x \wedge y] \vee (z] \\
&= ((x \wedge y) \vee z]
\end{aligned}$$

Thus $x \wedge (y \vee z) = (x \wedge y) \vee z$. Hence **S** is modular.

To prove that the converse is not necessarily true, consider the following Figure 2.5 of a JP-semilattice. Clearly, **B** is modular as it has no sublattice isomorphic to the pentagonal lattice. Observe that the lattice $\mathcal{I}(\mathbf{B})$ contains a sublattice $\{(0], (d], (d, c], (a, b], B\}$ (see the bullet elements) which is isomorphic to the pentagonal lattice and hence $\mathcal{I}(\mathrm{S})$ is non-modular.

□

We have the following useful characterization of modular JP-semilattice. We repeatedly use the Lemma 1.3.5.

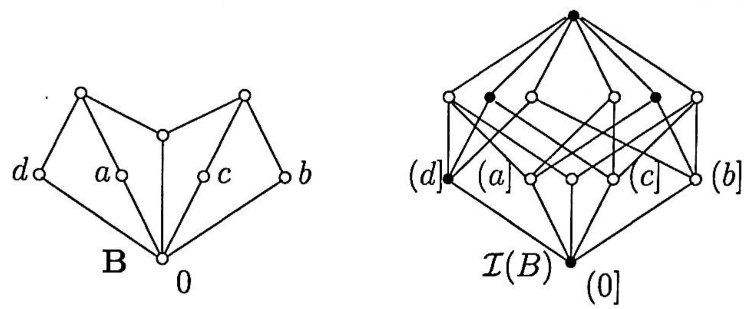


FIGURE 2.5. the butterfly and its lattice of ideals

Theorem 2.4.2 *Let S be a JP-semilattice. Then S is modular if and only if for any $x, y, z \in S$ with $z \leqslant x$ and $y \vee z$ exists implies $(x] \wedge ((y] \vee (z]) = ((x] \wedge (y]) \vee (z]$.*

Proof. Let S be modular and let $x, y, z \in S$ with $z \leqslant x$ and $y \vee z$ exists. Then $(x \wedge y) \vee z$ exists and $(x \wedge y) \vee z = x \wedge (y \vee z)$. Hence

$$(x] \wedge ((y] \vee (z]) = (x] \wedge (y \vee z] = (x \wedge (y \vee z)] = ((x \wedge y) \vee z]$$

$$= (x \wedge y] \vee (z] = ((x] \wedge (y]) \vee (z].$$

Conversely, let the condition holds. Let $x, y, z \in S$ with $z \leqslant x$ and $y \vee z$ exists. Then

$$(x \wedge (y \vee z)] = (x] \wedge (y \vee z] = (x] \wedge ((y] \vee (z]) = ((x] \wedge (y]) \vee (z]$$

$$= (x \wedge y] \vee (z] = ((x \wedge y) \vee z]$$

Thus $x \wedge (y \vee z) = (x \wedge y) \vee z$. Hence S is modular. □

Now we turn our attention to characterize the distributive JP-semilattices. First we have the following useful lemma.

Lemma 2.4.3 *Let I and J be two ideals of a distributive JP-semilattice* S. *Then*

$$I \vee J = \bigcup_{n=0}^{\infty} A_n$$

where $A_0 = I \cup J$ and for $n \geqslant 1$, and

$$A_n = \{x \in S \mid x = y \vee z \text{ for } y, z \in A_{n-1}\}.$$

Proof. By Theorem 1.3.4, we have

$$I \vee J = \bigcup_{n=0}^{\infty} A_n \text{ where } A_0 = I \cup J \text{ and for } n \geqslant 1,$$

$$A_n = \{x \in S \mid x \leqslant y \vee z \text{ for some } y, z \in A_{n-1}\}.$$

Let $x \in A_n$, we have $x \leqslant y \vee z$ for some $y, z \in A_{n-1}$. Then $x = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ as S is distributive. Since $x \wedge y, x \wedge z \in A_{n-1}$, we have $x = i \vee j$ for some $i, j \in A_{n-1}$. So the result holds. □

The following results are the characterizations of distributive JP-semilattices which also generalize the results of distributive lattices.

Theorem 2.4.4 *Let I and J be two ideals of a JP-semilattice* S. *Then the following are equivalent:*

(a) S *is distributive;*

(b) $I \vee J = \{a_1 \vee a_2 \vee \cdots \vee a_n \mid a_i \in I \cup J \text{ for all } i = 1, 2, \cdots, n\}$;

(c) $\mathcal{I}(S)$ *is a distributive lattice;*

(d) *for any $x, y, z \in S$ with $y \vee z$ exists implies*

$$(x] \wedge ((y] \vee (z]) = ((x] \wedge (y]) \vee ((x] \wedge (z]).$$

Proof. (a) $\Rightarrow$ (b). By using mathematical induction of the Lemma 2.4.3.

(b) $\Rightarrow$ (c). Let $I, J, K \in \mathcal{I}(S)$ and $x \in I \cap (J \vee K)$. Then $x \in I$ and $x = a_1 \vee a_2 \vee \cdots \vee a_n$, where $a_i \in J \cup K$ for all $i = 1, 2, \cdots, n$. Now for each $i = 1, 2, \cdots, n$, we have $a_i \leqslant x$ and hence $a_i \in I \cap J$ or $I \cap K$. Hence $a_i \in (I \cap J) \cup (I \cap K)$. Therefore, $x \in (I \cap J) \vee (I \cap K)$. The reverse inclusion is trivial and hence $\mathcal{I}(S)$ is a distributive lattice.

(c) $\Rightarrow$ (d). Trivial.

(d) $\Rightarrow$ (a). Let $x, y, z \in S$ with $y \vee z$ exists. Then

$$\begin{aligned}(x \wedge (y \vee z)] &= (x] \cap ((y] \vee (z]) \\ &= ((x \cap (y]) \vee ((x] \cap (z]) \\ &= (x \wedge y] \vee (x \wedge z] \\ &= ((x \wedge y) \vee (x \wedge z)].\end{aligned}$$

Hence $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Therefore, S is distributive. □

Let S be a JP-semilattice. An element $x \in S$ is said to be **join-irreducible** if $x = a \vee b$ for some $a, b \in S$, then either $x = a$ or $x = b$. The set of all join-irreducible elements of S is denoted by $\mathcal{J}(S)$. Recall that the set of all down-subsets of S is denoted by $\mathcal{O}(S)$. For $a \in S$, define

$$r(a) := \{x \in \mathcal{J}(S) \mid x \leqslant a\}.$$

We have the following result.

Theorem 2.4.5 *Let S be a finite distributive JP-semilattice. Then the map* $\varphi : \mathrm{S} \to \mathcal{O}(\mathcal{J}(S))$ *defined by*

$$\varphi(a) = r(a)$$

is a one-to-one JP-homomorphism.

Proof. First we show that every element of S is a join of join-irreducible elements. Let $a \in S$. If a is a join irreducible element, then there is nothing to prove. If a is not join-irreducible element, then there are $x, y \in S$ with $a = x \vee y$ such that $a \neq x$ and $a \neq y$. If both x and y are join-irreducible then we have the proof. If any of x and y is not join-irreducible, then we continue the process. Since S is finite, we obtain a set of join-irreducible elements whose join is a and hence we have the proof. Therefore, for each $a \in S$ we have

$$a = \bigvee r(a).$$

Let $x \in r(a) \cap r(b)$. Then $x \in \mathcal{J}(S)$ and $x \leqslant a, b$ and hence $x \leqslant a \wedge b$. Thus $x \in r(a \wedge b)$. This implies $r(a) \cap r(b) \subseteq r(a \wedge b)$. The reverse inclusion is trivial. Hence $r(a) \cap r(b) = r(a \wedge b)$. This shows that

$$\varphi(a \wedge b) = r(a \wedge b) = r(a) \cap r(b) = \varphi(a) \wedge \varphi(b).$$

Let $a, b \in S$ with $a \vee b$ exists. Let $x \in r(a \vee b)$. Then $x \in \mathcal{J}(S)$ and $x \leqslant a \vee b$. Hence $x = x \wedge (a \vee b) = (x \wedge a) \vee (x \wedge b)$ as S is distributive. Hence either $x = x \wedge a$ or $x = x \wedge b$. Thus either $x \leqslant a$ or $x \leqslant b$. Hence either $x \in r(a)$ or $x \in r(b)$. This implies $x \in r(a) \cup r(b)$. Therefore $r(a \vee b) \subseteq r(a) \cup r(b)$. The reverse inclusion is trivial. Hence $r(a \vee b) = r(a) \cup r(b)$. Thus

$$\varphi(a \vee b) = r(a \vee b) = r(a) \cup r(b) = \varphi(a) \vee \varphi(b).$$

Therefore, φ is a JP-homomorphism. To prove φ is one-to-one, let $\varphi(a) = \varphi(b)$. Then $r(a) = r(b)$. Hence $a = b$.

Therefore, φ is a one-to-one JP-homomorphism. □

Now we give a characterization of distributive JP-semilattices using down-subsets. First we have the following Lemma.

Lemma 2.4.6 *Let S be a distributive JP-semilattice and $K \in \mathcal{H}(S)$. Then*

$$(K] = \{x_1 \vee x_2 \vee \cdots x_n \mid x_i \in K \text{ for each } i = 1, 2, \cdots, n\}.$$

Proof. Let $x \in (K]$. If $x \in K_0 = K$, then by Theorem 1.3.3 the result is trivial. Suppose $x \in K_n$ for some $n \geqslant 1$. Then by Theorem 1.3.3, $x \leqslant y \vee z$ for some $y, z \in K_{n-1}$. This implies $x = x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ as S is distributive. Since K_{n-1} is a down-set (see Lemma 1.3.1), we have $x \wedge y, x \wedge z \in K_{n-1}$. Hence $x = k_1 \vee k_2$ for some $k_1, k_2 \in K_{n-1}$. By using mathematical induction we can show that $x = x_1 \vee x_2 \vee \cdots \vee x_n$ where $x_i \in K$ for each $i = 1, 2, \cdots, n$. □

Now we have the following result. This idea has been taken from [**10**, Theorem 2.3].

Theorem 2.4.7 *Let S be a JP-semilattice. For any $A, B, C \in \mathcal{O}(S)$ the following conditions are equivalent:*

(a) *S is distributive;*

(b) $(A] = \{a_1 \vee a_2 \vee \cdots \vee a_n \mid a_1, a_2, \cdots, a_n \in A\}$;

(c) $A \cap (B] \subseteq (A \cap B]$;

(d) $(A \cap B] = (A] \cap (B]$;

(e) $(A \cap (B \cap C]] = ((A \cap B] \cap C]$;

(f) *The map* $\varphi : \mathcal{O}(S) \to I(S)$ *defined by* $\varphi(A) = (A]$ *is a onto lattice-homomorphism.*

Proof. (a) $\Rightarrow$ (b). By the Lemma 2.4.6.

(b) $\Rightarrow$ (c). Let $x \in A \cap (B]$. Then $x \in A$ and by (b), $x = b_1 \vee b_2 \vee \cdots \vee b_n$ where $b_1, b_2, \cdots, b_n \in B$. Since $A \in \mathcal{O}(S)$ and $b_i \leqslant x$ for all $i = 1, 2, \cdots, n$ we have $b_i \in A$ for all $i = 1, 2, \cdots, n$. Hence $b_i \in A \cap B$ for all $i = 1, 2, \cdots, n$. Therefore, $x \in (A \cap B]$.

(c) $\Rightarrow$ (d). By (c), we have $(A] \cap (B] \subseteq ((A] \cap B] \subseteq (A \cap B]$ for any $A, B \in \mathcal{O}(S)$. Since $(A \cap B] \subseteq (A] \cap (B]$, we have $(A \cap B] = (A] \cap (B]$. Thus (d) holds.

(d) $\Rightarrow$ (e). Suppose (d) holds. Then

$$(A \cap (B \cap C]] = (A] \cap (B \cap C] = (A] \cap ((B] \cap (C])$$
$$= ((A] \cap (B]) \cap (C] = (A \cap B] \cap (C] = ((A \cap B] \cap C].$$

Thus (e) holds.

(e) $\Rightarrow$ (c). By taking $C = S$ in (e).

(d) $\Rightarrow$ (f). For any $A, B \in \mathcal{O}(S)$, we have $(A \cup B] = (A] \vee (B]$. Hence for $A, B \in \mathcal{O}(S)$,

$$\varphi(A \cap B) = (A \cap B] = (A] \cap (B] \text{ by } (d)$$
$$= \varphi(A) \cap \varphi(B)$$

and

$$\varphi(A \cup B) = (A \cup B] = (A] \vee (B] = \varphi(A) \cup \varphi(B).$$

Hence φ is a lattice homomorphism. Let $I \in \mathcal{I}(S)$. Then $\varphi(I) = (I] = I$. Thus φ is a onto lattice homomorphism. Therefore, (f) holds.

(f) $\Rightarrow$ (a). Since $\mathcal{O}(S)$ is always a distributive lattice, by (f) we have $I(S)$ is a distributive lattice and hence by Theorem 2.4.4, S is distributive. □

2.5. The Separation Theorem

Let S be a JP-semilattice. A non-empty subset F of S is said to be a **filter** (or **dual ideal**) if

(i) for $x \in F$ and $y \in S$ with $x \leqslant y$ implies $y \in F$, and

(ii) for $x, y \in F$ implies $x \wedge y \in F$.

The set of all filters of S is denoted by $\mathcal{F}(S)$. A filter F of S is called **prime** if $x, y \in S$ with $x \vee y$ exists and $x \vee y \in F$ implies either $x \in F$ or $y \in F$.

Lemma 2.5.1 *Let S be a JP semilattice. An ideal (filter) P is prime if and only if $S \setminus P$ is a prime filter (ideal).*

Proof. Let P be a prime ideal. If $x, y \in S \setminus P$, then $x, y \notin P$. Hence $x \wedge y \notin P$ which implies $x \wedge y \in S \setminus P$. Let $x \in S \setminus P$ and $x \leqslant y$. Then $x \notin P$ and hence $y \notin P$. Therefore $y \in S \setminus P$. This implies $S \setminus P$ is a filter. Let $x, y \in S$ with $x \vee y$ exists and $x \vee y \in S \setminus P$. Then $x \vee y \notin P$. This implies either $x \notin P$ or $y \notin P$ and consequently, either $x \in S \setminus P$ or $y \in S \setminus P$. Hence $S \setminus P$ is a prime filter.

By a reverse argument we have the converse of the above statement. □

We use the following famous lemma.

Lemma 2.5.2 (Zorn's Lemma) *In a partially ordered set* **P**, *if every chain of* **P** *has a largest element, then* **P** *has a maximal element.*

□

An ideal P of a JP-semilattice S is called **prime** if $a, b \in S$ with $a \wedge b \in P$ implies either $a \in P$ or $b \in P$. A prime ideal P containing an ideal J is called **a minimal prime ideal containing** J if for any prime ideal Q containing J with $Q \subseteq P$ implies $P = Q$. A minimal prime ideal containing $(0]$ is called a **minimal prime ideal.**

Lemma 2.5.3 *Let* S *be a JP-semilattice with* 0. *Then every prime ideal of S contains a minimal prime ideal.*

Proof. Let P be a prime ideal of S and let

$$\mathcal{X} = \{Q \subseteq P \mid Q \text{ is a prime ideal of } S\}.$$

Then $\mathcal{X}$ is nonempty since $P \in \mathcal{X}$. Let $\mathcal{C}$ be a chain in $\mathcal{X}$ and let $Q = \cap(X \mid X \in \mathcal{C})$. Then $Q \neq \emptyset$ since $0 \in Q$ and Q is an ideal. For clearly Q is a down-set since X is a down-set for all $X \in \mathcal{C}$. If $x, y \in Q$ and $x \vee y$ exists, then $x, y \in X$ for all $X \in \mathcal{C}$. Hence $x \vee y \in X$ for all $X \in \mathcal{C}$ as X is an ideal. Therefore $x \vee y \in Q$. Thus Q is an ideal of S. In fact, Q is prime. Indeed, if $x \wedge y \in Q$ for some $x, y \in S$, then $x \wedge y \in X$ for all $X \in \mathcal{C}$. Since X is prime, either $x \in X$ or $y \in X$. Thus either $Q = \cap(X \mid x \in X)$ or $Q = \cap(X \mid y \in X)$, providing that $x \in Q$ or

$y \in Q$. Therefore, by the dual form of Zorn's Lemma we have a minimal prime member of $\mathcal{X}$. □

Now we have the following Separation Theorem for distributive JP-semilattice.

Theorem 2.5.4 (The JP-separation Theorem) *Let* S *be a JP-semilattice. Then the following are equivalent:*

(a) S *is distributive;*

(b) *For any ideal I and any filter F of* S *such that $I \cap F = \emptyset$, there exists a prime ideal P containing I such that $P \cap F = \emptyset$.*

Proof. (a) $\Rightarrow$ (b). Let $\mathcal{I}$ be the set of all ideals containing I, but disjoint from F. Then $\mathcal{I} \neq \emptyset$ as $I \in \mathcal{I}$. Let $\mathcal{C}$ be a chain in $\mathcal{I}$ and let $M := \cup\{X \mid X \in \mathcal{C}$. We claim that M is the maximum element in $\mathcal{C}$.

Let $x \in M$ and $y \leqslant x$. Then $x \in X$ for some $X \in \mathcal{C}$. Hence $y \in X$ as X is an ideal. Therefore $y \in M$. Let $x, y \in M$. Then $x \in X$ and $y \in Y$ for some $X, Y \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, either $X \subseteq Y$ or $Y \subseteq X$. Suppose $X \subseteq Y$. So $x, y \in Y$. If $x \vee y$ exists, then $x \vee y \in Y$ as Y is an ideal. Hence $x \vee y \in M$. Moreover, M contains I and $F \cap M = \emptyset$. Therefore, M is the maximum element in $\mathcal{C}$.

Thus by Zorn's Lemma, $\mathcal{I}$ has a maximal element, say, P. We claim that P is prime. If P is not prime, there exists $a, b \in S$ such that $a, b \notin P$ but $a \wedge b \in P$. Then $(P \vee (a]) \cap F \neq \emptyset$ and $(P \vee (b]) \cap F \neq \emptyset$ as P is maximal. Hence there exists $x, y \in F$ such that $x \wedge y \in (P \vee (a]) \cap (P \vee (b]) = P \vee ((a] \wedge (b]) = P \vee (a \wedge b]$ as S is distributive implies $\mathcal{I}(S)$ is distributive. Thus $x \wedge y \in F$ and $x \wedge y \in P \vee (a \wedge b] = P$, which is a contradiction to $P \cap F = \emptyset$. Hence P is a prime ideal.

(b) $\Rightarrow$ (a). Let $a, b, c \in S$ such that $b \vee c$ exists. If $(a \wedge b) \vee (a \wedge c) \neq a \wedge (b \vee c)$, then $(a \wedge b) \vee (a \wedge c) < a \wedge (b \vee c)$. Consider $I = ((a \wedge b) \vee (a \wedge c)]$ and $F = [a \wedge (b \vee c))$. Then $I \cap F = \emptyset$ and hence by (b), there is a prime ideal P such that $I \subseteq P$ and $P \cap F = \emptyset$. Thus $(a \wedge b) \vee (a \wedge c) \in P$, this implies $a \wedge b \in P$ and $a \wedge c \in P$. So, either $a \in P$ or $b \vee c \in P$. Hence $a \wedge (b \vee c) \in P$, which is a contradiction. Therefore, $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$. Hence S is distributive. □

Corollary 2.5.5 *Let S be a distributive JP-semilattice and let I be an ideal of S. If $a \notin I$, then there exists a prime ideal P containing I such that $a \notin P$.* □

Theorem 2.5.6 *Let S be a distributive JP-semilattice. Then every ideal of S is the intersection of all prime ideals containing it.*

Proof. Let S be a JP-semilattice and let J be an ideal of S. We shall show that

$$J = \bigcap \{P \mid P \text{ is a prime ideal of } S \text{ and } J \subseteq P\}.$$

Clearly, $J \subseteq$ R.H.S. If $J \neq$ R.H.S., then there is $x \in$R.H.S. such that $x \notin J$. Hence by the Separation Theorem, there is a prime ideal Q of S such that $J \subseteq Q$ and $x \notin Q$, which is a contradiction. □

The following theorem is a characterization of a minimal prime ideal containing an ideal. This is also a generalization of [**22**, Lemma 3.1]

Theorem 2.5.7 *Let S be a distributive JP-semilattice and let J be an ideal of S. Then a prime ideal P containing J is a minimal prime ideal containing J if and only if for each $x \in P$ there is $y \in S \setminus P$ such that $x \wedge y \in J$.*

Proof. Let P be a prime ideal of S containing J such that the given condition holds. We shall show that P is a minimal prime ideal containing J. Let K be a prime ideal containing J such that $K \subseteq P$. Let $x \in P$. Then there is $y \in S \setminus P$ such that $x \wedge y \in J$. Hence $x \wedge y \in K$ as K contains J. Since K is prime and $y \notin K$ implies $x \in K$. Hence $P \subseteq K$. Thus $K = P$. Therefore P is a minimal prime ideal containing J.

Conversely, let P be a minimal prime ideal containing J. Let $x \in P$. Suppose for all $y \in S \setminus P$, $x \wedge y \notin J$. Set $D = (S \setminus P) \vee [x)$. We claim that $0 \notin D$. For if $0 \in D$, then $0 = q \wedge x$ for some $q \in S \setminus P$. Thus, $x \wedge q = 0 \in J$, which is a contradiction. Therefore, $0 \notin D$. Then by the JP-separation Theorem 2.5.4, there is a prime filter Q such that $D \subseteq Q$ and $0 \notin Q$. Let $M = S \setminus Q$. Then by Lemma 2.5.1, M is a prime ideal. We claim that $M \cap D = \emptyset$. If $a \in M \cap D$, then $a \in M$ and hence $a \notin Q$. Thus $a \notin D$ which is a contradiction. Hence $M \cap D = \emptyset$. Therefore, $M \cap (S \setminus P) = \emptyset$ and hence $M \subseteq P$. Also $M \neq P$, because $x \in D$ implies $x \in Q$ and hence $x \notin M$ but $x \in P$. This shows that P is not minimal, which is a contradiction. Hence the given condition holds. □

Theorem 2.5.8 *Let* S *be a JP-semilattice with* 0 *and let P be a prime ideal of S. Let C be a chain of all prime ideals X of S such that $X \subseteq P$. Then*

$$Q = \bigcap\{X \subseteq P \mid X \in C\}$$

is a prime ideal and hence it is a minimal prime ideal.

Proof. Clearly, C is non-empty as $P \in C$ and Q is non-empty as $0 \in Q$. Obviously, Q is an ideal. To show that Q is prime, let $x \wedge y \in Q$. Suppose $x \notin Q$.

This implies $x \notin X$ for some $X \in \mathcal{C}$. Now $x \wedge y \in Q$ implies $x \wedge y \in X$. Hence $y \in X$ as X is prime. We claim that $y \in Q$. If not, then $y \notin Y$ for some $Y \in C$ with $Y \subset X$. But $x \wedge y \in Q$ implies $x \wedge y \in Y$. Thus $x \in Y$ and so $x \in X$ as $Y \subset X$ which gives a contradiction. Therefore $y \in Q$. Hence Q is prime and in fact it is a minimal prime ideal. □

Thus we have the following extension of Stone's Separation Theorem.

Theorem 2.5.9 *Let J be an ideal and D be a filter of a distributive JP-semilattice* S *such that $J \cap D = \emptyset$. Then there exists a minimal prime ideal Q containing J such that $Q \cap D = \emptyset$.*

Proof. Let J be an ideal and D be a filter of a distributive JP-semilattice S such that $J \cap D = \emptyset$. Then by the Stone's JP-separation Theorem 2.5.4, there exists a prime ideal P containing J such that $P \cap D = \emptyset$. Choose any chain $\mathcal{C}$ of prime ideals X containing J such that $X \subseteq P$. Let $Q = \cap\{X \in \mathcal{C}\}$. Then by above Theorem 2.5.8, Q is a minimal prime ideal containing J and $Q \cap D = \emptyset$.
□

Let S be a JP-semilattice with 0 and let Q be a prime ideal of S. Define

$$O(Q) := \{x \in S \mid x \wedge y = 0 \text{ for some } y \in S \setminus Q\}.$$

The following theorem is a generalization of [8, Proposition 2.2]

Theorem 2.5.10 *Let* S *be a distributive JP-semilattice with* 0 *and let Q be a prime ideal of S. Then*

$$O(Q) = \bigcap\{P \mid P \text{ is a minimal prime ideal of } S \text{ such that } P \subseteq Q\}.$$

Proof. Suppose $X = \bigcap\{P \mid P$ is a minimal prime ideal of S such that $P \subseteq Q\}$. Let $x \in O(Q)$. Then $x \wedge y = 0$ for some $y \notin Q$. Let P be a minimal prime ideal contained in Q. Clearly, $y \notin P$. Since $x \wedge y = 0 \in P$ and P is prime, we have $x \in P$. Hence $x \in X$.

Conversely let, $x \in X$. If $x \notin O(Q)$. Then $x \wedge y \neq 0$ for all $y \in S \setminus Q$. Let $D = [x) \vee (S \setminus Q)$. Then $0 \notin D$. For if $0 \in D$, then $x \wedge q = 0$ for some $q \in S \setminus Q$ which is a contradiction. Therefore, $0 \notin D$. Consequently, there is a minimal prime ideal M such that $M \cap D = \emptyset$. Therefore, $M \cap (S \setminus Q) = \emptyset$. Hence $M \subseteq Q$. Also $M \neq Q$ because $x \in Q$. But $x \in D$ implies $x \notin M$. This shows that there is a minimal prime ideal $M \subset Q$ such that $x \notin M$ which is a contradiction to fact that $x \in X$. Hence $x \in O(Q)$. □

CHAPTER 3

Congruences on JP-Semilattices

3.1. Introduction

Let S be a JP-semilattice. An equivalence relation θ on S is said to be **compatible** with $\wedge$ if $a \equiv b(\theta)$ and $c \equiv d(\theta)$ implies $a \wedge c \equiv b \wedge d(\theta)$. Let θ be an equivalence relation on S. Then θ is said to be a **meet congruence** if it is compatible with $\wedge$. A meet congruence θ is said to be a **JP-congruence** if it is conditional compatible with $\vee$. That is, if $a \equiv b(\theta)$ and $c \equiv d(\theta)$, then $a \vee c \equiv b \vee d(\theta)$ whenever $a \vee c$ and $b \vee d$ exist.

Let θ be a JP-congruence on S. If $x \equiv y(\theta)$, then $x \wedge y \equiv y(\theta)$ and $x \wedge y \equiv x(\theta)$. So, x, y and $x \wedge y$ is in the same class. For this reason we can choose $x \equiv y(\theta)$ with $x \leqslant y$.

The set of all JP-congruences on S is denoted by Con(S). It is evident that Con(S), when ordered by set inclusion, is an algebraic lattice. We denote the algebraic lattice by $\mathbb{C}on(S)$.

In Section 3.2 we give some properties of JP-congruences which are useful for the calculation to show a binary relation is a JP-congruence. Here we also describe the largest and smallest JP-congruences containing an ideal as a class.

In Section 3.3 we give some characterizations of a distributive JP-semilattice. In Section 3.4 we prove the homomorphism theorem for JP-semilattices. In this section we also introduce a new notion of a filter. We call it by strong filter.

3.2. Some properties of congruences

For the computation to show that a binary relation θ is a JP-congruence the following result will be helpful.

Proposition 3.2.1 *Let S be a distributive JP-semilattice. For all $x, y, z \in S$ if $x \vee z$ and $y \vee z$ exist, then $(x \wedge y) \vee z$ exists and*

$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z).$$

Proof. By the axiom (vi) of JP-semilattice, $x \vee z$ and $y \vee z$ exists implies $((x \vee z) \wedge y) \vee ((x \vee z) \wedge z)$ exists. Then by (v), $((x \vee z) \wedge y) \vee z$ exists. Now $x \vee z$ exists implies $(x \wedge y) \vee (z \wedge y)$ exists and $(x \wedge y) \vee (z \wedge y) = (x \vee z) \wedge y$. Hence $((x \wedge y) \vee (z \wedge y)) \vee z$ exists. That is, $(x \wedge y) \vee z$ exists. Now

$$\begin{aligned}(x \vee z) \wedge (y \vee z) &= ((x \vee z) \wedge y) \vee ((x \vee z) \wedge z) \\ &= ((x \vee z) \wedge y) \vee z = ((x \wedge y) \vee (z \wedge y)) \vee z = (x \wedge y) \vee z\end{aligned}$$

□

We often use the following result.

Lemma 3.2.2 *Let S be a JP-semilattice, θ be a JP-congruence on S and $x, y, z \in S$. Then*

(a) *If* $x \equiv y(\theta)$, *then* $x \wedge z \equiv y \wedge z(\theta)$ *and* $x \vee z \equiv y \vee z(\theta)$ *whenever* $x \vee z$ *and* $y \vee z$ *exist.*

(b) *If* $x \equiv y(\theta)$ *and* $x \leqslant z \leqslant y$, *then* $x \equiv z(\theta)$.

(c) $x \equiv y(\theta)$ *if and only if* $x \wedge y \equiv x \vee y(\theta)$ *whenever* $x \vee y$ *exists.*

Proof. Assume that θ is a congruence on S.

(a) Let $x \equiv y(\theta)$. Since $z \equiv z(\theta)$ and θ is a congruence, we have $x \wedge z \equiv y \wedge z(\theta)$ and $x \vee z \equiv y \vee z(\theta)$ whenever $x \vee z$ and $y \vee z$ exist.

(b) Let $x \equiv y(\theta)$ and let $x \leqslant z \leqslant y$. Then $x = x \wedge z \equiv y \wedge z(\theta)$ by (a). Thus $x \equiv z(\theta)$.

(c) Let $x \equiv y(\theta)$ and let $x \vee y$ exists. Then by (a), $x \vee y \equiv y(\theta)$ and $x \wedge y \equiv y(\theta)$. Hence by symmetric and transitive property of θ, we have $x \wedge y \equiv x \vee y(\theta)$. Conversely, let $x \vee y$ exists and $x \wedge y \equiv x \vee y(\theta)$. Since $x \wedge y \leqslant x, y \leqslant x \vee y$, by (b) we have $x \wedge y \equiv x(\theta)$ and $x \wedge y \equiv y(\theta)$. Hence by symmetric and transitive property of θ, we have $x \equiv y(\theta)$. □

Theorem 3.2.3 *Let* S *be a distributive JP-semilattice and* I *be an ideal of* S. *Then the relation* $\Theta(I)$ *on* S *defined by*

$$x \equiv y(\Theta(I)) \Leftrightarrow (x] \vee I = (y] \vee I$$

is a JP-congruence having I *as a class. Moreover if each JP-congruence is compatible with any finite existing* $\vee$, *then* $\Theta(I)$ *is the the smallest JP-congruence having* I *as a class.*

Proof. Clearly, $\Theta(I)$ is an equivalence relation. Suppose $x \equiv y(\Theta(I))$ and $s \equiv t(\Theta(I))$. Then $(x] \vee I = (y] \vee I$ and $(s] \vee I = (t] \vee I$ and hence

$$(x \wedge s] \vee I = ((x] \wedge (s]) \vee I = ((x] \vee I) \wedge ((s] \vee I) \text{ as } \mathcal{I}(S) \text{ is distributive}$$

$$= ((y] \vee I) \wedge ((t] \vee I) = ((y] \wedge (t]) \vee I = (y \wedge t] \vee I.$$

Thus $x \wedge s \equiv y \wedge t(\Theta(I))$. Also if $x \vee s$ and $y \vee t$ exists, then

$$(x \vee s] \vee I = ((x] \vee (s]) \vee I = ((x] \vee I) \vee ((s] \vee I)$$

$$= ((y] \vee I) \vee ((t] \vee I) = ((y] \vee (t]) \vee I = (y \vee t] \vee I.$$

Thus $x \vee s \equiv y \vee t(\Theta(I))$. Therefore, $\Theta(I)$ is a JP-congruence. Clearly, $\Theta(I)$ contains I as a class. Finally, let Γ be a JP-congruence containing I as a class. Suppose $x \equiv y(\Theta(I))$ with $x \leqslant y$. Then $(x] \vee I = (y] \vee I$. Thus $y \in (x] \vee I$. Hence $y = x_1 \vee x_2 \vee \cdots \vee x_n \vee i_1 \vee \cdots \vee i_m$ for some $x_1, x_2, \cdots x_n \leqslant x$ and $i_1, \cdots, i_m \in I$. This implies

$$y = x \vee i_1 \vee \cdots \vee i_m$$

and trivialy

$$x = x \vee (x \wedge i_1) \vee \cdots \vee (x \wedge i_m).$$

Since $x \equiv x(\Gamma)$ and for each $j = 1, \cdots, m$, we have $i_j \equiv x \wedge i_j(\Gamma)$, so $x \equiv y(\Gamma)$. Thus $\Theta(I) \subseteq \Gamma$. $\square$

Now, we have a description of $\Theta(a, b)$.

Theorem 3.2.4 *Let S be a distributive JP-semilattice and $a, b, x, y \in S$ with $a \leqslant b$. Then*

$$x \equiv y(\Theta(a,b)) \Leftrightarrow x \wedge a = y \wedge a \text{ and } (x] \vee (b] = (y] \vee (b].$$

Proof. Let ψ denote the binary relation on S such that

$$x \equiv y(\psi) \Leftrightarrow x \wedge a = y \wedge a \text{ and } (x] \vee (b] = (y] \vee (b].$$

Then clearly ψ is an equivalence relation. Now let $x \equiv y(\psi)$ and $s \equiv t(\psi)$. Then $x \wedge a = y \wedge a$, $(x] \vee (b] = (y] \vee (b]$, $s \wedge a = t \wedge a$ and $(s] \vee (b] = (t] \vee (b]$. Hence $(x \wedge s) \wedge a = (y \wedge t) \wedge a$ and since S is distributive implies $\mathcal{I}(S)$ is distributive, so

$$(x \wedge s] \vee (b] = ((x] \wedge (s]) \vee (b] = ((x] \vee (b]) \wedge ((s] \vee (b])$$

$$= ((y] \vee (b]) \wedge ((t] \vee (b]) = ((y] \wedge (t]) \vee (b] = (y \wedge t] \vee (b].$$

Thus $x \wedge s \equiv y \wedge t(\psi)$. Also if $x \vee s$ and $y \vee t$ exists, then since S is distributive,

$$(x \vee s) \wedge a = (x \wedge a) \vee (s \wedge a) = (y \wedge a) \vee (t \wedge a) = (y \vee t) \wedge a$$

and

$$(x \vee s] \vee (b] = ((x] \vee (s]) \vee (b] = ((x] \vee (b]) \vee ((s] \vee (b])$$

$$= ((y] \vee (b]) \vee ((t] \vee (b]) = ((y] \vee (t]) \vee (b] = (y \vee t] \vee (b].$$

Thus $x \vee s \equiv y \vee t(\psi)$. Therefore, ψ is a JP-congruence. Clearly $a \equiv b(\psi)$. Let Γ be a congruence on S such that $a \equiv b(\Gamma)$. Let $x \equiv y(\psi)$ with $x \leqslant y$. Then $x \wedge a = y \wedge a$ and $(x] \vee (b] = (y] \vee (b]$. Since $a \equiv b(\Gamma)$ so, $x \wedge a \equiv x \wedge b(\Gamma)$ and

$y \wedge a \equiv y \wedge b(\Gamma)$. Thus $x \wedge b \equiv x \wedge a(\Gamma) = y \wedge a \equiv y \wedge b(\Gamma)$. Now we have

$$(y] = (y] \wedge ((y] \vee (b]) = (y] \wedge ((x] \vee (b]) = ((y] \wedge (x]) \vee ((y] \wedge (b]) = (x] \vee (y \wedge b].$$

This shows that $(x] \vee (y \wedge b]$ is a principal ideal and hence by Theorem 1.3.6 we have $y = x \vee (y \wedge b) \equiv x \vee (x \wedge b)(\Gamma) = x$. Hence ψ is the smallest congruence.

Therefore, $\psi = \Theta(a, b)$. □

It is well known that the binary relation $\psi(I)$ on a semilattice **S** defined by

$$x \equiv y(\psi(I)) \text{ if and only if } x \wedge a \in I \Leftrightarrow y \wedge a \in I \text{ for any } a \in S.$$

is a largest semilattice congruence containing an ideal I as a class. Now we have the following result for distributive JP-semilattices.

Theorem 3.2.5 *Let* **S** *be a distributive JP-semilattice and let I be an ideal of S. Then $\psi(I)$ is the largest JP-congruence containing I as a class.*

Proof. It is enough to show that $\psi(I)$ has the substitution property for partial operation $\vee$. Let $x \equiv y(\psi(I))$ and $s \equiv t(\psi(I))$ and $x \vee s$ and $y \vee t$ exist. Since **S** is a distributive JP-semilattice, for any $a \in S$ we have $(x \wedge a) \vee (s \wedge a)$, $(y \wedge a) \vee (t \wedge a)$ exist and $(x \vee s) \wedge a = (x \wedge a) \vee (s \wedge a)$, $(y \vee t) \wedge a = (y \wedge a) \vee (t \wedge a)$. Thus

$$\begin{aligned}(x \vee s) \wedge a \in I &\Leftrightarrow (x \wedge a) \vee (s \wedge a) \in I \\ &\Leftrightarrow x \wedge a \in I \text{ and } s \wedge a \in I \\ &\Leftrightarrow y \wedge a \in I \text{ and } t \wedge a \in I \\ &\Leftrightarrow (y \wedge a) \vee (t \wedge a) \in I \\ &\Leftrightarrow (y \vee t) \wedge a \in I.\end{aligned}$$

Thus $x \vee s \equiv y \vee t(\psi(I))$. Hence $\psi(I)$ is the largest JP-congruence. □

3.3. Kernel of a JP-homomorphism

Let $\varphi : \mathbf{S} \to \mathbf{P}$ be a JP-homomorphism. The **kernel** of φ is denoted by $\ker \varphi$ and defined by

$$\ker \varphi = \{(x, y) \in S^2 \mid \varphi(x) = \varphi(y)\}.$$

Lemma 3.3.1 *Let $\varphi : \mathbf{S} \to \mathbf{P}$ be a JP-homomorphism. Then $\ker \varphi$ is a JP-congruence on S.*

Proof. Clearly $\ker \varphi$ is an equivalence relation on S. Let $x_1 \equiv y_1(\ker \varphi)$ and $x_2 \equiv y_2(\ker \varphi)$. Then $\varphi(x_1) = \varphi(y_1)$ and $\varphi(x_2) = \varphi(y_2)$. Now $\varphi(x_1 \wedge x_2) = \varphi(x_1) \wedge \varphi(x_2) = \varphi(y_1) \wedge \varphi(y_2) = \varphi(y_1 \wedge y_2)$. Therefore, $x_1 \wedge x_2 \equiv y_1 \wedge y_2(\ker \varphi)$. To prove $\ker \varphi$ is conditional compatible with $\vee$, suppose $x_1 \vee x_2$ and $y_1 \vee y_2$ exist. Then by the definition of a JP-homomorphism, $\varphi(x_1) \vee \varphi(x_2)$ and $\varphi(y_1) \vee \varphi(y_2)$ exist and $\varphi(x_1 \vee x_2) = \varphi(x_1) \vee \varphi(x_2)$ and $\varphi(y_1 \vee y_2) = \varphi(y_1) \vee \varphi(y_2)$. Hence $\varphi(x_1 \vee x_2) = \varphi(x_1) \vee \varphi(x_2) = \varphi(y_1) \vee \varphi(y_2) = \varphi(y_1 \vee y_2)$. Thus $x_1 \vee x_2 \equiv y_1 \vee y_2(\ker \varphi)$.

Therefore $\ker \varphi$ is a JP-congruence. □

We have the following important result for distributive JP-semilattices.

Theorem 3.3.2 *Let $\mathbf{S}$ be a JP-semilattice. The following conditions are equivalent:*

(a) $\mathbf{S}$ *is distributive;*

(b) *for* $a \in S$, *the map* $\varphi : S \mapsto (a]$ *given by*

$$\varphi(x) = a \wedge x$$

is a JP-homomorphism of S *onto* $(a]$;

(c) *for* $a \in S$, *the binary relation* Θ_a *on* S *defined by*

$$x \equiv y(\Theta_a) \Longleftrightarrow x \wedge a = y \wedge a$$

is a congruence relation.

Proof. (a) $\Rightarrow$ (b). Let S be a distributive JP-semilattice. Then for any $x, y \in S$ we have

$$\varphi(x \wedge y) = a \wedge (x \wedge y) = (a \wedge x) \wedge (a \wedge y) = \varphi(x) \wedge \varphi(y).$$

Also if $x \vee y$ exists, then

$$\varphi(x \vee y) = a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y) = \varphi(x) \vee \varphi(y).$$

Thus φ is a JP-homomorphism. If $x \in (a]$, then $x \leqslant a$ and hence $x = a \wedge x = \varphi(x)$. Therefore, (b) holds.

(b) $\Rightarrow$ (c). Define a relation Θ_a on S given by $x \equiv y(\Theta_a) \Longleftrightarrow a \wedge x = a \wedge y$. If $\varphi : x \mapsto a \wedge x$ is a map from S to $(a]$, then we have $x \equiv y(\Theta_a) \Longleftrightarrow \varphi(x) = \varphi(y)$. Thus $\Theta_a = \ker \varphi$. Since by (b), φ is a JP-homomorphism, so by Lemma 3.3.1, $\ker \varphi$ is a congruence. Hence Θ_a is a congruence. Thus (c) holds.

(c) $\Rightarrow$ (a). Let $x, y \in S$ with $x \vee y$ exists. Then for any $a \in S$, we have $(a \wedge x) \vee (a \wedge y)$ exists. Since $a \wedge x = a \wedge (a \wedge x)$, so $x \equiv a \wedge x(\Theta_a)$. Similarly,

$y \equiv a \wedge y(\Theta_a)$. Thus $x \vee y \equiv (a \wedge x) \vee (a \wedge y)(\Theta_a)$. Hence

$$a \wedge (x \vee y) = a \wedge (a \wedge x) \vee (a \wedge y) = (a \wedge x) \vee (a \wedge y).$$

Thus (a) holds. □

3.4. Quotient JP-semilattice

For any $a \in S$ and $\theta \in \mathrm{Con}(S)$, the set

$$[a]\theta = \{x \in S \mid x \equiv a(\theta)\}$$

is said to be a **congruence class** containing a.

Lemma 3.4.1 *Every congruence class is a convex JP-subsemilattice.*

Proof. Let $[a](\theta)$ is a congruence class of a JP-semilattice **S**. Let $x, y \in [a](\theta)$ and $x \leqslant z \leqslant y$. Then $x \equiv a(\theta)$ and $y \equiv a(\theta)$. This implies

$$z = z \wedge y \equiv z \wedge a(\theta) \equiv z \wedge x(\theta) = x(\theta) \equiv a(\theta).$$

Hence $z \in [a](\theta)$. It is easy to check that $[a](\theta)$ is a JP-subsemilattice. Thus $[a](\theta)$ is a convex sublattice. □

For any $\theta \in \mathrm{Con}(S)$, the set of all congruence class under θ is denoted by $\frac{S}{\theta}$. That is,

$$\frac{S}{\theta} := \{[a](\theta) \mid a \in S\}.$$

Define $\wedge$ and conditional $\vee$ on $\frac{S}{\theta}$ given by

$$[a](\theta) \wedge [b](\theta) = [a \wedge b](\theta)$$

and $[a](\theta) \vee [b](\theta)$ exists if $c \vee d$ exists for some $c \in [a](\theta)$ and $d \in [b](\theta)$ and

$$[a](\theta) \vee [b](\theta) = [c \vee d](\theta).$$

Then this is a routine work to prove that $\langle \frac{S}{\theta}; \wedge, \vee \rangle$ is a JP-semilattice. The JP-semilattice $\langle \frac{S}{\theta}; \wedge, \vee \rangle$ is called a **quotient JP-semilattice** of S. We have the following result for distributive JP-semilattices.

Theorem 3.4.2 *Let S be a distributive JP-semilattice. Then for any congruence θ on S, the quotient JP-semilattice $\frac{S}{\theta}$ is a distributive JP-semilattice.*

Proof. Let S be a distributive JP-semilattice and let $[a](\theta) \vee [b](\theta)$ exists in $\frac{S}{\theta}$. Then there is $c \in [a](\theta)$ and $d \in [b](\theta)$ such that $c \vee d$ exists in S and $[a](\theta) \vee [b](\theta) = [c \vee d](\theta)$. Hence for any $s \in S$ we have

$$\begin{aligned}
[s](\theta) \wedge ([a](\theta) \vee [b](\theta)) &= [s](\theta) \wedge [c \vee d](\theta) \\
&= [s \wedge (c \vee d)](\theta) \\
&= [(s \wedge c) \vee (s \wedge d)](\theta) \quad \text{(as } S \text{ is distributive)} \\
&= [s \wedge c](\theta) \vee [s \wedge d](\theta) \\
&= ([s](\theta) \wedge [c](\theta)) \vee ([s](\theta) \wedge [d](\theta)) \\
&= ([s](\theta) \wedge [a](\theta)) \vee ([s](\theta) \wedge [b](\theta)).
\end{aligned}$$

Hence $\frac{S}{\theta}$ is a distributive JP-semilattice. □

Now we shall prove the homomorphism theorem for JP-semilattices.

Lemma 3.4.3 *If θ is a congruence on a JP-semilattice S, then the canonical map $\varphi : S \to \frac{S}{\theta}$ is a JP-epimorphism.*

Proof. Obviously, φ is a onto $\wedge$-homomorphism. Let $x \vee y$ exists, then $[x](\theta) \vee [y](\theta)$ exists. This implies $\varphi(x) \vee \varphi(y)$ exists. Now $\varphi(x \vee y) = [x \vee y](\theta) = [x](\theta) \vee [y](\theta) = \varphi(x) \vee \varphi(y)$. Hence φ is a JP-epimorphism. □

Theorem 3.4.4 (Homomorphism Theorem) *Every JP-homomorphic image of a JP-semilattice is JP-isomorphic to a suitable quotient JP-semilattice.*

Proof. Let $\varphi : \mathbf{S} \to \mathbf{P}$ be a onto JP-homomorphism. Then by the Lemma 3.3.1 $\ker\varphi$ is a congruence on S. Hence $\frac{\mathbf{S}}{\ker\varphi}$ is a quotient JP-semilattice of S. We prove that $\frac{\mathbf{S}}{\ker\varphi} \cong \mathbf{P}$.

Define a mapping $\eta : \frac{\mathbf{S}}{\ker\varphi} \to \mathbf{P}$ by

$$\eta([x](\ker\varphi)) = \varphi(x).$$

Clearly, the mapping η is a well defined and onto. To prove η is JP-homomorphism, let $[x](\ker\varphi), [y](\ker\varphi) \in \frac{S}{\ker\varphi}$. Then

$$\eta([x](\ker\varphi) \wedge [y](\ker\varphi)) = \eta([x \wedge y](\ker\varphi))$$

$$= \varphi(x \wedge y) = \varphi(x) \wedge \varphi(y) = \eta([x](\ker\varphi)) \wedge \eta([y](\ker\varphi)).$$

If $[x](\ker\varphi) \vee [y](\ker\varphi)$ exists, then $c \vee d$ exists for some $c \in [x](\ker\varphi)$ and $d \in [y](\ker\varphi)$. Hence

$$\begin{aligned}\eta([x](\ker\varphi) \vee [y](\ker\varphi)) &= \eta([c](\ker\varphi) \vee [d](\ker\varphi))\\ &= \eta([c \vee d](\ker\varphi)) \quad \text{since } c \vee d \text{ exists}\\ &= \varphi(c \vee d)\\ &= \varphi(c) \vee \varphi(d)\\ &= \eta([c](\ker\varphi)) \vee \eta([d](\ker\varphi))\\ &= \eta([x](\ker\varphi)) \vee \eta([y](\ker\varphi)).\end{aligned}$$

Therefore, η is JP-homomorphism. To prove η is one to one, let $\varphi(x) = \varphi(y)$. Then $x, y \in \ker\varphi$ and hence $[x](\ker\varphi) = [y](\ker\varphi)$. Hence η is one to one.

This complete the proof. □

Let S be a JP-semilattice and let F be a filter of S. Define a binary relation $\Theta(F)$ on S by

$$x \equiv y(\Theta(F)) \text{ if and only if } x \wedge f = y \wedge f \text{ for some } f \in F.$$

Theorem 3.4.5 *Let F be a filter of a distributive JP-semilattice* S *then the relation $\Theta(F)$ on S is a JP-congruence containing F as a class. Moreover, if* S *has the largest element* 1*, then $\Theta(F)$ is the smallest JP-congruence containing F as a class.*

Proof. Clearly $\Theta(F)$ is an equivalence relation. Let $x \equiv y(\Theta(F))$ and $s \equiv t(\Theta(F))$. Then $x \wedge f_1 = y \wedge f_1$ and $s \wedge f_2 = t \wedge f_2$ for some $f_1, f_2 \in F$. This implies

$$(x \wedge s) \wedge (f_1 \wedge f_2) = (x \wedge f_1) \wedge (s \wedge f_2) = (y \wedge f_1) \wedge (t \wedge f_2) = (y \wedge t) \wedge (f_1 \wedge f_2).$$

Since $f_1 \wedge f_2 \in F$, we have $x \wedge s \equiv y \wedge t(\Theta(F))$.

Also, if $x \vee s$ and $y \vee t$ exist, then

$$(x \vee s) \wedge (f_1 \wedge f_2) = (x \wedge f_1 \wedge f_2) \vee (s \wedge f_1 \wedge f_2)$$
$$= (y \wedge f_1 \wedge f_2) \vee (t \wedge f_1 \wedge f_2) = (y \vee t) \wedge (f_1 \wedge f_2).$$

Thus $\Theta(F)$ is a JP-congruence. Clearly, $\Theta(F)$ contains F as a class.

Moreover, suppose S has the largest element 1. Let θ be any congruence on S containing F as a class. If $x \equiv y(\Theta(F))$. Then $x \wedge f = y \wedge f$ for some $f \in F$. This implies $x = x \wedge 1 \equiv x \wedge f(\theta)$. Similarly, $y \equiv y \wedge f(\theta)$. Hence $x \equiv y(\theta)$. Thus $\Theta(F)$ is the smallest JP-congruence containing F as a class. □

Observe that in general (even for distributive JP-semilattice), $\frac{S}{\Theta(F)}$ is not a lattice. For example, consider the following (see Figure 3.1) distributive JP-semilattice S. Let F be a filter which is a proper subset of the above chain. Then

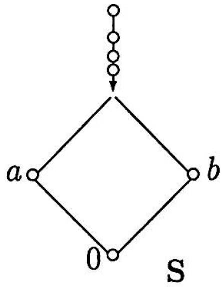


FIGURE 3.1

$\frac{S}{\Theta(F)}$, where $\Theta(F)$ is the smallest congruence containing F as a class, is isomorphic to the distributive JP-semilattice S and hence it is not a lattice. Now we turn our attention to apply some condition on the filter F such that $\frac{\mathrm{S}}{\Theta(F)}$ becomes a lattice.

3.5. Strong filters

A filter F of a JP-semilattice S is said to be **strong filter** if for any $x, y \in S$ such that $x \vee y$ does not exist implies either $x \in F$ or $y \in F$. In the Figure 3.1, the filters $[a)$ and $[b)$ are strong filters but the filters generated by any element of the chain are not strong filters.

Theorem 3.5.1 *Let* S *be a JP-semilattice with* 1 *and let* F *be a strong filter. Then* $\frac{\mathrm{S}}{\Theta(F)}$ *is a lattice.*

Proof. Let $[x](\Theta(F)), [y](\Theta(F)) \in \frac{\mathrm{S}}{\Theta(F)}$. If $x \vee y$ exists, then $[x](\Theta(F)) \vee [y](\Theta(F))$ exists and

$$[x](\Theta(F)) \vee [y](\Theta(F)) = [x \vee y](\Theta(F)).$$

If $x \vee y$ does not exist, then either $x \in F$ or $y \in F$. Without loss of generality, let $x \in F$. Then $[x](\Theta(F)) = [1](\Theta(F))$ and hence

$$[x](\Theta(F)) \vee [y](\Theta(F)) = [1](\Theta(F)) \vee [y](\Theta(F)) = [1](\Theta(F)).$$

Thus $\frac{\mathrm{S}}{\Theta(F)}$ is a lattice. □

The above theorem follows that if S is a JP-semilattice with 1 and if F is a strong filter, then for any $[x](\Theta(F)), [y](\Theta(F)) \in \frac{S}{\Theta(F)}$ we have

$$[x](\Theta(F)) \vee [y](\Theta(F)) = \begin{cases} [1](\Theta(F)) & \text{if } x \vee y \text{ does not exist} \\ [x \vee y](\Theta(F)) & \text{if } x \vee y \text{ exists.} \end{cases}$$

Theorem 3.5.2 *Let S be a distributive JP-semilattice with 1 and let F be a strong filter. Then $\frac{S}{\Theta(F)}$ is a distributive lattice.*

Proof. Let $[x](\Theta(F)), [y](\Theta(F)), [z](\Theta(F)) \in \frac{S}{\Theta(F)}$. If $y \vee z$ exists, then the result is trivial. Suppose $y \vee z$ does not exist. Then either $y \in F$ or $z \in F$. Without loss of generality, let $y \in F$. Then

$$[x](\Theta(F)) \wedge ([y](\Theta(F)) \vee [z](\Theta(F))) = [x](\Theta(F))$$

and

$$\begin{aligned}
&([x](\Theta(F)) \wedge [y](\Theta(F))) \vee ([x](\Theta(F)) \wedge [z](\Theta(F))) \\
&\qquad = [x](\Theta(F)) \vee ([x](\Theta(F)) \wedge [z](\Theta(F))) \\
&\qquad = [x \vee (x \wedge z)](\Theta(F)) \\
&\qquad = [x](\Theta(F)).
\end{aligned}$$

Therefore, $\frac{S}{\Theta(F)}$ is a distributive lattice. □

In rest of this section, by S we mean distributive JP-semilattice with 1 and by F we mean strong filter. Recall that the map $\varphi_F : S \to \frac{S}{\Theta(F)}$ given by $\varphi_F(x) = [x](\Theta(F))$ is the natural epimorphism.

Lemma 3.5.3 *For any ideals I and J of* $\mathbf{S}$, *the following hold.*

(i) $\varphi_F(I)$ *is an ideal of* $\frac{\mathbf{S}}{\Theta(F)}$;

(ii) $\varphi_F(I)$ *is a proper ideal of* $\frac{\mathbf{S}}{\Theta(F)}$ *if and only if* $I \cap F = \emptyset$;

(iii) $\varphi_F(I \vee J) = \varphi_F(I) \vee \varphi_F(J)$;

(iv) $\varphi_F(I \cap J) = \varphi_F(I) \cap \varphi_F(J)$.

Proof. (i) Let $\varphi_F(i), \varphi_F(j) \in \varphi_F(I)$. If $i \vee j$ exists, then

$$\varphi_F(i) \vee \varphi_F(j) = [i](\Theta(F) \vee [j](\Theta(F) = [i \vee j](\Theta(F)) \in \varphi_F(I).$$

Suppose $i \vee j$ does not exist. Without loss of generality let $i \in F$. Then

$$\varphi_F(i) \vee \varphi_F(j) = [1](\Theta(F) \vee [j](\Theta(F) = [1](\Theta(F)) = [i](\Theta(F)) \in \varphi_F(I).$$

Now let $\varphi_F(x) \in \varphi_F(I)$ and $\varphi_F(y) \subseteq \varphi_F(x)$. Then

$$\varphi_F(y) = \varphi_F(y) \cap \varphi_F(x) = \varphi_F(y \wedge x) \in \varphi_F(I).$$

Therefore, $\varphi_F(I)$ is an ideal.

(ii) Suppose $\varphi_F(I)$ is a proper ideal of $S/\Theta(F)$. Then there exists $x \in S$ such that $\varphi_F(x) \notin \varphi_F(I)$. Suppose $I \cap F \neq \emptyset$ and let $y \in I \cap F$. Then $y \equiv 1(\Theta(F)$ and hence $1 \in I$. This implies $x \in I$ which is a contradiction. Hence $I \cap F = \emptyset$.

Conversely, suppose $\varphi_F(I) = S/\Theta(F)$. Then there is $x \in I$ such that $[x]\Theta(F) = [1]\Theta(F)$. Hence $x \in F$ which implies that $I \cap F \neq \emptyset$.

(iii) Suppose $y \in \varphi_F(I \vee J)$. Then there is $x \in I \vee J$ such that $y = \varphi_F(x)$. Since $\mathbf{S}$ is a distributive JP-semilattice, we have $x = i_1 \vee i_2 \vee \cdots \vee i_n$ where $i_1, i_2, \cdots, i_n \in I \cup J$. If $i_j \vee i_k$ does not exist for any $1 \leqslant j, k \leqslant n$, then either $i_j \in F$ or $i_k \in F$. Thus either $I \cap F \neq \emptyset$ or $J \cap F \neq \emptyset$. Hence by (ii) either $\varphi_F(I)$

is a non proper ideal or $\varphi_F(J)$ is a non-proper ideal. Hence $y \in \varphi_F(I) \vee \varphi_F(J)$. Now suppose $i_j \vee i_k$ exists for all $1 \leqslant j, k \leqslant n$. Then $x = i \vee j$ for some $i \in I$ and $j \in J$. This implies $y = \varphi_F(i \vee j) = \varphi_F(i) \vee \varphi_F(j) \in \varphi_F(I) \vee \varphi_F(J)$. Hence $\varphi_F(I \vee J) \subseteq \varphi_F(I) \vee \varphi_F(J)$. The reverse inclusion is trivial. Hence (iii) holds.

(iv) is obvious. □

Theorem 3.5.4 *Let S be a distributive JP-semilattice with 1 and let F be a strong filter. Then for any ideal J of S, we have*

$$\varphi_F^{-1}\varphi_F(J) = \{x \in S \mid x \wedge f \in J \text{ for some } f \in F\}$$

$$= \bigcap\{P \mid P \text{ is a minimal prime ideal of } S \text{ with } J \subseteq P \text{ and } P \cap F = \emptyset\}.$$

Proof. We have

$$\varphi_F^{-1}\varphi_F(J) = \{y \in S \mid \varphi_F(y) \in \varphi_F(J)\}$$

$$= \{y \in S \mid \varphi_F(y) = \varphi_F(x) \text{ for some } x \in J\}$$

$$= \{y \in S \mid [y]\Theta(F) = [x]\Theta(F) \text{ for some } x \in J\}$$

$$= \{y \in S \mid y \wedge f = x \wedge f \in J \text{ for some } x \in J, f \in F\}$$

$$= \{y \in S \mid y \wedge f \in J \text{ for some } f \in F\}.$$

Now we consider two cases:

Case-1: $J \cap F \neq \emptyset$. Let P be any prime ideal containing to J such that $P \cap F \neq \emptyset$. Then

$$\{P \mid P \text{ is a minimal prime ideal of } S \text{ with } J \subseteq P \text{ and } P \cap F = \emptyset\} = \emptyset$$

and so,

$$\bigcap\{P \mid P \text{ is a minimal prime ideal of } S \text{ with } J \subseteq P \text{ and } P \cap F = \emptyset\}$$

$$= S = \{y \in S \mid y \wedge x \in J, x \in J \cap F\}.$$

Case-2: $J \cap F = \emptyset$. Clearly,

$$\{y \in S \mid y \wedge f \in J \text{ for some } f \in F\}$$

$$\subseteq \bigcap\{P \mid P \text{ is a prime ideal with } J \subseteq P \text{ and } P \cap F = \emptyset\}.$$

To prove the reverse inclusion, let $x \in S$ such that $x \wedge f \notin J$ for any $f \in F$ and let $G = [x) \vee F$. If $G \cap J \neq \emptyset$, then there is $t \in G \cap J$ such that $t = x_1 \wedge f$ for some $x \leqslant x_1$ and $f \in F$. This implies $x \wedge f \leqslant x_1 \wedge f = t$ and consequently, $x \wedge f \in J$ which is a contradiction. Therefore $J \cap G = \emptyset$. Then by the Separation Theorem there exists a prime ideal P of S such that $J \subseteq P$ and $G \cap P = \emptyset$. This implies that $x \notin P$ and $P \cap F = \emptyset$ as $F \subseteq G$.

This completes the proof of the theorem. □

We close the Chapter with the following important result.

Theorem 3.5.5 *Let* S *be a distributive JP-semilattice with* 1 *and let* F *be a strong filter. Suppose*

$$\mathcal{P} = \{P \mid P \textit{ is a prime ideal of } S \textit{ with } P \cap F = \emptyset\}$$

and

$$\mathcal{Q} = \{Q \mid Q \textit{ is a prime ideal of } S/\Theta(F)\}.$$

Then $\mathcal{P}$ and $\mathcal{Q}$ are order isomorphic.

Proof. Define a mapping $\psi : \mathcal{P} \to \mathcal{Q}$ given by

$$\psi(P) = \varphi_F(P).$$

Let $P \in \mathcal{P}$. Then by Lemma 3.5.3 (ii), $\varphi_F(P)$ is a proper ideal of $S/\Theta(F)$. To prove that $\varphi_F(P)$ is a prime ideal, let $\varphi_F(x) \wedge \varphi_F(y) \in \varphi_F(P)$. Then $[x](\Theta(F)) \wedge [y](\Theta(F)) = [x \wedge y](\Theta(F)) \in \varphi_F(P)$. This implies $[x \wedge y](\Theta(F)) = [p](\Theta(F))$ for some $p \in P$. Hence $x \wedge y \wedge f = p \wedge f$ for some $f \in F$. Thus $x \wedge y \wedge f \in P$. Since P is a prime ideal, $f \notin P$, we have $x \wedge y \in P$ and hence either $x \in P$ or $y \in P$. This implies either $\varphi_F(x) \in \varphi_F(P)$ or $\varphi_F(y) \in \varphi_F(P)$. Hence $\varphi_F(P)$ is a prime ideal of $S/\Theta(F)$. That is, $\varphi_F(P) \in \mathcal{Q}$. Therefore ψ is well defined. Clearly, ψ is an isotone. Since φ_F is a natural epimorphism, ψ is an epimorphism. To prove that ψ is one-one, let $\varphi_F(P) = \varphi_F(R)$. We shall show that $P = R$. Let $x \in P$. Then $\varphi_F(x) \in \varphi_F(R)$. This implies there is $y \in R$ such that $\varphi_F(x) = \varphi_F(y)$. Consequently, $[x](\Theta(F)) = [y](\Theta(F))$. Thus $x \wedge f = y \wedge f$ for some $f \in F$. This implies $x \wedge f = y \wedge f \in R$. Since $R \cap F = \emptyset$, we have $f \notin R$ and hence $x \in R$ as R is a prime ideal. Therefore, $P \subseteq R$. Similarly, we can show that $R \subseteq P$. Thus $P = R$. Hence ψ is one-one.

This complete the proof of the theorem. □

CHAPTER 4

Kernel Ideals of PJP-Semilattices

4.1. Introduction

Let S be a JP-semilattice with the smallest element 0. Let $a \in S$. An element $d \in S$ is called **pseudocomplement** of $a \in S$ if $a \wedge d = 0$ and for any $x \in S$ with $a \wedge x = 0$ implies $x \leqslant d$. Clearly the pseudocomplement of an element is unique. The pseudocomplement of an element $a \in S$ is denoted by a^*. A JP-semilattice is said to be **Pseudocomplemented JP-semilattice** or (simply **PJP-semilattice**) if every element has a pseudocomplement. A JP-semilattice $\langle S; \wedge, \vee, ^*, 0, 1\rangle$ with * (called **pseudocomplementation**) is said to be a **JP-semilattice with pseudocomplementation**. Like as a distributive lattice, the distributivity of a JP-semilattice does not guarantee that it is pseudocomplemented. For example, consider the distributive JP-semilattice given by the following Figure 4.1. This is not pseudocomplemented as a has no pseudocomplement.

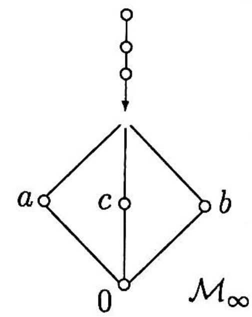


FIGURE 4.1

4.2. Congruence kernels and cokernels

Congruence kernels and cokernels have been studied by Cornish [9] for pseudocomplemented distributive lattices and by Blyth [2] for pseudocomplemented semilattices. In this Chapter we study the congruence kernels and cokernels for distributive PJP-semilattices. First we prove some identities which we need in this thesis.

Theorem 4.2.1 *Let* S *be a PJP-semilattice. Then for any* $a, b \in S$ *we have*

(i) $a \leqslant a^{**}$,

(ii) $a \leqslant b$ *implies* $b^* \leqslant a^*$,

(iii) $a^* = a^{***}$,

(iv) $0^* = 1$, *the largest element of* S.

(v) $a \wedge b^* = a \wedge (a \wedge b)^*$

Proof. (i) For all $a \in S$ we have $a^* \wedge a = 0$. Hence by the definition, $a \leqslant a^{**}$.

(ii) Let $a \leqslant b$. Then $a \wedge b^* \leqslant b \wedge b^* = 0$. This implies $a \wedge b^* = 0$. Hence by the definition, $b^* \leqslant a^*$.

(iii) By (i) $a \leqslant a^{**}$ and hence by (ii) $a^{***} \leqslant a^*$. Also by (i) $a^* \leqslant a^{***}$. Hence $a^* = a^{***}$.

(iv) is by the definition.

(v) Using (ii), this is trivial that $a \wedge b^* \leqslant a \wedge (a \wedge b)^*$. To prove the reverse inequality, let $x = a \wedge (a \wedge b)^*$. Then $x \leqslant a$ and $x \wedge b = (a \wedge b) \wedge (a \wedge b)^* = 0$. This implies $x \leqslant a$ and $x \leqslant b^*$. Hence $x \leqslant a \wedge b^*$. Therefore, $a \wedge b^* = a \wedge (a \wedge b)^*$. □

The above theorem shows that every PJP-semilattice **S** has the greatest element 1 and hence it is directed above, that is, every pair of elements of S has a common upper bound. Thus every PJP-semilattice is directed above.

Let S be a pseudocomplemented JP-semilattice. The set

$$\mathrm{Sk}(S) = \{a^* \mid a \in S\}$$

is called the **skeleton** of S. The elements of $\mathrm{Sk}(S)$ are called **skeletal**. It is evedient that $\sup\{a^*, b^*\}$ in $\mathrm{Sk}(S)$ always exists and we denote it by $a^* \veebar b^*$. That is, for any $a, b \in \mathrm{Sk}(S)$, we have $a \veebar b = \sup\{a, b\}$ in $\mathrm{Sk}(S)$.

Now we have the following lemma.

Theorem 4.2.2 *Let* **S** *be a PJP-semilattice. Then*

(i) $a \in \mathrm{Sk}(S) \Leftrightarrow a = a^{**}$;

(ii) $a, b \in \mathrm{Sk}(S) \Rightarrow a \wedge b = (a \wedge b)^{**}$;

(iii) $a, b \in \mathrm{Sk}(S) \Rightarrow a \veebar b = (a^* \wedge b^*)^*$.

Proof. (i) Let $a \in \mathrm{Sk}(S)$, then $a = b^*$ for some $b \in S$. Hence by Theorem 4.2.1 (iii), we have $a = b^{***} = a^{**}$. The converse is by the definition of $\mathrm{Sk}(S)$.

(ii) Let $a, b \in \mathrm{Sk}(S)$. We have $a \geqslant a \wedge b$. This implies $a = a^{**} \geqslant (a \wedge b)^{**}$. Similarly, $b \geqslant (a \wedge b)^{**}$. Thus $(a \wedge b)^{**}$ is a lower bound of a and b. Hence $a \wedge b \geqslant (a \wedge b)^{**}$. But by Theorem 4.2.1 (i), we have $a \wedge b \leqslant (a \wedge b)^{**}$. Therefore $a \wedge b = (a \wedge b)^{**}$. This implies $a \wedge b \in \mathrm{Sk}(S)$.

(iii) Let $a, b \in \mathrm{Sk}(S)$. Then $a^* \wedge b^* \leqslant a^*, b^*$. This implies $a = a^{**}, b = b^{**} \leqslant (a^* \wedge b^*)^*$. Then $(a^* \wedge b^*)^*$ is an upper bound of a, b in $\mathrm{Sk}(S)$. Let $x \in \mathrm{Sk}(S)$ such that $a, b \leqslant x$. Then $x^* \leqslant a^* \wedge b^*$ and hence $(a^* \wedge b^*)^* \leqslant x^{**} = x$. Thus $a \veebar b = (a^* \wedge b^*)^*$. □

4.3. PJP-congruence kernels

A JP-congruence θ of a PJP-semilattice S is said to be **PJP-congruence** on S, if it is compatible with *, that is,

$$x \equiv y(\theta) \Rightarrow x^* \equiv y^*(\theta).$$

Theorem 4.3.1 *Let S be a PJP-semilattice. Then a JP-congruence θ on S is a PJP-congruence if and only if*

$$x \equiv 0(\theta) \Rightarrow x^* \equiv 1(\theta).$$

Proof. If θ is a PJP-congruence, then clearly the condition holds. Conversely, let θ be a JP-congruence such that the condition holds. Let $x \equiv y(\theta)$. Then

$x^* \wedge y \equiv x^* \wedge x = 0(\theta)$ and so $(x^* \wedge y)^* \equiv 1(\theta)$. This implies

$$\begin{aligned} x^* &= x^* \wedge 1 \\ &\equiv x^* \wedge (x^* \wedge y)^*(\theta) \\ &= x^* \wedge y^* \quad \text{(by Theorem 4.2.1 (v))}. \end{aligned}$$

Similarly, we have $y^* \equiv x^* \wedge y^*(\theta)$. Hence $x^* \equiv y^*(\theta)$ and therefore θ is a PJP-congruence. □

Theorem 4.3.2 *Let S be a distributive PJP-semilattice and let I be an ideal of S such that for $i, j \in I$ implies $(i^* \wedge j^*)^* \in I$. Define a binary relation $\Theta(I)$ on S by*

$$x \equiv y(\Theta(I)) \text{ if and only if } x \wedge i^* = y \wedge i^* \text{ for some } i \in I.$$

Then $\Theta(I)$ is the smallest PJP-congruence containing I as a class.

Proof. Clearly, $\Theta(I)$ is both reflexive and symmetric. To prove that it is transitive, let $x \equiv y(\Theta(I))$ and $y \equiv z(\Theta(I))$. Then $x \wedge i^* = y \wedge i^*$ and $y \wedge j^* = z \wedge j^*$ for some $i, j \in I$. Then by the condition $k = (i^* \wedge j^*)^* \in I$. We have

$$\begin{aligned} x \wedge k^* &= x \wedge (i^* \wedge j^*)^{**} = x \wedge (i^* \wedge j^*) \quad \text{(by Theorem 4.2.2 (ii))} \\ &= (x \wedge i^*) \wedge j^* = (y \wedge i^*) \wedge j^* = (y \wedge j^*) \wedge i^* \\ &= (z \wedge j^*) \wedge i^* = z \wedge (i^* \wedge j^*) = z \wedge (i^* \wedge j^*)^{**} \\ &= z \wedge k^*. \end{aligned}$$

Hence $x \equiv z(\Theta(I)$. Thus $\Theta(I)$ is transitive.

Let $x \equiv y(\Theta(I)$ and $s \equiv t(\Theta(I)$. Then there are $i, j \in I$ with $k = (i^* \wedge j^*)^* \in I$ such that $x \wedge i^* = y \wedge i^*$ and $s \wedge j^* = t \wedge j^*$. Hence

$$\begin{aligned}(x \wedge s) \wedge k^* &= (x \wedge s) \wedge (i^* \wedge j^*)^{**} \\ &= (x \wedge s) \wedge (i^* \wedge j^*) \quad \text{(by Theorem 4.2.2 (ii))} \\ &= (x \wedge i^*) \wedge (s \wedge j^*) = (y \wedge i^*) \wedge (t \wedge j^*) \\ &= (y \wedge t) \wedge (i^* \wedge j^*)^{**} \\ &= (y \wedge t) \wedge k^*.\end{aligned}$$

Also if $x \vee s$ and $y \vee t$ exists, then

$$\begin{aligned}(x \vee s) \wedge k^* &= (x \wedge k^*) \vee (s \wedge k^*) \text{ as } S \text{ is a distributive JP-semilattice} \\ &= (x \wedge i^* \wedge j^*) \vee (s \wedge i^* \wedge j^*) \quad \text{(by Theorem 4.2.2 (ii))} \\ &= (y \wedge i^* \wedge j^*) \vee (t \wedge i^* \wedge j^*) \\ &= (y \wedge k^*) \vee (t \wedge k^*) \\ &= (y \vee t) \wedge k^* \text{ as } S \text{ is a distributive JP-semilattice} \\ &= (y \vee t) \wedge k^*.\end{aligned}$$

Hence $\Theta(I)$ is a JP-congruence. To prove that $\Theta(I)$ is a PJP-congruence, let $x \equiv 0(\Theta(I))$. Then $x \wedge i^* = 0 \wedge i^* = 0$. This implies $i^* \leqslant x^*$. Hence $x^* \wedge i^* = i^* = 1 \wedge i^*$. This implies $x^* \equiv 1(\Theta(I)$. Hence by Theorem 4.3.1, $\Theta(I)$ is a PJP-congruence.

Finally, let θ be a PJP-congruence containing I as a class and let $x \equiv y(\Theta(I)$. Then $x \wedge i^* = y \wedge i^*$ for some $i \in I$. Since θ be a PJP-congruence containing I as a class. We have $i \equiv 0(\theta)$. This implies $i^* \equiv 1(\theta)$. Hence

$$x = x \wedge 1 \equiv x \wedge i^*(\theta) = y \wedge i^* \equiv y \wedge 1(\theta) = y.$$

Therefore $\Theta(I)$ is the smallest congruence containing I as a class. □

Kernel Ideals. Let θ be a PJP-congruence on S. Then

$$\ker(\theta) = \{x \in S \mid x \equiv 0(\theta)\}$$

is called the **kernel** of θ. Clearly, $\ker(\theta)$ is an ideal. A subset J of S is said to be **congruence kernel** if $J = \ker(\theta)$ for some PJP-congruence θ on S.

Observe that in the PJP-semilattice M given in Figure 4.2, the ideal $I = \{0, a, b\}$ is not a kernel of any PJP-congruence on M. For $0 \equiv a(\theta)$ for any PJP-congruence θ on M, then $1 \equiv a^* = b(\theta)$, that is, $0 \equiv 1(\theta)$. Thus I is not a PJP-congruence kernel. An ideal I of a PJP-semilattice S is said to be a **kernel ideal** if $I = \ker(\theta)$ for some PJP-congruence θ on S. The set of all kernel ideals will be denoted by $\mathrm{KI}(S)$.

For a distributive lattice, we have the following result.

Theorem 4.3.3 *Let* L *be a distributive lattice. Then an ideal I of* L *is a kernel ideal for some lattice congruence θ if and only if*

$$x \in I \Rightarrow x^{**} \in I.$$

□

Now we are interested to characterize the kernel ideal for distributive PJP-semilattices. Observe that the above result is not true for distributive PJP-semilattices. For counterexample, consider the distributive PJP-semilattice given in Figure 4.2. Let $I = \{0, a, b\}$. Then I is an ideal of M and for any $x \in I$ we have $x = x^{**} \in I$. But I is not a kernel ideal of any PJP-congruence θ on M, for $0 \equiv a(\theta)$ implies $1 \equiv b(\theta)$.

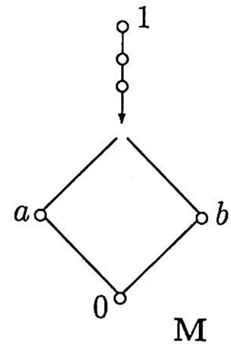


FIGURE 4.2

The following result is the generalization of Theorem 2.2 [2].

Theorem 4.3.4 *An ideal I of a distributive PJP-semilattice* S *is a kernel ideal of* S *if and only if*

$$i, j \in I \Rightarrow (i^* \wedge j^*)^* \in I.$$

Proof. Let I be a kernel ideal of S. Then $I = \ker\theta$ for some PJP-congruence θ. If $i, j \in I$, then $i \equiv 0(\theta)$ and $j \equiv 0(\theta)$. Hence by Theorem 4.3.1, $i^* \equiv 1(\theta)$ and $j^* \equiv 1(\theta)$. Hence $i^* \wedge j^* \equiv 1(\theta)$. This implies $(i^* \wedge j^*)^* \equiv 0(\theta)$. Thus $(i^* \wedge j^*)^* \in I$.

Conversely, let I be an ideal of S and suppose the condition holds. Then by Theorem 4.3.2, the binary relation $\Theta(I)$ on S defined by

$$x \equiv y(\Theta(I)) \text{ if and only if } x \wedge i^* = y \wedge i^* \text{ for some } i \in I.$$

is a PJP-congruence containing the ideal I as a class. So it is enough to show that I is a kernel ideal of $\Theta(I)$. For all $i \in I$, by taking $i = j$ in the condition we have $i^{**} \in I$. Hence

$$\begin{aligned} x \equiv 0(\Theta(I)) &\Leftrightarrow x \wedge i^* = 0 \text{ for some } i \in I \\ &\Leftrightarrow x \leqslant i^{**} \text{ for some } i \in I \\ &\Leftrightarrow x \in I. \end{aligned}$$

Thus I is a kernel ideal. □

Theorem 4.3.5 *Let* S *be a distributive PJP-semilattice. An ideal I of S is a kernel ideal if and only if*

(i) $i \in I$ *implies* $i^{**} \in I$;

(ii) *for all* $i, j \in I$ *there is* $k \in I$ *such that* $i^* \wedge j^* = k^*$.

Proof. Let I be a kernel ideal. Then by taking $i = j$ in Theorem 4.3.4 we have $i \in I \Rightarrow i^{**} \in I$. Thus (i) holds. Let $i, j \in I$. Put $k = (i^* \wedge j^*)^*$, then by Theorem 4.3.4, $k \in I$. Also $k^* = i^* \wedge j^*$. Thus (ii) holds.

Conversely, let I be an ideal and $i, j \in I$. Then by (ii), there is $k \in I$ such that $k^* = i^* \wedge j^*$. Thus by (i), $k^{**} = (i^* \wedge j^*)^* \in I$. Hence by Theorem 4.3.4, I is a kernel ideal. □

Theorem 4.3.6 *Let* S *be a distributive PJP-semilattice. A principal ideal* $I = (x]$ *of S is a kernel ideal if and only if* $x \in \mathrm{Sk}(S)$.

Proof. Suppose $I = (x]$ is a kernel ideal, then $x^{**} \in I$. This implies $x^{**} \leqslant x$. But $x \leqslant x^{**}$. Hence $x = x^{**} \in \mathrm{Sk}(S)$.

Conversely, let $I = (x]$ be a principal ideal and $x \in \mathrm{Sk}(S)$. Then by Theorem 4.2.2 (i), we have $x = x^{**}$. Let $i, j \in I$. Then $i, j \leqslant x$. This implies $x^{*} \leqslant i^{*} \wedge j^{*}$. Thus $(i^{*} \wedge j^{*})^{*} \leqslant x^{**} = x$. This implies $(i^{*} \wedge j^{*})^{*} \in I$. Hence by Theorem 4.3.4, I is a kernel ideal. □

***-ideals.** An ideal I of a JP-semilattice is said to be ***-ideal** if $i \in I$ implies $i^{**} \in I$.

Theorem 4.3.7 *Every kernel ideal of a distributive PJP-semilattice is *-ideal but the converse is not true.*

Proof. By Theorem 4.3.5 (i) it is immediate that every kernel ideal of a distributive PJP-semilattice is *-ideal. To prove the converse is not true consider the distributive PJP-semilattice **M** given in Figure 4.2. Here the ideal $I = \{0, a, b\}$ is a *-ideal but not a kernel ideal. □

Theorem 4.3.8 *Let* S *be a distributive PJP-semilattice. Every principal *-ideal I of* S *can be written as $(a^{**}]$ for some $a \in I$. Moreover, for any $a \in S$ the principal ideal $I = (a^{**}]$ is a kernel ideal.*

Proof. Let I be a principal *-ideal of S. Then $I = (a]$ for some $a \in S$. Since I is a *-ideal, $a \in I$ we have $a^{**} \in I$. Thus $a^{**} \leqslant a$. But $a \leqslant a^{**}$. Hence $I = (a^{**}]$ for some $a \in S$.

For any $a \in S$, since $a^{**} \in \mathrm{Sk}(S)$, so by Theorem 4.3.6, $I = (a^{**}]$ is a kernel ideal.

□

Theorem 4.3.9 *A *-ideal I of a distributive PJP-semilattice is a kernel ideal if and only if $i^{**} \veebar j^{**} \in I$ for all $i, j \in I$.*

Proof. Since for any $i, j \in I$ we have

$$(i^* \wedge j^*)^* = (i^{***} \wedge j^{***})^* \text{ by Theorem 4.2.1 (iii)}$$

$$= i^{**} \veebar j^{**} \text{by Theorem 4.2.2 (iii).}$$

By Theorem 4.3.4, I is a kernel ideal if and only if for any $i, j \in I$ implies $i^{**} \veebar j^{**} \in I$.

□

Glivenko Congruence. Let S be a JP-semilattice and let I be an ideal of S. We have proved that the binary relation $\psi(I)$ on S defined by

$$x \equiv y(\psi(I)) \text{ if and only if } x \wedge a \in I \Leftrightarrow y \wedge a \in I \text{ for any } a \in S$$

is a largest JP-congruence containing I as a class (see Theorem 3.2.5).

Now we have the following result.

Theorem 4.3.10 *Let S be a distributive PJP-semilattice. If I is a kernel ideal of S, then $\psi(I)$ is the largest PJP-congruence containing I as a class.*

Proof. By Theorem 3.2.5, $\psi(I)$ is a largest JP-congruence. Let $x \equiv 0(\psi(I))$. Then $x \in I$. Now for any $a \in S$,

$$\begin{aligned} x^* \wedge a \in I &\Rightarrow (x^* \wedge (x^* \wedge a)^*)^* \in I, \text{ by Theorem 4.3.4} \\ &\Rightarrow (x^* \wedge a^*)^* \in I, \text{ by Theorem 4.2.1 (v)} \\ &\Rightarrow a \in I, \text{ since } a \leqslant a^{**} \leqslant (x^* \wedge a^*)^* \\ &\Rightarrow 1 \wedge a \in I. \end{aligned}$$

Also

$$1 \wedge a = a \in I \Rightarrow x^* \wedge a \in I.$$

Thus $x^* \equiv 1(\psi(I))$. Hence by Theorem 4.3.1, $\psi(I)$ is a PJP-congruence. □

Theorem 4.3.11 *If I is a kernel ideal of a distributive PJP-semilattice* S *and if $x \equiv y(\psi(I))$, then $[(x \wedge y^*)^* \wedge (x^* \wedge y)^*]^* \in I$.*

Proof. Let $x \equiv y(\psi(I))$. Then $x \wedge x^* = 0 \equiv y \wedge x^*(\psi(I))$. Therefore $y \wedge x^* \in I$. Similarly, $x \wedge y^* \in I$. Hence $[(x \wedge y^*)^* \wedge (x^* \wedge y)^*]^* \in I$ as I is a kernel ideal. □

Let S be a distributive PJP-semilattice. A binary relation G on S defined by

$$x \equiv y(G) \Leftrightarrow x^{**} = y^{**}$$

is a semilattice congruence called **Glivenko congruence**. It is evident that G is compatible with *. We shall show that G is a PJP-congruence.

Let I be an ideal. Define

$$I^0 = \{x \in S \mid x \wedge i = 0 \text{ for all } i \in I\}.$$

Theorem 4.3.12 *I^0 is a kernel ideal.*

Proof. Let $x, y \in I^0$. Then $x \wedge i = y \wedge i = 0$ for all $i \in I$. Hence $i \leqslant x^*, y^*$ and consequently, $(x^* \wedge y^*)^* \leqslant i^*$. This implies $(x^* \wedge y^*)^* \wedge i \leqslant i^* \wedge i = 0$. Hence $(x^* \wedge y^*)^* \in I^0$. Thus by Theorem 4.3.4, I^0 is a kernel ideal. □

Theorem 4.3.13 *Let I be a kernel ideal of a distributive PJP-semilattice* S. *Then $\psi(I) \wedge \psi(I^0) = G$.*

Proof. Let $x \equiv y(\psi(I) \wedge \psi(I^0))$. Then by Theorem 4.3.11, we have $[(x \wedge y^*)^* \wedge (x^* \wedge y)^*]^* \in I$ and $[(x \wedge y^*)^* \wedge (x^* \wedge y)^*]^* \in I^0$ whence $[(x \wedge y^*)^* \wedge (x^* \wedge y)^*]^* = 0$. This implies

$$x \wedge y^* \leqslant (x \wedge y^*)^{**} \leqslant [(x \wedge y^*)^* \wedge (x^* \wedge y)^*]^* = 0.$$

Thus $x \wedge y^* = 0$. Hence $y^* \leqslant x^*$. Similarly, $x^* \leqslant y^*$. This implies $x^* = y^*$ and consequently, $x^{**} = y^{**}$. Hence $x \equiv y(G)$

Conversely, let $x \equiv y(G)$. Since $a \equiv a^{**}(G)$ for any $a \in S$, we have $x \wedge a \equiv x \wedge a^{**}(G)$, $y \wedge a \equiv y \wedge a^{**}(G)$ and $x \wedge a \equiv y \wedge a^{**}(G)$. Hence $(x \wedge a)^{**} = (x \wedge a^{**})^{**}$, $(y \wedge a)^{**} = (y \wedge a^{**})^{**}$ and $(x \wedge a)^{**} = (y \wedge a^{**})^{**}$. Now for any $a \in S$,

$$\begin{aligned} x \wedge a \in I &\Leftrightarrow (x \wedge a)^{**} \in I \text{ as } I \text{ is a kernel ideal of } S \\ &\Leftrightarrow (y \wedge a^{**})^{**} \in I \\ &\Leftrightarrow (y \wedge a)^{**} \in I \\ &\Leftrightarrow y \wedge a \in I \end{aligned}$$

Also, for all $i \in I$,

$$\begin{aligned} x \wedge a \in I^0 &\Leftrightarrow (x \wedge a) \wedge i = 0 \\ &\Leftrightarrow x \wedge (a \wedge i) = 0 \\ &\Leftrightarrow x \leqslant (a \wedge i)^* \\ &\Leftrightarrow x^{**} \leqslant (a \wedge i)^* \\ &\Leftrightarrow y^{**} \leqslant (a \wedge i)^* \\ &\Leftrightarrow y \leqslant (a \wedge i)^* \\ &\Leftrightarrow y \wedge (a \wedge i) = 0 \\ &\Leftrightarrow y \wedge a \in I^0. \end{aligned}$$

Hence $x \equiv y(\psi(I) \wedge \psi(I^0))$. Therefore $G = \psi(I) \wedge \psi(I^0)$. □

Corollary 4.3.14 *G is a PJP-congruence.*

Proof. This is obvious since $\psi(I) \wedge \psi(I^0)$ is a PJP-congruence. □

4.4. Congruence Cokernels

We already have proved that if F is a filter of a distributive JP-semilattice **S**, then the congruence $\Theta(F)$ defined by

$$x \equiv y(\Theta(F)) \Longleftrightarrow x \wedge f = y \wedge f \text{ for some } f \in F$$

is the smallest congruence containing F as a class (see Theorem 3.4.5). Now we have the following result for PJP-semilattices.

Theorem 4.4.1 *Let S be a PJP-semilattice and let F be a filter of S. Then $\Theta(F)$ is the smallest PJP-congruence containing F as a class.*

Proof. By Theorem 3.4.5, $\Theta(F)$ is a JP-congruence containing F as a class. Let $x \equiv 0(\Theta(F))$. Then $x \wedge f = 0$ for some $f \in F$. This implies $f \leqslant x^*$. Thus $x^* \in F$. Hence $x^* \equiv 1(\Theta(F)$. Hence by Theorem 4.3.1, we have $\Theta(F)$ is a PJP-congruence. □

Let θ be a PJP-congruence on S. Then

$$\text{Coker}(\theta) = \{x \in S \mid x \equiv 1(\theta)\}$$

is called the **cokernel** of θ. A subset J of S is said to be **congruence cokernel** if $J = \text{Coker}(\theta)$ for some PJP-congruence θ on S.

Lemma 4.4.2 *Every cokernel is a filter.*

Proof. Let $F = \text{Coker}(\theta)$ for some PJP-congruence θ. If $x, y \in F$, then $x \equiv 1(\theta)$ and $y \equiv 1(\theta)$. Hence $x \wedge y \equiv 1(\theta)$. Thus $x \wedge y \in F$. Now let $x \in F$ and $x \leqslant y$. Then $x = x \wedge y \equiv 1 \wedge y(\theta) = y$. Thus $y \equiv 1(\theta)$. Hence $y \in F$. Therefore F is a filter.

□

Corollary 4.4.3 *Every filter of a PJP-semilattice is a cokernel.*

Proof. It is clear from the fact that for any filter F of S we have

$$x \in F \Leftrightarrow x \equiv 1(\Theta(F)).$$

□

Let S be a PJP-semilattice. A filter F of S is said to be a ***-filter** if

$$f^{**} \in F \Rightarrow f \in F.$$

Lemma 4.4.4 *Let* S *be a distributive PJP-semilattice. If $a \vee b$ exists, then*

$$(a \vee b)^* = a^* \wedge b^*.$$

Proof. We have $(a \vee b) \wedge (a^* \wedge b^*) = (a \wedge a^* \wedge b^*) \vee (b \wedge a^* \wedge b^*) = 0 \vee 0 = 0$. Let $(a \vee b) \wedge x = 0$. Then $(a \wedge x) \vee (b \wedge x) = 0$. Hence $a \wedge x = 0$ and $b \wedge x = 0$. This implies $x \leqslant a^*, b^*$. Hence $x \leqslant a^* \wedge b^*$. Therefore $(a \vee b)^* = a^* \wedge b^*$. □

For every filter F of S define

$$F_* = \{x \in S \mid x^* \in F\}.$$

Lemma 4.4.5 *Let* S *be a distributive PJP-semilattice and F be a filter of* S*. Then F_* is a kernel ideal of* S*.*

Proof. Let $x, y \in F_*$. Then $x^*, y^* \in F$. If $x \vee y$ exists, then by Lemma 4.4.4, we have $(x \vee y)^* = x^* \wedge y^* \in F$ as F is a filter. Hence $x \vee y \in F_*$. Let $x \in F_*$ and $y \leqslant x$. Then $y^* \geqslant x^* \in F$. This implies $y^* \in F$. Thus $y \in F_*$. Hence F_* is an ideal.

To prove that F_* is a kernel ideal, let $x, y \in F_*$. Then $x^*, y^* \in F$ so that $(x^* \wedge y^*)^{**} = x^* \wedge y^* \in F$ and consequently $(x^* \wedge y^*)^* \in F_*$. Hence by Theorem 4.3.4, F_* is a kernel ideal.

□

For every $I \in KI(S)$ define

$$I_* = \{x \in S \mid x^* \in I\}.$$

Lemma 4.4.6 *Let S be a distributive PJP-semilattice and I be a kernel ideal of S. Then I_* is a *-filter of S.*

Proof. Let $x, y \in I_*$. Then $x^*, y^* \in I$. So that by Theorem 4.3.4, we have $(x \wedge y)^* = (x \wedge y)^{***} = (x^{**} \wedge y^{**})^* \in I$. Hence $x \wedge y \in I_*$. Now let $x \in I_*$ and $y \geqslant x$. Then $y^* \leqslant x^* \in I$ so that $y^* \in I$ and consequently, $y \in I_*$. Hence I_* is a filter. Let $x^{**} \in I_*$. Then $x^* = x^{***} \in I$ and hence $x \in I_*$. Therefore I_* is a *-filter.

□

Theorem 4.4.7 *For any filter F of a distributive PJP-semilattice, $(F_*)_* = F$ if and only if F is a *-filter.*

Proof. Let $(F_*)_* = F$ and let $x^{**} \in F$. Since F is a filter, F_* is a kernel ideal. Hence $x^* \in F_*$ and so $x \in (F_*)_* = F$. Thus F is a *-filter.

Conversely, let F be a *-filter. Then

$$\begin{aligned} x \in (F_*)_* &\Leftrightarrow x^* \in F_* \\ &\Leftrightarrow x^{**} \in F \\ &\Leftrightarrow x \in F \quad (\Rightarrow \text{ as } F \text{ is a *-filter and } \Leftarrow \text{ as } F \text{ is a filter }). \end{aligned}$$

□

Boolean Congruences. Let S be a PJP-semilattice. A congruence θ on S is said to be **boolean congruence** if $\frac{S}{\theta}$ is a Boolean lattice.

Theorem 4.4.8 *A PJP-congruence θ is a boolean congruence if and only if for all $x \in X$, $x \equiv x^{**}(\theta)$.*

Proof. This is immediate from the fact that $([x](\theta))^* = [x^*](\theta)$. □

A filter F of a PJP-semilattice S is called **D-filter** if it contains the dense filter $D = \{x \in S \mid x^* = 0\}$.

Theorem 4.4.9 *Every *-filter is a D-filter but the converse is not true.*

Proof. Let F be a *-filter and let $d \in D$. Then $d^{**} = 1 \in F$ which implies that $d \in F$. Hence F contains D. Thus F is a D-filter.

To prove the converse is not true, consider the distributive PJP-semilattice N given in Figure 4.3. If $F = [c)$, then F is a D-filter but not *-filter. □

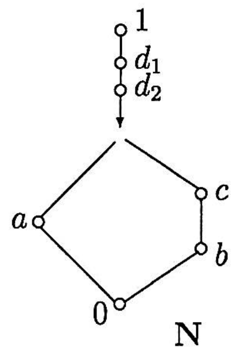


FIGURE 4.3

Theorem 4.4.10 *Let S be a distributive PJP-semilattice. Then the following are equivalent:*

(i) *every D-filter is a *-filter;*

(ii) $\Theta(D)$ *is a boolean congruence.*

Proof. (i)$\Rightarrow$ (ii). For each $x \in S$ we have $F = [x^{**}) \vee D$ is a D-filter and hence F is a *-filter. Since $x^{**} \in F$, we have $x \in F$. Thus $x = x^{**} \wedge d$ for some $d \in D$. This implies $x \wedge d = x^{**} \wedge d$. Hence $x \equiv x^{**}\Theta(D)$. Therefore, by Theorem 4.4.8, $\Theta(D)$ is a boolean congruence.

(ii)$\Rightarrow$ (i). Let F be a D-filter. By (ii), $\Theta(D)$ is a boolean congruence. Hence by Theorem 4.4.8 $x \equiv x^{**}(\Theta(D))$. Thus $x \wedge d = x^{**} \wedge d$ for some $d \in D$. If $x^{**} \in F$, then $x^{**} \wedge d \in F$ as $D \subseteq F$. Hence $x \wedge d \in F$ and consequently, $x \in F$. Thus F is a *-filter. □

CHAPTER 5

Stone JP-semilattices

5.1. Introduction

Stone lattices is a well known subclass of pseudocomplemented lattices. In this chapter we intend to explore the Stone's property in PJP-semilattices. In fact, we generalize some results of Stone lattices for Stone JP-semilattices. For Stone lattices we refer the reader to [**16**, **17**]. A distributive PJP-semilattice $\mathbf{S} = \langle S; \wedge, \vee, ^*, 0, 1\rangle$ is said to be a **Stone JP-semilattice** if for any $a \in S$,

$$a^* \vee a^{**} \text{ exists and } a^* \vee a^{**} = 1.$$

In Section 5.2 we give an example of a Stone JP-semilattice. This is a distributive JP-semilattice but the underlying semilattice is not distributive in the sense of distributivity in semilattices. We give a characterization of Stone JP-semilattices. In Section 5.3 we study the kernel ideals of Stone JP-semilattices. In section 5.4 we study the kernel preserving JP-homomorphism of Stone JP-semilattices.

5.2. A characterization of Stone JP-semilattices

Consider the JP-semilattice M given in the Figure 5.1. The Figure represents that for any $n \geqslant 1$, a_n is an upper bound of a, b and there is $b_n \geqslant b$ such that $a_n \wedge b_n = b_n$. Observe that M is not a distributive semilattice, for $b \geqslant a \wedge b_0$ but there are no $a_r \geqslant a$ and $b_r \geqslant b_0$ such that $b = a_r \wedge b_r$.

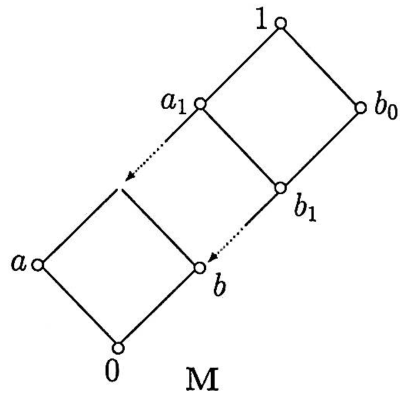


FIGURE 5.1. a distributive JP-semilattice

Theorem 5.2.1 M *is a distributive JP-semilattice.*

Proof. Let $x, y, z \in M$ with $y \vee z$ exists. Without loss of generality if we assume that $y \leqslant z$, then $x \wedge (y \vee z) = x \wedge z = (x \wedge y) \vee (x \wedge z)$. Thus M is a distributive JP-semilattice. Now suppose $y \| z$.

Case I. Without loss of generality assume $y = a$ and $z = b_n$ for some n.

Subcase 1. Suppose $a_n \leqslant x \leqslant 1$. Then $b_n \leqslant x$ and hence

$$x \wedge (y \vee z) = x \wedge (a \vee b_n) = x \wedge a_n = a_n$$

and

$$(x \wedge y) \vee (x \wedge z) = (x \wedge a) \vee (x \wedge b_n) = a \vee b_n = a_n.$$

Subcase 2. Suppose $a < x < a_n$. Then $x = a_r$ for some $r > n$ and hence

$$x \wedge (y \vee z) = x \wedge (a \vee b_n) = a_r \wedge a_n = a_r$$

and

$$(x \wedge y) \vee (x \wedge z) = (a_r \wedge a) \vee (a_r \wedge b_n) = a \vee b_r = a_r.$$

Subcase 3. Suppose $x = a$. Then

$$x \wedge (y \vee z) = a \wedge (a \vee b_n) = a \wedge a_n = a$$

and

$$(x \wedge y) \vee (x \wedge z) = (a \wedge a) \vee (a \wedge b_n) = a \vee 0 = a.$$

Subcase 4. Suppose $b_n \leqslant x \leqslant b_0$. Then

$$x \wedge (y \vee z) = x \wedge (a \vee b_n) = x \wedge a_n = b_n$$

and

$$(x \wedge y) \vee (x \wedge z) = (x \wedge a) \vee (x \wedge b_n) = 0 \vee b_n = b_n.$$

Subcase 5. Suppose $x < b_n$. Then $x < a_n$ and hence

$$x \wedge (y \vee z) = x \wedge (a \vee b_n) = x \wedge a_n = x$$

and

$$(x \wedge y) \vee (x \wedge z) = (x \wedge a) \vee (x \wedge b_n) = 0 \vee x = x.$$

Case II. Without loss of generality assume $y = a_n$ for some n and $z = b_r$ for some $0 \leqslant r < n$.

Subcase 1. Suppose $a_r \leqslant x \leqslant 1$. Then $b_n \leqslant x$ and hence

$$x \wedge (y \vee z) = x \wedge (a_n \vee b_r) = x \wedge a_r = a_r$$

and

$$(x \wedge y) \vee (x \wedge z) = (x \wedge a_n) \vee (x \wedge b_r) = a_n \vee b_r = a_r.$$

Subcase 2. Suppose $a_n \leqslant x = a_m < a_r$. Then

$$x \wedge (y \vee z) = a_m \wedge a_r = a_m$$

and

$$(x \wedge y) \vee (x \wedge z) = (a_m \wedge a_n) \vee (a_m \wedge b_r) = a_n \vee b_m = a_m.$$

Subcase 3. For $x \leqslant a_n$ or b_r. We have

$$x \wedge (y \vee z) = x = (x \wedge y) \vee (x \wedge z)$$

Subcase 4. Suppose $b_r \leqslant x \leqslant b_0$. Then

$$x \wedge (y \vee z) = x \wedge a_r = b_r$$

and

$$(x \wedge y) \vee (x \wedge z) = (x \wedge a_n) \vee (x \wedge b_r) = b_n \vee b_r = b_r.$$

Thus M is a distributive JP-semilattice. □

Theorem 5.2.2 M *is a stone JP-semilattice.*

Proof. Observe that for any $x \geqslant a, b$, we have $x^* = 0$ and for any $x \in [b, b_0]$ we have $x^* = a$ and $a^* = b_0$. Hence, $x^* \vee x^{**} = 1$ for each $x \in M$. Therefore M is a Stone JP-semilattice. □

Now we give a characterization of a Stone JP-semilattice which is a generalization of an important result for Stone lattices in Lattice Theory.

Theorem 5.2.3 *For a distributive PJP-semilattice S, the following conditions are equivalent:*

(a) S *is a Stone JP-semilattice.*

(b) $a^* \vee b^*$ *exists and* $(a \wedge b)^* = a^* \vee b^*$ *for any* $a, b \in S$.

Proof. (a) $\Longrightarrow$ (b). Let S be a Stone JP-semilattice. We shall show that $(a \wedge b)^*$ is the least upper bound of a^* and b^*.

By Theorem 4.2.1 (ii), $a^*, b^* \leqslant (a \wedge b)^*$. Let $a^*, b^* \leqslant x$. We have to show that $(a \wedge b)^* \leqslant x$. It is enough to show that $x \wedge (a \wedge b)^* = (a \wedge b)^*$, that is, $x \wedge (a \wedge b)^*$ is the pseudocomplement of $a \wedge b$.

Clearly, $(a \wedge b) \wedge (x \wedge (a \wedge b)^*) = x \wedge (a \wedge b) \wedge (a \wedge b)^* = 0$. Let $(a \wedge b) \wedge y = 0$. We shall show that $y \leqslant x \wedge (a \wedge b)^*$. Now $y \wedge a \wedge b = 0$ implies $y \leqslant (a \wedge b)^*$ and $y \wedge b \leqslant a^*$. Hence $y \wedge a^{**} \wedge b \leqslant a^* \wedge a^{**} = 0$. Thus $y \wedge a^{**} \wedge b = 0$ and hence $y \wedge a^{**} \leqslant b^* \leqslant x$. Also $y \wedge a^* \leqslant a^* \leqslant x$. Therefore,

$$y = y \wedge 1 = y \wedge (a^* \vee a^{**}) = (y \wedge a^*) \vee (y \wedge a^{**}) \leqslant x.$$

Hence $y \leqslant x \wedge (a \wedge b)^*$. This implies $x \wedge (a \wedge b)^* = (a \wedge b)^*$.

(b) $\Longrightarrow$ (a). By (b), $a^* \vee a^{**}$ exists and $a^* \vee a^{**} = (a \wedge a^*)^* = 0^* = 1$. Hence (a) holds. □

5.3. Kernel Ideals of Stone JP-semilattices

We have seen that in a distributive PJP-semilattice the condition $x \in I$ implies $x^{**} \in I$ for an ideal I is necessary for the ideal to be a kernel ideal but not sufficient. We now prove that the condition is sufficient for a Stone JP-semilattice. Of course, the following result is also a generalization of [**9**, Theorem 1.5].

Theorem 5.3.1 *Let* S *be a Stone JP-semilattice. If I is an ideal of* S, *then the following are equivalent:*

(a) *I is a kernel ideal;*

(b) *$x \in I$ implies $x^{**} \in I$.*

Proof. (a) $\Rightarrow$ (b). Since S is a distributive PJP-semilattice, by Theorem 4.3.5 we have (b) holds.

(b) $\Rightarrow$ (a). Let $i, j \in I$. Then $i^{**}, j^{**} \in I$. Since S is a Stone JP-semilattice, by Theorem 5.2.3, $i^{**} \vee j^{**}$ exists and $i^{**} \vee j^{**} = (i^* \wedge j^*)^*$. This implies $(i^* \wedge j^*)^* = i^{**} \vee j^{**} \in I$. Therefore, by Theorem 4.3.4, I is a kernel ideal. Thus (a) holds. □

We have seen that the description of join of two ideals of a distributive JP-semilattice is not so handy. The life is easier as expected for the join of two kernel ideals of a Stone JP-semilattice.

Theorem 5.3.2 *Let* S *be a Stone JP-semilattice. If I and J are two kernel ideals of* S, *then $I \vee J$ is a kernel ideal and*

$$I \vee J = \{x \in S \mid x \leqslant (i^* \wedge j^*)^* \text{ for some } i \in I, j \in J\}.$$

Indeed,

$$I \vee J = \{x \in S \mid x = i \vee j \text{ for some } i \in I, j \in J\}.$$

Proof. Let

$$K = \{x \in S \mid x \leqslant (i^* \wedge j^*)^* \text{ for some } i \in I \text{ and } j \in J\}.$$

We show that K is the smallest kernel ideal containing I and J. Clearly K is a down set. Let $x, y \in K$ with $x \vee y$ exists. Then $x \leqslant (i_1^* \wedge j_1^*)^*$ and $y \leqslant (i_2^* \wedge j_2^*)^*$ for some $i_1, i_2 \in I$ and $j_1 \in j_2 \in J$. Since S is Stone, $(i_1^* \wedge j_1^*)^* \vee (i_2^* \wedge j_2^*)^*$ exists and $(i_1^* \wedge j_1^*)^* \vee (i_2^* \wedge j_2^*)^* = (i_1^* \wedge i_2^* \wedge j_1^* \wedge j_2^*)^*$. Since I and J are kernel ideals, by Theorem 4.3.5, there is $k_1 \in I$ and $k_2 \in J$ such that $k_1^* = i_1^* \wedge i_2^*$ and $k_2^* = j_1^* \wedge j_2^*$. Hence $x \vee y \leqslant (k_1^* \wedge k_2^*)^*$. Thus $x \vee y \in K$. Moreover

$$(x^* \wedge y^*)^* \leqslant ((i_1^* \wedge j_1^*)^{**} \wedge (i_2^* \wedge j_2^*)^{**})^* = (i_1^* \wedge i_2^* \wedge j_1^* \wedge j_2^*)^* = (k_1^* \wedge k_2^*)^*.$$

Hence $(x^* \wedge y^*)^* \in K$. Therefore, K is a kernel ideal. For each $i \in I$ we have $i \leqslant i^{**} \leqslant (i^* \wedge j^*)^*$ for any $j \in J$. Hence $i \in K$ which implies that K contains I. Similarly, K contains J.

Indeed, since S is a Stone JP-semilattice, by Theorem 5.2.3, $i^{**} \vee j^{**}$ exists and $i^{**} \vee j^{**} = (i^* \wedge j^*)^*$. Thus $x \in I \vee J$ implies $x \leqslant (i^* \wedge j^*)^*$. So,

$$\begin{aligned} x &= x \wedge (i^* \wedge j^*)^* \quad \text{for some } i \in I, j \in J \\ &= x \wedge (i^{**} \vee j^{**}) \\ &= (x \wedge i^{**}) \vee (x \wedge j^{**}) \quad \text{as S is a distributive JP-semilattice.} \end{aligned}$$

Now $i \in I$ implies $i^{**} \in I$ as I is a kernel ideal. Hence $x \wedge i^{**} \in I$. Similarly, $x \wedge j^{**} \in J$. Hence $x = i \vee j$ for some $i \in I$ and $j \in J$. Thus

$$I \vee J \subseteq K = \{x \in S \mid x \leqslant (i^* \wedge j^*)^* \quad \text{for some } i \in I, j \in J\}$$

$$\subseteq \{x \in S \mid x = i \vee j \quad \text{for some } i \in I, j \in J\}$$

$$\subseteq I \vee J.$$

Hence

$$I \vee J = \{x \in S \mid x \leqslant (i^* \wedge j^*)^* \quad \text{for some } i \in I, j \in J\}$$

$$= \{x \in S \mid x = i \vee j \quad \text{for some } i \in I, j \in J\}$$

$$= K.$$

Therefore $I \vee J$ is a kernel ideal. □

The set of all kernel ideals of a Stone JP-semilattice S is denoted by KI(S).

Corollary 5.3.3 *$KI(S)$ is a sublattice of $I(S)$.* □

Theorem 5.3.4 *Let* S *be Stone JP-semilattice. Then* KI(S) *is a distributive sublattice of $I(S)$.*

Proof. Let $I, J, K \in$ KI(S). Let $x \in I \wedge (J \vee K)$. Then $x \in I$ and $x \in J \vee K$. This implies $x = j \vee k$ for some $j \in J$ and $k \in K$. Hence $j \in I$ and $k \in I$. Thus $j \in I \wedge J$ and $k \in I \wedge K$ consequently $x = j \vee k \in (I \wedge J) \vee (I \wedge K)$.

The reverse inclusion is trivial. □

Theorem 5.3.5 *Let S be a Stone JP-semilattice. Then* KI(S) *is a complete lattice.*

Proof. Let $\{I_k\}$ be a family of kernel ideals of S. Let $i, j \in \bigcap_{k=1} I_k$. Then by Theorem 4.3.4, $(i^* \wedge j^*)^* \in \bigcap_{k=1} I_k$. Thus $\bigcap_{k=1} I_k$ is a kernel ideal. Hence $\bigwedge_{k=1} I_k = \bigcap_{k=1} I_k$. We show that

$$\bigvee_{k=1} I_k = \{x \leqslant (x_1^* \wedge x_2^* \wedge \cdots \wedge x_n^*)^* \text{ for some } x_i \in I_{k_i}, 1 \leqslant i \leqslant n\}.$$

Clearly, R.H.S is a down-set. Let $x, y \in R.H.S.$ with $x \vee y$ exists. Since S is a Stone JP-semilattice, we clearly have

$$x \vee y \leqslant (x_1^* \wedge x_2^* \wedge \cdots \wedge x_n^*)^* \vee (y_1^* \wedge y_2^* \wedge \cdots \wedge y_m^*)^*$$

$$= (x_1^* \wedge x_2^* \wedge \cdots \wedge x_n^* \wedge y_1^* \wedge y_2^* \wedge \cdots \wedge y_m^*)^*$$

where $x_i \in I_{k_i}, 1 \leqslant i \leqslant n$ and $y_j \in I_{k_j}, 1 \leqslant j \leqslant m$. Thus $x \vee y \in$R.H.S. and hence R.H.S. is an ideal. Moreover,

$$(x^* \wedge y^*)^* \leqslant ((x_1^* \wedge x_2^* \wedge \cdots \wedge x_n^*)^{**} \wedge (y_1^* \wedge y_2^* \wedge \cdots \wedge y_m^*)^{**})^*$$

$$\leqslant (x_1^* \wedge x_2^* \wedge \cdots \wedge x_n^* \wedge y_1^* \wedge y_2^* \wedge \cdots \wedge y_m^*)^*.$$

Hence $(x^* \wedge y^*)^* \in$ R.H.S. Therefore, R.H.S. is a kernel ideal. For each $a \in I_i$ we clearly have $a \in$R.H.S. which implies that R.H.S. contains each I_i. Let M be any kernel ideal containing each I_i. Let $x \in$ R.H.S. Then $x \leqslant (x_1^* \wedge x_2^* \wedge \cdots \wedge x_n^*)^*$ for some $x_i \in I_{k_i}, 1 \leqslant i \leqslant n$. This implies $x_i \in M$ for each $1 \leqslant i \leqslant n$. Since M is a kernel ideal, for each $1 \leqslant i \leqslant n$ we have $x_i \equiv 0(\theta)$ for congruence θ on S. Hence

$x_i^* \equiv 1(\theta)$. Thus $x_1^* \wedge x_2^* \wedge \cdots \wedge x_n^* \equiv 1(\theta)$. This implies $(x_1^* \wedge x_2^* \wedge \cdots \wedge x_n^*)^* \equiv 0(\theta)$. Hence $(x_1^* \wedge x_2^* \wedge \cdots \wedge x_n^*)^* \in M$ which implies that $x \in M$. Thus

$$\bigvee_{k=1} I_k = \{x \leqslant (x_1^* \wedge x_2^* \wedge \cdots \wedge x_n^*)^* \text{ for some } x_i \in I_{k_i}, 1 \leqslant i \leqslant n\}.$$

□

Let S be a Stone JP-semilattice and let $I \in \mathrm{KI(S)}$. Define

$$I^{\sqcap} = \{x \in S \mid x^{**\downarrow} \cap I = (0]\}.$$

Then clearly, $(0]^{\sqcap} = S$ and $S^{\sqcap} = (0]$.

Theorem 5.3.6 *For any $I \in \mathrm{KI(S)}$ we have*

(a) $I^{\sqcap} \in \mathrm{KI(S)}$;

(b) $I^{\sqcap}$ *is the pseudocomplement of I in* KI(S).

Proof. (a) Clearly, $I^{\sqcap}$ is a down-set. Let $x, y \in I^{\sqcap}$ with $x \vee y$ exists. Then $x^{**} \wedge i = y^{**} \wedge i = 0$ for all $i \in I$. For any $i \in I$, we have

$$\begin{aligned}(x \vee y)^{**} \wedge i &= (x^* \wedge y^*)^* \wedge i \\ &= (x^{**} \vee y^{**}) \wedge i \text{ (as S is a Stone JP-semilattice)} \\ &= (x^{**} \wedge i) \vee (y^{**} \wedge i) \\ &= 0.\end{aligned}$$

Hence $x \vee y \in I^{\cap}$. Moreover, for all $i \in I$ we have

$$\begin{aligned}[(x^* \wedge y^*)^*]^{**} \wedge i &= (x^* \wedge y^*)^* \wedge i \\ &= (x^{**} \vee y^{**}) \wedge i \text{ (as S is a Stone JP-semilattice)} \\ &= (x^{**} \wedge i) \vee (y^{**} \wedge i) \\ &= 0.\end{aligned}$$

Hence $(x^* \wedge y^*)^* \in I^{\cap}$.

Therefore, $I^{\cap}$ is a kernel ideal.

(b) Clearly $I \cap I^{\cap} = (0]$. Let $J \in$ KI(S) with $I \cap J = (0]$. Suppose $x \in J$. Then $x^{**} \in J$ as $J \in$ KI(S). Thus $x^{**} \wedge i = 0$ for all $i \in I$. This implies $x \in I^{\cap}$. Therefore, $I^{\cap}$ is the pseudocomplement of I in KI(S). □

For $Q \subseteq S$, define

$$Q^{\downarrow} = \{x \in S \mid x \leqslant y \text{ for some } y \in Q\}.$$

For $x \in S$, we write $x^{\downarrow}$ in stead of $\{x\}^{\downarrow}$. Thus $x^{\downarrow} = (x]$.

Theorem 5.3.7 *$I \in$ KI(S) has a complement if and only if I is principal.*

Proof. Suppose I has a complement. Then $S = I \vee I^{\cap}$ and since S is Stone, by Theorem 5.3.2, we have $1 = (x^* \wedge y^*)^*$ for some $x \in I$ and $y \in I^{\cap}$. This implies $y^{**\downarrow} \cap I = (0]$. Thus $y^{**} \wedge i = 0$ and hence $i \leqslant y^*$ for all $i \in I$. Therefore, for all

$i \in I$ we have

$$i^{**} = i^{**} \wedge 1 = i^{**} \wedge (x^* \wedge y^*)^* = [i \wedge (x^* \wedge y^*)^*]^{**}$$
$$\leqslant [i \wedge (x^* \wedge i)^*]^{**} = (i \wedge x^{**})^{**} = i^{**} \wedge x^{**}.$$

Thus we obtain $i^{**} = i^{**} \wedge x^{**}$ and so $i \leqslant i^{**} \leqslant x^{**} \in I$ (as I is a kernel ideal). Hence $I = x^{**\downarrow}$.

Conversely, let I be principal. Then $I = x^{\downarrow}$ for some $x \in S$ and hence $I = x^{**\downarrow}$ as $x \leqslant x^{**} \in I$. This implies

$$I^{\sqcap} = \{y \in S \mid y^{**} \wedge x^{**} = 0\} = \{y \in S \mid y \wedge x = 0\} = \{y \in S \mid y \leqslant x^*\} = x^{*\downarrow}.$$

Thus, by Theorem 5.3.2,

$$I \vee I^{\sqcap} = \{y \in S \mid y \leqslant (x^* \wedge x^{**})^*\} = S.$$

Hence $I^{\sqcap}$ is the complement of I. □

Recall that a filter F of a PJP-semilattice S is said to be a ***-filter** if

$$f^{**} \in F \Rightarrow f \in F.$$

The set of all *-filters of a PJP-semilattice S will be denoted by $F^*(S)$.

First we give a description of the join of two *-filters.

Theorem 5.3.8 *Let* S *be a Stone JP-semilattice. If F_1 and F_2 are two *-filters of* S*, then $F_1 \vee F_2$ is a *-filter and*

$$F_1 \vee F_2 = \{x \in S \mid x^* \leqslant (i \wedge j)^* \text{ for some } i \in F_1, j \in F_2\}.$$

Proof. Let

$$K = \{x \in S \mid x^* \leqslant (i \wedge j)^* \text{ for some } i \in F_1 \text{ and } j \in F_2\}.$$

We show that K is the smallest *-filter containing F_1 and F_2. Since $x \leqslant y$ implies $y^* \leqslant x^*$ we have K is an up-set. Let $x, y \in K$. Then $x^* \leqslant (i_1 \wedge j_1)^*$ and $y^* \leqslant (i_2 \wedge j_2)^*$ for some $i_1, i_2 \in F_1$ and $j_1 \in j_2 \in F_2$. Since S is Stone, $x^* \vee y^*$ exists and $(x \wedge y)^* = x^* \vee y^* \leqslant (i_1 \wedge j_1)^* \vee (i_2 \wedge j_2)^* = (i_1 \wedge i_2 \wedge j_1 \wedge j_2)^*$ where $i_1 \wedge i_2 \in F_1$ and $j_1 \wedge j_2 \in F_2$. Hence $x \wedge y \in K$. Moreover, if $x^{**} \in K$, since $x^* = x^{***}$ we have $x \in K$. Therefore, K is a *-filter. Let $i \in F_1$. Since for any $j \in F_2$, we have $i^* \vee j^*$ exists and $i^* \leqslant i^* \vee j^* = (i \wedge j)^*$. Hence $i \in K$ which implies that K contains F_1. Similarly, K contains F_2. Let M be any *-filter containing F_1 and F_2. Let $x \in K$. Then $x^* \leqslant (i \wedge j)^*$ for some $i \in F_1, j \in F_2$. This implies $i, j \in M$. Since M is a filter, $i \wedge j \in M$ and hence $(i \wedge j)^{**} \in M$. Thus $x^{**} \in M$. This implies $x \in M$ as M is a *-filter. Thus $I \vee J = K$. □

Corollary 5.3.9 *Let* S *be a Stone JP-semilattice. For any* $F_1, F_2 \in F^*(S)$ *we have*

$$F_1 \vee F_2 = \{x \in S \mid x^* \leqslant i^* \vee j^* \textit{ for some } i \in F_1, j \in F_2\}.$$

Proof. This is immediate from the fact that in Stone JP-semilattice $(i \wedge j)^* = i^* \vee j^*$. □

Now we have the following result.

Theorem 5.3.10 *Let* **S** *be a Stone JP-semilattice. Then* $\mathrm{KI}(S) \cong F^*(S)$.

Proof. Define a map $f : \mathrm{KI}(S) \to F^*(S)$ by

$$f(I) = \{x \in S \mid x^* \in I\}.$$

By Lemma 4.4.6, $f(I) \in F^*(S)$. Clearly, f is well defined, one to one and preserves the $\cap$ on $\mathrm{KI}(S)$. Let $I, J \in \mathrm{KI}(S)$. Then

$$\begin{aligned} f(I \vee J) &= \{x \in S \mid x^* \in I \vee J\} \\ &= \{x \in S \mid x^* = i \vee j \text{ for some } i \in I \text{ and } j \in J\} \\ &= \{x \in S \mid x^{**} = i^* \wedge j^* \text{ for some } i^{**} \in I \text{ and } j^{**} \in J\} \\ &= \{x \in S \mid x^* = i^{**} \vee j^{**} \text{ for some } i^{**} \in I \text{ and } j^{**} \in J\} \\ &= \{x \in S \mid x^* = i^{**} \vee j^{**} \text{ for some } i^* \in f(I) \text{ and } j^* \in f(J)\} \\ &= f(I) \vee f(J). \end{aligned}$$

Let $F \in F^*(S)$. Define

$$I = \{x \in S \mid x^* \in F\}.$$

Then by Lemma 4.4.5, I is a kernel ideal. Hence

$$\begin{aligned} f(I) &= \{x \in S \mid x^* \in I\} \\ &= \{x \in S \mid x^{**} \in F\} \\ &= F. \end{aligned}$$

Hence f is onto.

Therefore f is an isomorphism. □

5.4. Kernel homomorphisms

Let S_1 and S_2 be two JP-semilattices. Recall that a semilattice homomorphism $\varphi : S_1 \to S_2$ is called JP-homomorphism if for all $x, y \in S$ with $x \vee y$ exists in S_1 implies $\varphi(x) \vee \varphi(y)$ exists in S_2 and $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$. A JP-homomorphism $\varphi : S_1 \to S_2$ is said to be a **strong JP-homomorphism** if

$$x \vee y \text{ exists in } S_1 \text{ if and only if } \varphi(x) \vee \varphi(y) \text{ exists in } S_2.$$

The other definitions are analogous. Remark that every one-to-one strong JP-homomorphism is a JP-embedding. A JP-homomorphism φ is said to be a **PJP-homomorphism** if

$$\varphi(x^*) = \varphi(x)^*.$$

Let $\varphi : S_1 \to S_2$ be a PJP-homomorphism. Define

$$\varphi_0 = \{x \in S_1 \mid \varphi(x) = 0\}.$$

Lemma 5.4.1 *φ_0 is a kernel ideal.*

Proof. Clearly, φ_0 is an ideal. To prove φ_0 is a kernel ideal let $x, y \in \varphi_0$. Then

$$\varphi((x^* \wedge y^*)^*) = (\varphi(x^* \wedge y^*))^* = (\varphi(x)^* \wedge \varphi(y)^*)^* = 0.$$

Hence $(x^* \wedge y^*)^* \in \varphi_0$. Thus by Theorem 4.3.4 we have φ_0 is a kernel ideal. □

Let S_1 and S_2 be semilattices and let $f : S_1 \to S_2$ be a mapping. For each $X \subseteq S_1$ and for each $Y \subseteq S_2$ define

$$f(X) = \{f(x) \mid x \in X\} \quad \text{and} \quad f^{\leftarrow}(Y) = \{x \in X \mid f(x) \in Y\}.$$

Theorem 5.4.2 *Let* S_1 *and* S_2 *be two PJP-semilattices and let* $f : S_1 \to S_2$ *be a mapping. Then*

(a) *if* f *is a PJP-epimorphism, then* $f^{\leftarrow}(Y)$ *is a kernel ideal of* S_1 *for each kernel ideal* Y *of* S_2*;*

(b) *if* f *is a strong PJP-epimorphism, then* $f(X)$ *is a kernel ideal of* S_2 *for each kernel ideal* X *of* S_1.

Proof. (a) Let Y be a kernel ideal of S_2. Since $0 \in Y$ and f is an PJP-epimorphism, we have $f^{\leftarrow}(Y)$ is non-empty. Let $x \in f^{\leftarrow}(Y)$ and $t \leqslant x$. Then $f(t) = f(t \wedge x) = f(t) \wedge f(x) \leqslant f(x) \in Y$. Thus $f(t) \in Y$ as Y is an ideal. Hence $t \in f^{\leftarrow}(Y)$.

Let $x_1, x_2 \in f^{\leftarrow}(Y)$ with $x_1 \vee x_2$ exists. Then $f(x_1), f(x_2) \in Y$. Since f is a JP-homomorphism we have $f(x_1) \vee f(x_2)$ exists and $f(x_1 \vee x_2) = f(x_1) \vee f(x_2)$. Since Y is an ideal, we have $f(x_1 \vee x_2) = f(x_1) \vee f(x_2) \in Y$. Thus $x_1 \vee x_2 \in f^{\leftarrow}(Y)$. Therefore $f^{\leftarrow}(Y)$ is an ideal.

Finally, let $x_1, x_2 \in f^{\leftarrow}(Y)$. Then $f(x_1), f(x_2) \in Y$ and hence $[f(x_1)^* \wedge f(x_2)^*]^* \in Y$ as Y is a kernel ideal. Since f is a PJP-homomorphism, we have $[f(x_1^*) \wedge f(x_2^*)]^* \in Y$. Thus $[f(x_1^* \wedge x_2^*)]^* \in Y$. Consequently, $[f(x_1^* \wedge x_2^*)^*] \in Y$. Hence $(x_1^* \wedge x_2^*)^* \in f^{\leftarrow}(Y)$. Therefore, $f^{\leftarrow}(Y)$ is a kernel ideal of S_1.

(b) Let X be a kernel ideal of S_1. Since $0 \in X$, we have $f(X)$ is non-empty. Let $y \in f(X)$ and $t \in S_2$ with $t \leqslant y$. Then there is $x \in X$ such that $f(x) = y$ and since f is a PJP-epimorphism, there is $x_1 \in S_1$ such that $f(x_1) = t$. Now

$$t = t \wedge y = f(x_1) \wedge f(x) = f(x_1 \wedge x).$$

Since $x_1 \wedge x \in X$, we have $t \in F(X)$.

Let $y_1, y_2 \in f(X)$ with $y_1 \vee y_2$ exists. Then $y_1 = f(x_1)$ and $y_2 = f(x_2)$ for some $x_1, x_2 \in X$. Since f is a strong JP-homomorphism, we have $x_1 \vee x_2$ exists and $f(x_1 \vee x_2) = f(x_1) \vee f(x_2) = y_1 \vee y_2$. Since X is an ideal, we have $x_1 \vee x_2 \in X$. Thus $y_1 \vee y_2 \in f(X)$. Therefore $f(X)$ is an ideal.

Finally, let $y_1, y_2 \in f(X)$. Then $y_1 = f(x_1), y_2 = f(x_2)$ for some $x_1, x_2 \in X$. Since f is a PJP-homomorphism, we have

$$(y_1^* \wedge y_2^*)^* = [f(x_1)^* \wedge f(x_2)^*]^* = [f(x_1^*) \wedge f(x_2^*)]^* = [f(x_1^* \wedge x_2^*)]^* = f((x_1^* \wedge x_2^*)^*).$$

Since X is a kernel ideal, we have $(x_1^* \wedge x_2^*)^* \in X$. Thus $(y_1^* \wedge y_2^*)^* \in f(X)$. Therefore, $f(X)$ is a kernel ideal of S_2. □

Let S_1 and S_2 be two PJP-semilattices. Then every strong PJP-epimorphism $f : S_1 \to S_2$ induces mappings $f_K : \mathrm{KI}(S_1) \to \mathrm{KI}(S_2)$ defined by

$$f_K(I) = f(I)$$

and $f_K^{\leftarrow} : \mathrm{KI}(S_2) \to \mathrm{KI}(S_1)$ defined by

$$f_K^{\leftarrow}(J) = f^{\leftarrow}(J).$$

Now we have the following result.

Theorem 5.4.3 *Let S_1 and S_2 be two Stone JP-semilattices. If $f : S_1 \to S_2$ is a Strong PJP-epimorphism, then f_K is a lattice epimorphism.*

Proof. Let $I, J \in \mathrm{KI}(S_1)$ and let $x \in f_K(I) \cap f_K(J)$. Then for some $i \in I$ and $j \in J$ we have $x = f(i) = f(j) = f(i) \wedge f(j) = f(i \wedge j) \in f_K(I \cap J)$. Thus $f_K(I) \cap f_K(J) \subseteq f_K(I \cap J)$. The reverse inclusion is trivial. Hence f_K preserves the $\cap$.

Now let $x \in f_K(I \vee J)$. Then $x = f(y)$ for some $y \in I \vee J$. Since S_1 is a Stone JP-semilattice, by Theorem 5.3.2, $y = i \vee j$ for some $i \in I$ and $j \in J$. Thus $x = f(i \vee j) = f(i) \vee f(j) \in f_K(I) \vee f_K(J)$. Hence $f_K(I \vee J) \subseteq f_K(I) \vee f_K(J)$. The reverse inclusion is trivial. Hence f_K preserves the $\vee$. □

Corollary 5.4.4 *If $f : \mathrm{S}_1 \to \mathrm{S}_2$ is a PJP-isomorphism, then* $\mathrm{KI}(\mathrm{S}_1) \cong \mathrm{KI}(\mathrm{S}_2)$.

Now we have the following result.

Theorem 5.4.5 *Let S_1 and S_2 be two Stone JP-semilattices. If $f : \mathrm{S}_1 \to \mathrm{S}_2$ is a strong PJP-epimorphism, then for every $I \in \mathrm{KI}(\mathrm{S}_1)$,*

$$f_K^{\leftarrow}(f_K(I)) = I \vee f_0.$$

Proof. First suppose $x \in I \vee f_0$. Then $x = i \vee j$ for some $i \in I$ and $f(j) = 0$. Therefore, $f(x) = f(i \vee j) = f(i) \vee f(j) = f(i)$. Hence $f(x) \in f_K(I)$. Consequently, $x \in f_K^{\leftarrow}(f_K(I))$.

Conversely, let $x \in f_K^{\leftarrow}(f_K(I))$. Then for some $i \in I$ we have

$$f(x) = f(i) \leqslant f(i^{**}) = [f(i^*)]^*.$$

So that $f(x \wedge i^*) = f(x) \wedge f(i^*) = 0$. This implies $x \wedge i^* \in f_0$. Consequently,

$$x \leqslant x^{**} \leqslant (i^* \wedge x^*)^* = [i^* \wedge (i^* \wedge x)^*]^*.$$

It follows by Theorem 5.3.2 that $x \in I \vee f_0$. □

Now the following result gives a characterization of a pseudocomplemented lattice isomorphism.

Theorem 5.4.6 *Let S_1 and S_2 be two Stone JP-semilattices. If $f : S_1 \to S_2$ is a Strong PJP-epimorphism, then the following statements are equivalent:*

(a) *f_K is a pseudocomplemented lattice epimorphism;*

(b) *f_0 is a principal ideal.*

Proof. (a) $\Rightarrow$ (b). By (a) we have

$$f_K(f_0 \vee f_0^{\sqcap}) = f_K(f_0^{\sqcap}) = f_K(f_0)^{\sqcap} = S_2.$$

Hence $f_0 \vee f_0^{\sqcap} = S_1$. Thus f_0 is a complemented element of $\mathrm{KI}(S_1)$. Hence by Theorem 5.3.7, f_0 is a principal ideal.

(b) $\Rightarrow$ (a). For every $I \in \mathrm{KI}(S_1)$, let $x \in f_K(I^{\sqcap})$. Then $x = f(y)$ for some $y \in I^{\sqcap}$. So that $y^{**} \wedge i = 0$ for all $i \in I$. This implies $f(y)^{**} \wedge f(i) = 0$ for all $i \in I$. Thus $x = f(y) \in f_K(I)^{\sqcap}$. Hence $f_K(I^{\sqcap}) \subseteq f_K(I)^{\sqcap}$.

Conversely, let $x \in f_K(I)^{\sqcap}$. Then $x \wedge f(i) = 0$ for all $i \in I$. Since $x \in S_2$ and f is an epimorphism, there is $z \in S_1$ such that $x = f(z)$. Thus $f(z) \wedge f(i) = 0$. Hence $z \wedge i \in f_0$. Since f_0 is a principal kernel ideal, by Theorem 4.3.6 there is $t \in \mathrm{Sk}(S_1)$ such that $f_0 = t^{**\downarrow}$. This implies $z \wedge i \leqslant t^{**}$ and hence $z \wedge i \wedge t^* = 0$ and

$$f(z \wedge t^*) = f(z) \wedge f(t)^* = f(z) \wedge 1 = f(z).$$

Putting $z \wedge t^* = a$ we thus have $x = f(a)$ such that for all $i \in I$ we have $a \wedge i = 0$. Hence $x \in f_K(I^{\sqcap})$. □

CHAPTER 6

JP Distributive Semilattices

6.1. Introduction

In this Chapter we study the JP-semilattice such that the underlying semilattice is a distributive semilattice.

Recall that a JP-semilattice is said to be a **JP distributive semilattice** if its underlying semilattice is a distributive semilattice. We already have shown in Chapter 2 that the class of distributive JP-semilattices properly contain the class of JP distributive semilattices.

It is well known that in a semilattice S a non-empty subset I of S is an ideal of S if it is a down-set and every pair of elements of I has a common upper bound in I. First we like to mention that an ideal of a distributive JP-semilattice need not be an ideal of a distributive semilattice. For a counter example consider the semilattice S as given by the following Figure 6.1. If we choose $I = \{0, a, b, c\}$, then I is an ideal of S as a distributive JP-semilattice but not an ideal of S as a distributive semilattice. In Chapter 2 we proved the Stone's Separation Theorem for a distributive JP-semilattice. As every JP distributive

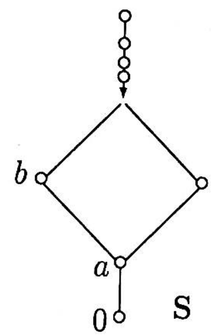


FIGURE 6.1. a JP Stone semilattice which is not a lattice

semilattice is a distributive JP-semilattice, so the Stone's Separation Theorem for JP distributive semilattice is obvious. Still in Section 6.2 we give another proof of the theorem by using a different technique. Here we use maximal prime filter instead of minimal prime ideal.

A pseudocomplemented distributive semilattice S with 1 is called a **Stone semilattice** if for any $c, x \in S$ with $c \geqslant x^*, x^{**}$ implies $c = 1$. In this chapter we study the Stone JP-semilattice such that the underlying semilattice is distributive. We call this semilattice by **JP Stone semilattice.** By definition, in a Stone JP-semilattice S we have $x^* \vee x^{**}$ exists for each $x \in S$. So, a PJP distributive semilattice S is a JP Stone semilattice if $x^* \vee x^{**} = 1$ for all $x \in S$. The example given in Figure 6.1 shows that every JP Stone semilattice need not be a lattice even not a near lattice.

In Section 6.3, it is shown that in a JP Stone semilattice S we have $x \vee y^*$ always exists for any $x, y \in S$. This observation turns that we have a straightforward generalization of the famous result of C.C. Chen and G. Grätzer for lattices. In Section 6.4 we give some characterizations of minimal prime ideals for a JP Stone semilattice. Here we also give a characterization of a JP Stone semilattice in terms of minimal prime ideals. In Section 6.5 we study the kernel ideals of JP Stone semilattices with some characterizations.

6.2. The Separation Theorem for JP distributive semilattices

Let H be a non-empty subset of S. The smallest filter containing H is called **filter generated by** H. It is denoted by $[H)$. If $H = \{a\}$, then we write $[a)$ for $[\{a\})$. The filter $[a)$ is said to be **principal filter**. The following results are similar to meet semilattices.

Lemma 6.2.1 *Let S be a JP-semilattice. Then*

(a) *$F = [H)$ if and only if for all $f \in F$ there exists $h_1, h_2, \cdots h_n \in H$ such that*

$$f \geqslant h_1 \wedge h_2 \wedge \cdots \wedge h_n.$$

(b) *For any $F, G \in \mathcal{F}(S)$, we have*

$$F \vee G = \{x \in S \mid x \geqslant f \wedge g \text{ for some } f \in F \text{ and } g \in G\}.$$

(c) *For any $a \in S$, we have*

$$[a) = \{x \in S \mid x \geqslant a\}.$$

□

For a distributive JP-semilattice S every element $x \in F \vee G$ where $F, G \in \mathcal{F}(S)$ can not be written as $x = f \wedge g$ for some $f \in F$ and $g \in G$. For example consider the JP-pentagon $\mathcal{N}_\infty$. It is a distributive JP-semilattice. If $F = [b)$ and $G = [c)$, then $a \in F \vee G$ but a can not be written as $a = f \wedge g$ for some $f \in F$ and $g \in G$.

The following theorem is immediate from the definition of distributive semilattice.

Theorem 6.2.2 *Let S be a JP distributive semilattice. Then for any $F_1, F_2 \in \mathcal{F}(S)$ we have*

$$F_1 \vee F_2 = \{f_1 \wedge f_2 \mid f_1 \in F_1, f_2 \in F_2\}.$$

□

A prime filter F is called **maximal** if there is a prime filter T such that $F \subseteq T$, then $F = T$. We have the following separation theorem for JP distributive semilattice.

Theorem 6.2.3 (The JP-Separation Theorem) *Let S be a JP distributive semilattice. Then for any ideal I and any filter F of S such that $I \cap F = \emptyset$, there exists a prime filter P containing F such that $P \cap I = \emptyset$.*

Proof. Let $\mathcal{F}$ be the set of all filters containing F, but disjoint from I. Then $\mathcal{F} \neq \emptyset$ as $F \in \mathcal{F}$. Let $\mathcal{C}$ be a chain in $\mathcal{F}$ and let $M := \cup\{X \mid X \in \mathcal{C}$. We claim that M is a maximal element in $\mathcal{C}$.

Let $x \in M$ and $x \leqslant y$. Then $x \in X$ for some $X \in \mathcal{C}$. Hence $y \in X$ as X is a filter. Therefore $y \in M$. Let $x, y \in M$. Then $x \in X$ and $y \in Y$ for some $X, Y \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, either $X \subseteq Y$ or $Y \subseteq X$. Suppose $X \subseteq Y$. So $x, y \in Y$ and hence $x \wedge y \in Y$ as Y is a filter. Hence $x \wedge y \in M$. Thus M is a filter. Clearly, M is the maximum filter containing F and $I \cap M = \emptyset$. Thus by Zorn's Lemma, $\mathcal{F}$ has a maximal element, say, P. We claim that P is a prime filter.

If P is not prime, there exists $a, b \in S$ such that $a \vee b$ exists and $a, b \notin P$ but $a \vee b \in P$. Then $(P \vee [a)) \cap I \neq \emptyset$ and $(P \vee [b)) \cap I \neq \emptyset$ as P is maximal. Hence there exist $p, q \in P$ such that $p \wedge a, q \wedge b \in I$ and hence $p \wedge q \wedge a, p \wedge q \wedge b \in I$ as I is an ideal. Since $p, q \in P$ and P is a filter, we have $r = p \wedge q \in P$. Since $a \vee b$ exists and S is JP distributive semilattice and hence distributive JP-semilattice. We have $(r \wedge a) \vee (r \wedge b)$ exists and $r \wedge (a \vee b) = (r \wedge a) \vee (r \wedge b) \in P \cap I$, a contradiction. Hence P is a prime filter. □

Corollary 6.2.4 *In a JP distributive semilattice* S, *if F is a filter of S and $a \in S$ with $a \notin F$ then there is a prime filter $P \supseteq F$ such that $a \notin P$.* □

Corollary 6.2.5 *In a JP distributive semilattice* S, *if F is a filter of S and $a, b \in S$ with $a \neq b$ then there is a prime filter P containing exactly one of a and b.* □

Corollary 6.2.6 *In a JP distributive semilattice* S *with* 1, *if $a, b \in S$ with $a \vee b$ does not exist then there is a prime filter F such that $a, b \notin F$.*

Proof. If $a \vee b$ does not exist, then there is $d \geqslant a, b$ such that $d < 1$. Then there is a prime filter F such that $d \notin F$ and hence $a, b \notin F$. □

Corollary 6.2.7 *In a JP distributive semilattice* S, *every filter F is a intersection of all prime filters P containing F.*

Proof. Let S be JP distributive semilattice and let F be a filter of S. Let

$$M = \bigcap\{X \mid X \text{ is a prime filter of } S \text{ and } F \subseteq X\}.$$

Clearly, $F \subseteq M$. We shall prove that $F = M$. If $F \neq M$, then there is $a \in M$ such that $a \notin F$. Then by Corollary 6.2.4, there is a prime filter P such that $F \subseteq P$ and $a \notin P$. This implies $a \notin M$, a contradiction. Hence $F = M$. □

6.3. JP Stone semilattices

Ramana and Rama Rao [26] proved that in a Stone semilattice $\mathbf{S} = \langle S; \wedge \rangle$, the least upper bound of $\{x, y^*\}$ exists for any $x, y \in S$. Here we modify the following crucial result for JP Stone semilattices.

Lemma 6.3.1 *Let $\langle S; \wedge, \vee, ^*, 0, 1 \rangle$ be a JP Stone semilattice. Then $x^* \vee y$ exists for any $x, y \in S$.*

Proof. Since $y \geqslant 0 = x^* \wedge x^{**}$, we have $y = x_1 \wedge x_2$ for some $x_1 \geqslant x^*$ and $x_2 \geqslant x^{**}$. Hence $x_1 \geqslant y, x^*$. We show that x_1 is the least upper bound of y and x^*. Let $z \geqslant y, x^*$. Then $z \geqslant x_1 \wedge x_2$ and hence $z = a_1 \wedge a_2$ for some $a_1 \geqslant x_1$ and $a_2 \geqslant x_2$. Thus $a_2 \geqslant x_2 \geqslant x^{**}$ and $a_2 \geqslant z \geqslant x^*$. Hence $a_2 \geqslant x^* \vee x^{**} = 1$ as S is a Stone JP-semilattice. This implies $a_2 = 1$. Thus $z = a_1 \wedge a_2 = a_1 \wedge 1 = a_1 \geqslant x_1$. This implies $x_1 = x^* \vee y$. □

Remark. In the above Lemma the distributivity of the underlying semilattice can not be relaxed. For example consider the distributive JP-semilattice **M** given in the Figure 6.2. Then **M** is a Stone JP-semilattice, that is, distributive JP-semilattice such that $x^* \vee x^{**} = 1$ for each $x \in M$. It is shown in the Section 5.2 that **M** is not a distributive semilattice. Observe that $b_0^* \vee b = a \vee b$ does not exist. On the other hand Stone is necessary, for example, consider the distributive semilattice $\mathbf{M}_2$ given in the Figure 6.2. Here $a \vee a^*$ does not exist.

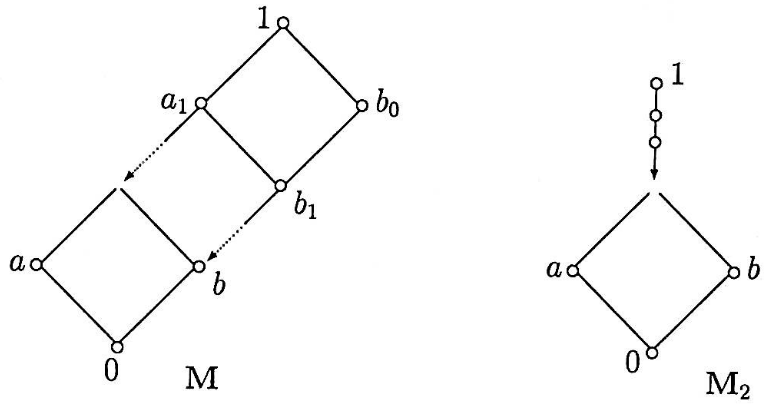


FIGURE 6.2. a distributive JP-semilattice

Define $D(S) = \{x \in S \mid x^* = 0\}$. The set $D(S)$ is called the **dense set**. The element of $D(S)$ is called the **dense element**.

Lemma 6.3.2 *Let* $\mathbf{S}$ *be a PJP-semilattice. Then* $D(S)$ *is a filter. Moreover if* $a \vee a^*$ *exists then* $a \vee a^* \in D(S)$.

Proof. Let $x, y \in D(S)$. Then we have $(x \wedge y)^{**} = x^{**} \wedge y^{**} = 1 \wedge 1 = 1$. This implies $(x \wedge y)^* = 0$. Hence $x \wedge y \in D(S)$.

Let $x \in D(S)$ and $y \in S$ with $y \geqslant x$. Then $y^* \leqslant x^* = 0$. This implies $y^* = 0$ and hence $y \in D(S)$. Therefore $D(S)$ is a filter of $\mathbf{S}$.

Moreover, if $a \vee a^*$ exists, then

$$(a \vee a^*)^* = a^* \wedge a^{**} = 0.$$

Thus, $a \vee a^* \in D(S)$. $\square$

Corollary 6.3.3 *Let S be a JP Stone semilattice. Then for all $x \in S$ we have $x \vee x^* \in D(S)$.*

Proof. Since the underlying semilattice is distributive then then by Lemma 6.3.1, $x \vee x^*$ exists and hence $x \vee x^* \in D(S)$. □

Let S be a JP Stone Semilattice. Then $a \vee a^*$ exists, for each $a \in S$ and $a \vee a^* \in D(S)$. Hence like as a lattice we can interpret the identity

$$a = a^{**} \wedge (a \vee a^*).$$

This shows that every $a \in S$ can be written as $a = b \wedge c$ where $b \in \mathrm{Sk}(S)$ and $c \in D(S)$. This observation turns our attention to straightforward generalization of the following result due to C.C. Chen and G.Grätzer [**3**, Theorem 14.5]. Define

$$\varphi(S) : a \to \{x \in D(S) \mid x \geqslant a^*\}.$$

Theorem 6.3.4 *Let S be a JP Stone semilattice. Then* $\mathrm{Sk}(S)$ *is a Boolean algebra,* $D(S)$ *is a distributive JP-semilattice with* 1, *and* $\varphi(S)$ *is a* $\{0,1\}$-*homomorphism of* $\mathrm{Sk}(S)$ *into* $\mathcal{D}(D(S))$. *The triple* $\langle \mathrm{Sk}(S), D(S), \varphi(S)\rangle$ *characterizes* S *up to isomorphism.* □

6.4. Minimal prime ideals for JP Stone semilattices

In this section we discuss the minimal prime ideals of a JP Stone semilattice. First we have the following useful characterization of minimal prime ideals for a JP Stone semilattice.

Theorem 6.4.1 *Let S be a JP Stone semilattice and let P be a prime ideal of S. Then the following are equivalent:*

(a) *P is minimal.*

(b) *$x \in P$ implies $x^* \notin P$.*

(c) *$x \in P$ implies $x^{**} \in P$.*

(d) *$P \cap D(S) = \emptyset$.*

Proof. (a) $\Rightarrow$ (b). Let P be minimal and $x \in P$. Suppose $x^* \in P$. Set $D = (S \setminus P) \vee [x)$. We claim that $0 \notin D$. For if $0 \in D$, then $0 = q \wedge x$ for some $q \in S \setminus P$. This implies $q \leqslant x^*$ and hence $q \in P$ which is a contradiction. Therefore, $0 \notin D$. By JP-separation Theorem, there is a prime filter Q such that $D \subseteq Q$ and $0 \notin Q$. Let $M = S \setminus Q$. Then by Lemma 2.5.1, M is a prime ideal. We claim that $M \cap D = \emptyset$. If $a \in M \cap D$. Then $a \notin Q$ and consequently $a \notin D$ which is a contradiction. Hence $M \cap D = \emptyset$. Therefore $M \cap (S \setminus P) = \emptyset$ and hence $M \subseteq P$. Also $M \neq P$ because $x \in D$ implies $x \in Q$ and hence $x \notin M$. This shows that P is not minimal. Hence $x^* \notin P$.

(b) $\Rightarrow$ (c). Let $x \in P$. We have $0 = x^* \wedge x^{**} \in P$. By (b), since $x^* \notin P$ and P is prime, we have $x^{**} \in P$.

(c) $\Rightarrow$ (d). Let $x \in P \cap D(S)$. Then $x \in P$ and $x^* = 0$. Thus $x \in P$ and $x^{**} = 1 \notin P$ which contradict (c).

(d) $\Rightarrow$ (a). If P is not minimal, then there is a prime ideal $Q \subset P$ (that is, Q is a proper subset of P). Let $x \in P \setminus Q$. Since $x \wedge x^* = 0 \in Q$ and $x \notin Q$, we have $x^* \in Q \subset P$. Since S is a JP Stone semilattice we have by Lemma 6.3.1,

$x \vee x^*$ exists and hence $x \vee x^* \in P$. By Lemma 6.3.2, we have $x \vee x^* \in D(S)$. This contradicts (d). □

Observe that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) hold if S is PJP distributive semilattice. But for (d) $\Rightarrow$ (a) we need the Stone property.

The following theorem is a generalization of Grätzer and Schmidt [**20**].

Theorem 6.4.2 *Let S be a PJP distributive semilattice. Then S is a JP Stone semilattice if and only if $P \vee Q = S$ for any two distinct minimal prime ideals P and Q.*

Proof. Let S be a JP Stone semilattice and let P and Q be two distinct minimal prime ideals of S. Choose $a \in P \setminus Q$. Then $a \notin Q$ and hence $a^* \in Q$ as $a \wedge a^* = 0 \in Q$. Since P is minimal, by Lemma 6.4.1 we have $a^{**} \in P$. Hence $a^* \vee a^{**} = 1 \in P \vee Q$. Thus $P \vee Q = S$.

Conversely, suppose S is not Stone. If $a^* \vee a^{**} \neq 1$, then there is a prime ideal R such that $a^* \vee a^{**} \in R$. We claim that $(S \setminus R) \vee [a^*) \neq S$. Suppose $(S \setminus R) \vee [a^*) = S$. Then there is $x \in S \setminus R$ such that $x \wedge a^* = 0$. Hence $x \leqslant a^{**}$ which implies $a^{**} \in S \setminus R$. Thus $a^{**} \notin R$ which is a contradiction. Similarly, we can show that $(S \setminus F) \vee [a^{**}) \neq S$. By the dual of Lemma 2.5.1 there are maximal prime filters F and G such that $(S \setminus R) \vee [a^*) \subseteq F$ and $(S \setminus R) \vee [a^{**}) \subseteq G$. Set $P = S \setminus F$ and $Q = S \setminus G$. Then P and Q are minimal prime ideals of S. We claim that $P \neq Q$. Indeed, $a \in P$, otherwise $a \in F$. Hence $a \wedge a^* = 0 \in F$ (as $a^* \in F$), a contradiction. Thus by Theorem 6.4.1, we have $a^{**} \in P$ while $a^{**} \notin Q$ (as $a^{**} \in G$). We now show that $P \vee Q \neq S$. Let $x \in P$. If $x \notin R$, then $x \in S \setminus R$

and hence $x \in F$, a contradiction. Hence $x \in R$. Thus $P \subseteq R$. Similarly, $Q \subseteq R$. This implies $P \vee Q \subseteq R$. Hence $P \vee Q \neq S$. □

6.5. Kernel ideals for JP Stone semilattices

Theorem 6.5.1 *Let S be a JP Stone semilattice and let J be an ideal of S. Then the following are equivalent:*

(a) *J is a kernel ideal;*

(b) *$x \in J$ implies $x^{**} \in J$;*

(c) *each minimal prime ideal containing J is a minimal prime ideal;*

(d) *J is an intersection of minimal prime ideals of S.*

Proof. (a) is equivalent to (b) by Theorem 5.3.1.

(b) $\Rightarrow$ (c). Suppose P is a minimal prime ideal containing J and $x \in P$. Then by Theorem 2.5.7 we have $x \wedge y \in J$ for some $y \in S \setminus P$. Thus by (b) we have $x^{**} \wedge y^{**} = (x \wedge y)^{**} \in J \subseteq P$. Since P is a prime ideal, $x^{**} \in P$. Hence by Theorem 6.4.1, we have P is a minimal prime ideal of S. Thus (c) holds.

(c) $\Rightarrow$ (d). Clearly, (d) follows from (c) by Lemma 2.5.3 and Theorem 2.5.6.

(d) $\Rightarrow$ (b). Let $x \in J$. Then by (d), $x \in P$ for all minimal prime ideal P and hence by Theorem 6.4.1, we have $x^{**} \in P$ for all minimal prime ideal P. Therefore $x^{**} \in J$. □

Recall that for any kernel ideal J of a PJP semilattice S, the equivalence relation $\psi(I)$ defined by

$$x \equiv y(\psi(J)) \text{ if and only if } x \wedge a \in J \Leftrightarrow y \wedge a \in J \text{ for any } a \in S$$

is the largest PJP congruence containing J as a class. Now we have the following result.

Theorem 6.5.2 *Let S be a distributive PJP-semilattice and let J be a kernel ideal of S. Then*

$$\psi(J) = \bigcap\{\psi(P) \mid P \text{ is a minimal prime ideal of } S \text{ and } J \subseteq P\}.$$

Proof. Suppose $x, y \in S$ with $x \equiv y(\psi(J))$ and let $x \in P$. Then by Theorem 2.5.7 $x \wedge z \in J$ for some $z \in S \setminus P$. Hence $y \wedge z \in J$ and so $y \in P$. By symmetry we can prove that $y \in P$ implies $x \in P$. Thus $x \in P$ if and only if $y \in P$. Hence $x \equiv y(\psi(P))$. Therefore, $\psi(J) \subseteq RHS$. But we have $J = \bigcap\{P \mid P$ is a minimal prime ideal of S and $J \subseteq P\}$ and so J is a congruence class of S modulo $\bigcap\{\psi(P) \mid P$ is a minimal prime ideal of S and $J \subseteq P\}$. Hence $\bigcap\{\psi(P) \mid P$ is a minimal prime ideal of S and $J \subseteq P\} \subseteq \psi(J)$. This completes the proof. □

Theorem 6.5.3 *Let S be a JP Stone semilattice and let J be a prime kernel ideal of S. Then the following are equivalent:*

(a) $x \equiv y(\psi(J))$;

(b) $x^{**} \wedge a^* = y^{**} \wedge a^*$ *for some* $a \in J$;

(c) $x \wedge (b \vee b^*) \wedge a^* = y \wedge (b \vee b^*) \wedge a^*$ *for some* $b \in S$ *and* $a \in J$.

Proof. (a) $\Rightarrow$ (b). $x \equiv y(\psi(J))$. Then either $x, y \in J$ or $x, y \notin J$. Suppose $x, y \in J$. Since J is a kernel ideal of S, we have $(x^* \wedge y^*)^* \in J$. Now

$$x^{**} \wedge [(x^* \wedge y^*)^*]^* = x^{**} \wedge (x^* \wedge y^*)^{**} = x^{**} \wedge x^* \wedge y^* = 0.$$

Similarly, $y^{**} \wedge [(x^* \wedge y^*)^*]^* = 0$. Hence $x^{**} \wedge a^* = y^{**} \wedge a^*$ where $a = (x^* \wedge y^*)^* \in J$.

Now suppose $x, y \notin J$. Since J is prime ideal, $x \wedge y \notin J$. Hence $(x \wedge y)^* \in J$. Now

$$x^{**} \wedge ((x \wedge y)^*)^* = x^{**} \wedge (x \wedge y)^{**} = x^{**} \wedge y^{**} = y^{**} \wedge ((x \wedge y)^*)^*.$$

Thus (b) holds.

(b) $\Rightarrow$ (c). Since S is a JP Stone semilattice, we have $x \vee x^*$ and $y \vee y^*$ exist for any $x, y \in S$. Also $x \vee x^*, y \vee y^* \in D(S)$, the dense set of S. Hence $(x \vee x^*) \wedge (y \vee y^*) \in D(S)$ as $D(S)$ is a filter. Hence $(x \vee x^*) \wedge (y \vee y^*) = b \vee b^*$ for some $b \in S$. Now for some $a \in J$ we have

$$\begin{aligned} x \wedge (b \vee b^*) \wedge a^* &= (x^{**} \wedge (x \vee x^*)) \wedge (b \vee b^*) \wedge a^* \\ &= x^{**} \wedge ((x \vee x^*) \wedge (b \vee b^*)) \wedge a^* \\ &= x^{**} \wedge (b \vee b^*) \wedge a^* \\ &= y^{**} \wedge (b \vee b^*) \wedge a^* \\ &= y \wedge (b \vee b^*) \wedge a^*. \end{aligned}$$

Thus (c) holds.

(c) $\Rightarrow$ (b). Suppose $x \wedge (b \vee b^*) \wedge a^* = y \wedge (b \vee b^*) \wedge a^*$ for some $b \in S$ and $a \in J$. Then $(x \wedge (b \vee b^*) \wedge a^*)^{**} = (y \wedge (b \vee b^*) \wedge a^*)^{**}$ so that $x^{**} \wedge a^* = y^{**} \wedge a^*$. Hence (b) holds.

(b) ⇒ (a). Let $x \wedge a \in J$ for any $a \in S$. If $a \in J$, then $y \wedge a \in J$. If $a \notin J$, then $a^* \in J$ as J is a prime ideal. Since J is a kernel ideal, $(x \wedge a)^{**} \in J$. Now

$$\begin{aligned} (x \wedge a)^{**} &= x^{**} \wedge a^{**} \in J \\ &\Rightarrow x^{**} \wedge (a^*)^* \in J \\ &\Rightarrow y^{**} \wedge (a^*)^* \in J \\ &\Rightarrow (y \wedge a)^{**} \in J \\ &\Rightarrow y \wedge a \in J. \end{aligned}$$

Similarly, if $y \wedge a \in J$, then $x \wedge a \in J$. Hence (a) holds. □

Let J be an ideal of a PJP semilattice S. Define

$$J_* = \{x \in S \mid x \geqslant a^* \text{ for some } a \in J\}.$$

If S is a pseudocomplemented lattice, then by Cornish [**9**], J_* is a filter of S. But if S is a PJP-semilattice, then we do not know whether J_* is a filter or not. Now we have the following result.

Lemma 6.5.4 *Let* S *be a JP Stone semilattice and* J *be a kernel ideal of* S*, then* J_* *is a filter.*

Proof. By the definition, J_* is an up-set. Let $x, y \in J_*$. Then $x \geqslant a^*$ and $y \geqslant b^*$ for some $a, b \in J$. Since J is a kernel ideal we have $b^{**} \in J$ and since S is a JP Stone semilattice we have $x \wedge y \geqslant a^* \wedge b^* = (a \vee b^{**})^*$. Now $a \vee b^{**} \in J$ as J is an ideal and $a \vee b^{**}$ exists. Hence $x \wedge y \in J_*$. Therefore J_* is a filter. □

Recall that if F is a filter, then

$$x \equiv y(\Theta(F)) \text{ if and only if } x \wedge f = y \wedge f \text{ for some } f \in F.$$

Lemma 6.5.5 *Let* S *be a JP Stone semilattice and J is a kernel ideal of S. Then* $\ker(\Theta(J_*)) = J$.

Proof. Let $x \in \ker(\Theta(J_*))$. Then $x \equiv 0(\Theta(J_*))$. Thus $x \wedge f = 0 \wedge f = 0$ for some $f \in J_*$. This implies $x \leqslant f^*$ for some $f \geqslant a^*$ where $a \in J$. Hence $x \leqslant a^{**}$ for some $a \in J$. Since J is a kernel ideal we have $a^{**} \in J$ and hence $x \in J$. Therefore $\ker(\Theta(J_*)) \subseteq J$.

Conversely, let $x \in J$. Then $x^{**} \in J$. Since $x^* \geqslant x^{***}$, we have $x^* \in J_*$. Now $x \wedge x^* = 0 = 0 \wedge x^*$ implies $x \in \ker(\Theta(J_*))$. Therefore $J \subseteq \ker(\Theta(J_*))$. □

For any ideal J of a JP-semilattice S and $a \in S$, define

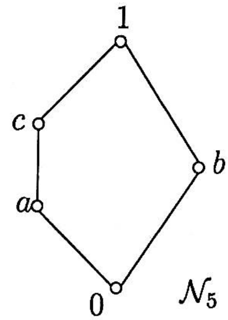
$$J_a = \{x \in S \mid a \wedge x \in J\}.$$

Clearly, $J \subseteq J_a$ and $a \in J$ implies $J_a = S$. For any JP-semilattice S and $a \in S$ the set J_a may not be an ideal of S. For, if we consider the pentagonal lattice $\mathcal{N}_5$ (see Figure 6.3) as a JP-semilattice and $J = (a]$, then $J_c = \{0, a, b\}$ which is not an ideal.

Now we have the following result.

Lemma 6.5.6 *Let* S *be a distributive JP-semilattice and J be an ideal of S. Then for any $a \in S$ we have J_a is an ideal of S.*

Proof. Since J is an ideal, J_a is down-set. Let $x, y \in J_a$ with $x \vee y$ exists. Then $(x \vee y) \wedge a = (x \wedge a) \vee (y \wedge a) \in J$. Hence $x \vee y \in J_a$. Thus J_a is an ideal of S. □



$\mathcal{N}_5$

FIGURE 6.3

By the definition it is clear that if $a \geqslant b$, then $J_a \subseteq J_b$. Define

$$D(J) = \{a \in S \mid J_a = J\}.$$

Clearly, $D((0]) = D(S)$, the dense set. Recall that the congruence $\psi(J)$ is defined by

$$x \equiv y(\psi(J)) \text{ if and only if } x \wedge i \in J \Leftrightarrow y \wedge i \in J.$$

We have the following result.

Lemma 6.5.7 *Let* S *be a JP Stone semilattice and* J *be a kernel ideal of* S. *Then*

(a) $D(J)$ *is a filter;*

(b) Coker $\psi(J) = D(J)$.

Proof. (a) Let $x \in D(J)$ and $x \leqslant y$. Then $J_x = J$ and $J_y \subseteq J_x = J$. But $J \subseteq J_y$ is trivial. Thus $J_y = J$ and hence $y \in D(J)$. Now let $x, y \in D(J)$. Then $J_x = J_y = J$. Let $a \in J_{x \wedge y}$. Then $x \wedge y \wedge a \in J$ and hence $y \wedge a \in J_x = J$. Consequently, $a \in J_y = J$. Hence $J_{x \wedge y} \subseteq J$. But $J \subseteq J_{x \wedge y}$ is trivial. Thus $J_{x \wedge y} = J$ and hence $x \wedge y \in D(J)$. This implies $D(J)$ is a filter.

(b) Let $x \equiv 1(\psi(J))$ and let $a \in J_x$. Then $a \wedge x \in J$ and $a \wedge x \equiv a(\psi(J))$. Thus $a \in J$. Hence $x \in D(J)$. This implies that $\mathrm{Coker}(\psi(J)) \subseteq D(J)$. Now let $x \in D(J)$. Then $J_x = J$. This implies $x \wedge a \in J$ if and only if $a = 1 \wedge a \in J$ for any $a \in S$. Hence $x \equiv 1(\psi(J))$. Thus $x \in \mathrm{Coker}(\psi(J))$. □

Theorem 6.5.8 *Let S be a JP Stone semilattice and J be a prime ideal of S. Then the following are equivalent.*

(a) *J is a kernel ideal;*

(b) $\psi(J) = \Theta(D(J))$;

(c) $D(J) = D(S) \vee J_*$;

(d) $D(S) \subseteq D(J)$.

Proof. (a) ⇒ (b). Let $x \equiv y(\psi(J))$. Since J is a prime ideal, by Theorem 6.5.3, we have $x \wedge (b \vee b^*) \wedge a^* = y \wedge (b \vee b^*) \wedge a^*$, for some $b \in S$ and $a \in J$. Now $a \in J$ implies $a^* \equiv 1(\psi(J))$, and since $(b \vee b^*)^{**} \wedge a^* = 1^{**} \wedge a^*$ for any $a \in J$, by Theorem 6.5.3 we have, $b \vee b^* \equiv 1(\psi(J))$. This implies $(b \vee b^*) \wedge a^* \equiv 1(\psi(J))$. Thus $(b \vee b^*) \wedge a^* \in \mathrm{Coker}(\psi(J))$. Hence $x \equiv y\Theta(\mathrm{Coker}(\psi(J)))$. Hence by Lemma 6.5.7, $x \equiv y(\Theta(D(J)))$. Conversely, suppose $x \equiv y(\Theta(D(J)))$. That is, $x \equiv y\Theta(\mathrm{Coker}(\psi(J)))$. Then $x \wedge f = y \wedge f$ for some $f \in \mathrm{Coker}(\psi(J))$. Now

$$x = x \wedge 1 \equiv x \wedge f = y \wedge f \equiv y \wedge 1 = y(\psi(J)).$$

Thus (b) holds.

(b) ⇒ (c). Let $x \in D(S) \vee J_*$. Then $x = a \wedge b$ for some $a \in D(S)$ and $b \in J_*$. This implies $x^{**} = (a \wedge b)^{**} = a^{**} \wedge b^{**}$ where $a^{**} = 1$ and $b \geqslant c^*$for some $c \in J$. Thus $x^{**} = b^{**} \geqslant c^*$ and hence $x^{**} \wedge c^* = 1 \wedge c^*$. Consequently, by Theorem 6.5.3

we have $x \equiv 1(\psi(J))$. Hence $x \equiv 1(\Theta(D(J)))$. This implies $x \in D(J)$, that is $D(S) \vee J_* \subseteq D(J)$.

Conversely, let $x \in D(J) = \text{Coker}(\psi(J))$ (by Lemma 6.5.7). Then $x \equiv 1(\psi(J))$. Hence by Theorem 6.5.3, $x \wedge (b \vee b^*) \wedge a^* = 1 \wedge (b \vee b^*) \wedge a^* = (b \vee b^*) \wedge a^*$ for some $b \in S$ and $a \in J$. This implies $x \geqslant (b \vee b^*) \wedge a^*$ where $b \vee b^* \in D(S)$ and $a^* \in J_*$. So, $x \in D(S) \vee J_*$ and hence $D(J) \subseteq D(S) \vee J_*$.

(c) $\Rightarrow$ (d) is trivial.

(d) $\Rightarrow$ (a). Let $x \in J$. Since $x \vee x^* \in D(S) \subseteq D(J)$ (by (d)), we have $J_{x \vee x^*} = J$. Since $x^{**} \wedge (x \vee x^*) = x \in J$, we have $x^{**} \in J_{x \vee x^*} = J$. Hence J is a kernel ideal of S. □

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